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Extension of Complex Algebra to Three-Fold Space

T. Proctor Hall

Cystopteris fragilis Bernh. Union, Page, Pottawattamie, and Fremont counties. Rich woods; common.

Onoclea struthiopteris Hoffm. Decatur county. Rich woods; infrequent.

OPHIOGLOSSACEÆ.

Botrychium virginianum Swartz. Pottawattamie county. Rich woods; frequent.

EXTENSION OF COMPLEX ALGEBRA TO THREE-FOLD SPACE.

BY T. PROCTOR HALL.

Taking rectangular coördinates in a plane, let x, y, be unit vectors along the axes of X, Y, respectively; and let A be any unit vector from O in the plane, making an angle a with X.

Let i be a rotor such that i^n rotates any vector, A, through n. 90° in the positive direction. Then

$$i\,x = y.$$
$$i^2 \mathrm{A} = -\mathrm{A}$$
$$\therefore i^2 = -1.$$

Equating vectors from O:

$$\mathrm{A} = x \cos a + y \sin a.$$
$$= (\cos a + i \sin a)\, x.$$
$$= e^{ia}\, x, \text{ (by expansion in series).}$$

A vector (A) is here considered to be composed of three distinct components or factors; a unit direction (x), a length (which, for the sake of simplicity, is here considered unity), and a rotor ($\cos a + i \sin a$, or e^{ia}) which has rotated the vector from unit position (x) to any other position (A) in the plane.

The product of two vectors is

$$\mathrm{A}_1\,\mathrm{A}_2 = (\cos a_1 + i \sin a_1)(\cos a_2 + i \sin a_2)\, x.x.$$
$$= [\cos (a_1 + a_2) + i \sin (a_1 + a_2)]\, x.$$
$$= e^{\,i(a_1 + a_2)}\, x.$$

Since x is unity in every one of its capacities, $x\,x = x$, as given above.

The unit vector, x, is a factor of every term of this algebra, and may be dropped, leaving an algebra of rotors only, which has the laws of operation of ordinary algebra, and which combines with it to form an algebra of tensors and rotors—the ordinary complex algebra.

The kinds of number involved in this algebra are:

(1) reals, a, b, c, ...

(2) plane imaginaries, *i*a, *i*b, ...

(3) the plane complex, $z=a+i\,b$.

Next let the *xy*-plane be the equatorial plane of a sphere of which Z is the pole. Let the power of the rotor i be extended so as to rotate any vector, whether in the *xy*-plane or not, about the *z*-axis. Let j be a new rotor, such that j^m rotates any vector through m. 90° in a direction from the plane of x, y, toward the pole z.

By means of these two rotors i^n, j^m, a vector may be turned from the unit position (x) to any other position (A); and the order of the rotations is indifferent.

$$j\,i^n\,x = i^n\,j\,x = j\,x = z.$$

$$j^2\,A = -A$$

$$\therefore j^2 = -1.$$

It follows that j^m may be expressed in the forms $\cos b + j \sin b$, and e^{jb}. Any unit vector, A, is therefore of the form

$$A = (\cos a + i \sin a)(\cos b + j \sin b)\,x$$
$$= e^{ia+jb}\,x.$$

From either of these forms the product or the quotient of two vectors is evident.

Dropping x, as before, and introducing tensors we obtain a tensor-rotor algebra which, *when the i and j binary factors are kept separate*, has the laws of operation of common algebra, and has many of the advantages of a vector algebra without its limitations.

The most general quantity in this algebra is the double complex

$$(a+i\,b)\,(c+j\,d),$$

in which a, b, c, d, are connected by one relation. The double complex may be expressed in the form

$$(a+i\,b)\left(1+j\frac{c}{\sqrt{a^2+b^2}}\right),$$

which is identical with

$$a+i\,b+j\,c.$$

But unfortunately in the latter form it does not obey the laws of common algebra, except in addition, subtraction, and multiplication by reals.

The double complex

$$a+i\,b+j\,c$$

is related to points in three-fold space in the same way that the plane complex

$$a+i\,b$$

is related to points in a plane, and in the form

$$(a+i\,b)\left(1+j\frac{c}{\sqrt{a^2+b^2}}\right),$$

or in the general forms

$$(a+i\,b)\,(c+j\,d),\ e^{r+ia+jb},$$

it may be treated as an ordinary algebraic quantity.

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A REVIEW OF THE CERCOPIDÆ OF NORTH AMERICA NORTH OF MEXICO.

BY E. D. BALL.

The family cercopidæ, though of world-wide distribution, has comparatively few representatives within our borders, and those few have been but imperfectly known, the literature on the subject being scattered and fragmentary, the generic references often incorrect, and the specific determinations, owing to the extreme variability in color of some species, and the striking similarity of color between others, rendered very questionable. Scarcely one of the more common forms but what has been referred to under at least four different genera and several have more than that number of specific names.

This paper recognizes twenty species as included within our fauna, of which Say described six, Fitch three, Germar one, Uhler four, two were introduced from Europe, and four are here described as new; besides these there have been twenty more described, which have been referred to the first twenty as synonyms or varieties.

Except for the isolated descriptions and a few lists, the first systematic work done on the American species was Uhler's article on the family in the Standard Nat. Hist. (1884).

In 1889 Provancher, in his Hemip. du Canada, published the first synopsis of the group; he divided the family into