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THE TAXONOMY OF ALGEBRAIC SURFACES

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The problems of taxonomy are problems of order. Any discrete set can be arranged in linear order but it does not follow that any linear order is satisfactory. The separation of natural neighbors may be inevitable. Examples are Linnaeus' botanical classification, or the arrangement of logical classes (*abcd*, $\bar{a}bcd$, ----) where a natural arrangement applies in general surfaces of connectivity greater than one, or *n*-dimensional space. Any number of interrelations of a discrete finite set can be indicated by a three dimensional model where the elements are points and the relations say colored lines, as in a Cayley color group abstracted from a surface.

When the set of elements is infinite the only apparent arrangement is in a space of *n* dimensions. For the cubic surfaces every real point in a space of nineteen dimensions corresponds to a real cubic.

To condense this two fundamental schemes are used, either a classification by projections (real or complex) or by a birational reduction.

The literature of the cubic surface contains three different projective attacks. For the 'general form' Cayley[1] gave a four parameter form (*lmnp*) having in general twenty-seven lines rational in (*lmnp*). Schläfli[2] reduced the set to a four parameter double trihehedral form. Rodenburg[3] took the reduction to sum of five cubes and classified by the coefficients.

The comparison of these forms is rendered difficult by the fact that one of two algebraid equations (neither of which has been explicitly written) is encountered: a quintic for the pentahedron of the five cubes, discussed by Clebsch[4] and the well known equation of order 27 for the lines.

The groups are in general of order 120 and 51840 respectively. By a theorem of Jordan the adjunction of the quintic roots is ineffective as to the resolution of the line equation, and the ad-

1 Phil. Trans., 1869.
2 Phil. Trans., 1863.
3 Math. Ann., 14.
4 Crelle, 59.

junction of the lines is similarly ineffective as regards the pentahedron.

The desirable connection would be given by a four parameter form which has rational lines and rational pentahedron in a domain as simple as possible.

In discussing this problem the question of a group of transformations of the surface into itself is met at the outset. Some easily attainable results are: —

1. If the group is continuous of non-multiplicative type the surface is ruled.
2. If continuous of multiplicative type is is singular. A list can be made and the examples correlated with Schläfli's classes.
3. For the general case, since the pentahedron must transform into itself there must be equalities among the coefficients of Rodenburg or the group is the identity. In this case we should have the truly general form.

The possible groups are of order 120 as in Clebsch's diagonal surface, 24, 12, 6 or 2. The Clebsch surface is not contained in Cayley's form nor are some of the others, the trouble being that Cayley's reduction of the general quadric is a special one. The diagonal surface is rational in the domain ($\sqrt{5}$) for both lines and pentahedron.

Since Klein[5] takes small distortions of the diagonal surface as the general type, whereas Cayley's form is finitely removed, some doubt may be expressed as to the equivalence of the forms thus obtained with the general one. Further his remark that a model should be symmetrical is unfortunate since the general cubic has no transformation into itself and hence it is in no way symmetrical.

The problem may be attacked by assuming the pentahedron and a triangle which involve the adjunction of cubic and quadratic irrationalities. The reduction of Schläfli's form requires adjunction of the roots of a quartic and cubic.

These adjunctions are possible by rationalizing conics and cubics. The detailed formulæ are naturally complicated and restrictions arise which differ according to the group of transformations into itself. Finally a complete exploration of the four dimensional space is required to show that the forms approach every point.

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5 Math. Ann. 6.