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A SHORT SOLUTION OF THE DIOPHANTINE EQUATION $2X^4 - Y^4 = Z^2$

J. S. TURNER

Let X, Y, Z be any positive integral solutions of this equation, and let H be the G.C.D. of X, Y . Then $X = Hx, Y = Hy, Z = H^2z$, and

(1) $2x^4 - y^4 = z^2$, where x, y, z are odd and co-prime in pairs. Hence it suffices to find primitive solutions x, y, z .

From any primitive solution except $x = y = z = 1$, a primitive solution with smaller x is derived as follows:

From (1),

$$\left[\frac{y^2 + z}{2} \right]^2 + \left[\frac{y^2 - z}{2} \right]^2 = x^4, \text{ or } \left[\frac{z + y^2}{2} \right]^2 + \left[\frac{z - y^2}{2} \right]^2 = x^4,$$

hence

(2) $x^2 = l^2 + m^2$, where l, m are co-prime positive integers, not both odd, with one of the following pairs of equations,

$$(3) \quad \frac{1}{2}(y^2 + z) = l^2 - m^2, \quad \frac{1}{2}(y^2 - z) = 2lm,$$

$$(4) \quad \frac{1}{2}(y^2 + z) = 2lm, \quad \frac{1}{2}(y^2 - z) = l^2 - m^2,$$

$$(5) \quad \frac{1}{2}(z + y^2) = l^2 - m^2, \quad \frac{1}{2}(z - y^2) = 2lm,$$

$$(6) \quad \frac{1}{2}(z + y^2) = 2lm, \quad \frac{1}{2}(z - y^2) = l^2 - m^2.$$

(4) becomes (3) on changing the sign of z , (5) so reduces on changing the sign of m , and (6) reduces to (4) on interchanging l, m . Hence it is only necessary to consider (3) and to allow l, m to be interchanged, or z, m to become negative. From (3),

(7) $y^2 = l^2 + 2lm - m^2, z = l^2 - 2lm - m^2$, hence l is odd. Also $(l + m)^2 - y^2 = 2m^2$, and the G.C.D. of $l + m + y, l + m - y$ is 2, hence either $l + m + y = 2f^2, l + m - y = 4g^2$ or $l + m + y = 4g^2, l + m - y = 2f^2$, where f, g , are co-prime positive integers and f is odd. Therefore

$$(8) \quad l + m = f^2 + 2g^2, y = \pm (f^2 - 2g^2), m = 2fg.$$

From (2),

(9) $l = r^2 - s^2, m^2 = 2rs, x = r^2 + s^2$, where r, s are co-prime positive integers, not both odd. From (7), (8), (9),

$$(10) \quad z = 2(r^2 - s^2)^2 - (f^2 + 2g^2)^2,$$

(11) $fg = rs, f^2 + 2g^2 - 2fg = r^2 - s^2$. The same equations (11) are obtained from (4), (5), (6), except that (5) leads to negative values of f and r .

Eliminate r from (11) and express the result as a quadratic in f .

Thus $f^2(g^2 - s^2) + 2fgs^2 - s^4 - 2s^2g^2 = 0$, $f(g^2 - s^2) = -gs^2 + \sqrt{2g^2 - s^4}$. Hence

$$(12) \quad 2g^4 - s^4 = t^2$$

(13) $rg - fs + gs = \pm t$. If $t = 0$, then from (12) $g = s = 0$, hence $m = 0$, $x = y = z = 1$, contrary to hypothesis; hence t may be taken > 0 . Let h be the G.C.D. of g, s , and write

$$(14) \quad g = hx_1, s = hy_1, t = h^2z_1, \text{ then}$$

(15) $2x_1^4 - y_1^4 = z_1^2$, where x_1, y_1, z_1 are odd and co-prime in pairs. Also $x = r^2 + s^2 = (f - g)^2 + g^2 + 2s^2 > g^2 \geq x_1 > 0$.

Unless $x_1 = y_1 = z_1 = 1$, the process can be repeated; also this solution must ultimately be reached since there cannot be an infinite sequence of decreasing x 's.

To reverse the process, suppose first that $x_1 = y_1 = z_1 = 1$. Then from (14) and (11) we have $g = s, f = r, 2rs = 3s^2$. Now r is prime to s , hence $s = 2, r = 3, x = 13, y = 1, z = 239$. Next suppose that $x_1 \pm y_1$; then from (14), (11) and (13), $fx_1 = ry_1, fy_1 - rx_1 = h(x_1y_1 \pm z_1)$, hence $r(y_1^2 - x_1^2) = h(x_1y_1 \pm z_1)$. Choose the least positive integral h for which $h(x_1y_1 \pm z_1) / (y_1^2 - x_1^2)$ is an integer λ , then for each determination of h and $\lambda, x_1, y_1, h, \lambda$ are co-prime in pairs. Then $r = \lambda x_1, f = \lambda y_1$, and x, y, z are given by (9), (8), (10). It readily follows that z is prime to x and y .

The next three solutions of (1) in order of magnitude are $x = 1525, y = 1343, z = 2750257$; $x = 2165017, y = 2372159, z = 3503833734241$; $x = 42422452969, y = 9788425919, z = 2543305831910011724639$.

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