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A Physical Representation of the Summation of Certain Types of Series

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A PHYSICAL REPRESENTATION OF THE SUMMATION OF CERTAIN TYPES OF SERIES.

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Most of us are on the lookout for concrete illustrations of abstract ideas. To be able to visualize a mathematical process is to many of us a step toward the better understanding of that process. The following brief discussion, although having no pretentions to absolute originality, is offered as a physical illustration of the summation of certain simple geometric series.

In the accompanying four diagrams of figure 17 we have a series of four sketches of combinations of simple machines. In each case a weightless platform supporting a man of weight W is suspended from the point a of the weightless, frictionless lever Fab. The fulcrum of this lever is at F. The point of application of the force f which the man exerts, in the manner shown by the arrow in each case, is at the point b. The force f is transmitted to the point b by the frictionless fixed pulleys P_1 and P_2 in diagrams *1* and *3*, and by the frictionless fixed pulley P_1 in diagrams *2* and *4*. The mechanical advantage of the lever is considered to be m, in order to make the problem general, where m is greater than unity in diagrams *1* and *2*, and less than unity in diagrams 3 and 4.

The problem in each case is to determine the force f that will place the system in equilibrium. This is of course a very simple physical problem. However, there are at least two ways of approaching the solution of the problems, and it is in the results from these two methods of approach that we find the ideas involved in this paper.

Consider diagram *1*. We can solve this problem algebraically by equating the weight of the man plus the reaction of the force f, which he exerts, to the upward force f multiplied by m, the mechanical advantage of the machine, Fab. This gives us

$$W + f = mf \qquad (1)$$

or

$$f = W / (m - 1) \qquad (2)$$

The second manner of attacking the problem is in an approach by an infinite series. The man can be considered as in readiness to exert the proper force, and he indulges in the following reasoning. First he knows that if his weight is W he must pull with a force of W/m in order to support himself. But this force will create an additional thrust on the platform

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of W/m, and so he must exert an added force of W/m^2 to overcome this. This added force in turn causes an extra thrust on the platform of W/m^2, and so he must exert an additional pull

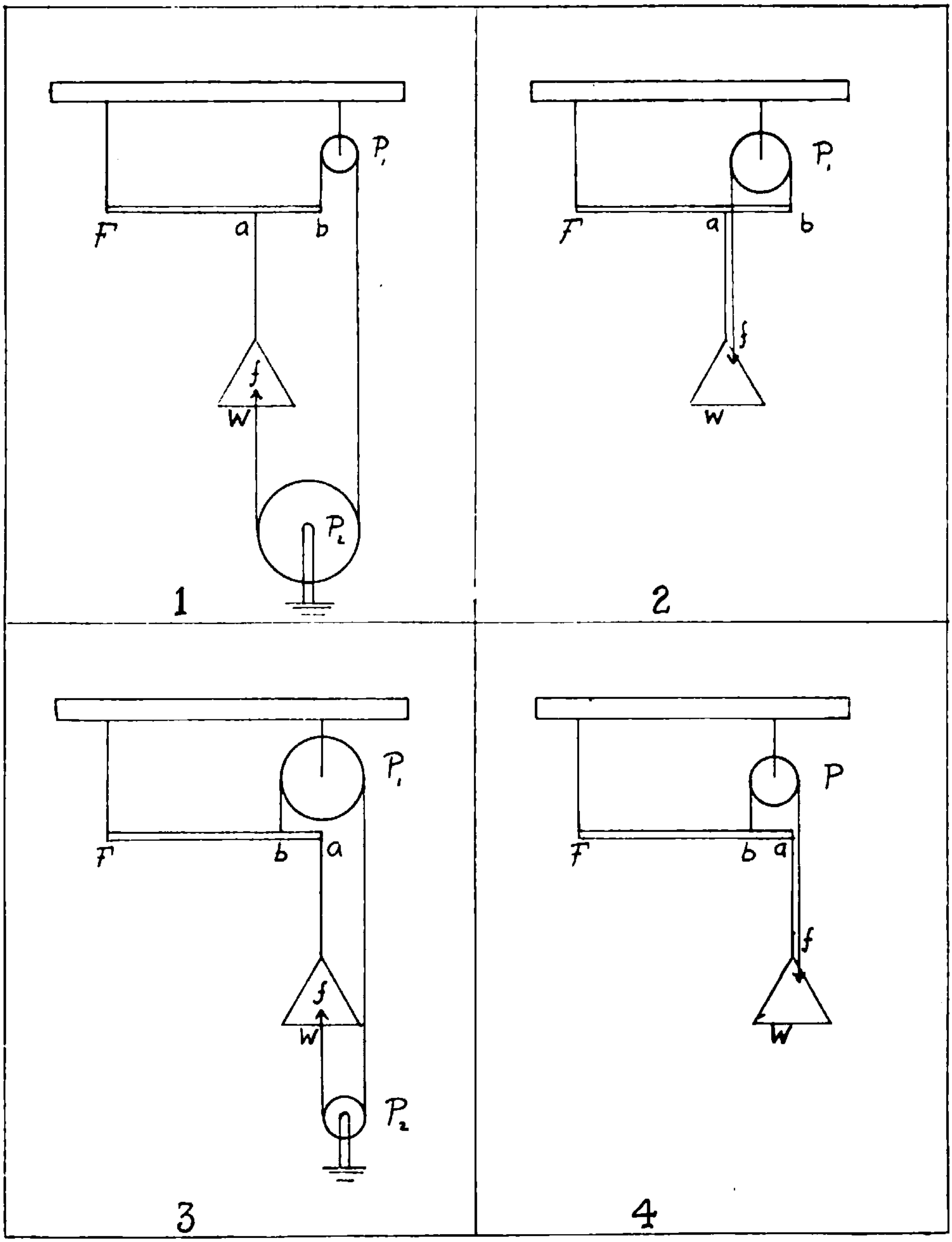

FIG. 17.

of W/m^3. Without going further we see that the total force he must exert is represented by the infinite series

$$f = W/m + W/m^2 + W/m^3 + W/m^4 + \dots\dots\dots \quad (3)$$

By equating equations (2) and (3), and cancelling the W's, we obtain the following series:

$$1/m+1/m^2+1/m^3+1/m^4+\ldots\ldots=1/(m-1) \quad (4)$$

This is the correct summation of the series and the series is convergent, since we assumed m to be greater than unity. Hence the two methods of approach are equally good, and both lead to the correct answer.

It is a matter of some interest to speculate as to which method would be used by a man on an actual platform of this kind. It seems that the algebraic method would certainly not be used. Either his muscles would gradually exert tension in the manner represented by equation (3), or else he would approach the correct force by an oscillatory muscular pull, the oscillations gradually getting smaller and smaller until the correct force f has been reached. This type of series will be found in the discussion below. Such a problem as this, aside from these psychological aspects, cannot help but be of some value to a teacher of elementary physics or mathematics in that it gives a tangible meaning to an infinite series. Of course there are many other problems that will illustrate this particular point.

Consider now the arrangement shown in diagram 2. The reaction is now opposite in direction to W, and the algebraic solution is given by

$$W-f=mf \quad (5)$$

or

$$f=W/(m+1) \quad (6)$$

The other method of obtaining f is somewhat similar to the preceding one. A first pull of W/m is necessary. This pull, however, decreases the load by W/m, and therefore the tension in the rope must be slacked by an amount W/m^2. This in turn adds to the thrust on the platform of W/m^2, and an additional pull of W/m^3 must be exerted. In short the force is determined by the following series:

$$f=W/m-W/m^2+W/m^3-W/m^4+\ldots\ldots \quad (7)$$

Equating equations (6), and (7), and cancelling the W's we obtain

$$1/m-1/m^2+1/m^3-1/m^4+\ldots=1/(m+1) \quad (8)$$

As long as m is greater than unity this is a convergent series and is correctly summed. Here again then we have the two methods of attack leading in one case to a simple answer, and in the other to an infinite converging series, the series being a correct representation of the algebraic result.

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Turning now to the arrangement shown in diagram *3* we arrive, by the two methods of approach to equations identical with equations (2) and (3), respectively. However, in this case, m is less than unity, so that equation (2) leads to a negative value for f, which means that no positive pull will yield equilibrium, and hence that the physical solution is impossible. The discussion of the two cases when m equals unity is obvious. The series (3) becomes now a divergent series, and cannot be summed. Here then the second method of approach fails to yield any result, whereas the algebraic method does yield a result although it has no physical reality. A glance at the divergent series (3) shows that the man is forced to exert a greater and greater pull, which situation would no doubt correspond with the facts in an actual situation. But the more the man pulls the more certain he is of falling to the ground.

The most interesting case, however, is the last one, represented in diagram *4*. Here we arrive by the two methods of approach at equations (6) and (7), respectively. The algebraic solution (6), is perfectly definite and physically possible, even when m is less than unity. However, the series (7) is divergent, when m is less than unity, and ordinarily considered it has no sum. A glance at the series will show that the man first pulls with a certain force, then relaxes the tension by a greater amount, next pulls with a still greater force, and so on, pulling and relaxing with forces ever increasing in magnitude. It is evident that this latter method would not be the actual one, and it again becomes a matter of interesting speculation as to with what rhythmical, or other, muscular efforts the man arrives at the correct force for equilibrium. It is possible that the terms of the divergent series (7) could be grouped in a certain fashion to yield a convergent series which would have the correct sum.

It is evident that the second method of analysis of the problem succeeds in cases shown in diagrams 1 and 2, for all values of m greater than unity, fails in *3*, for m is less than unity, but fails because the solution is impossible, and fails utterly in *4* for values of m less than unity, although the solution of the problem is possible and perfectly definite.

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