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A SHORT SOLUTION OF THE DIOPHANTINE EQUATION $2X^4 - Y^4 = Z^2$

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Let X, Y, Z be any positive integral solutions of this equation, and let H be the G.C.D. of X, Y. Then $X = Hx$, $Y = Hy$, $Z = H^2z$, and

(1) $2x^4 - y^4 = z^2$, where x, y, z are odd and co-prime in pairs. Hence it suffices to find primitive solutions x, y, z.

From any primitive solution except $x = y = z = 1$, a primitive solution with smaller x is derived as follows:

From (1),

$$\left[\frac{y^2+z}{2}\right]^2 + \left[\frac{y^2-z}{2}\right]^2 = x^4, \text{ or} \left[\frac{z+y^2}{2}\right]^2 + \left[\frac{z-y^2}{2}\right]^2 = x^4,$$

hence

(2) $x^2 = l^2 + m^2$, where l, m are co-prime positive integers, not both odd, with one of the following pairs of equations,

(3) $\frac{1}{2}(y^2+z) = l^2 - m^2$, $\quad \frac{1}{2}(y^2-z) = 2lm$,
(4) $\frac{1}{2}(y^2+z) = 2lm$, $\quad \frac{1}{2}(y^2-z) = l^2 - m^2$,
(5) $\frac{1}{2}(z+y^2) = l^2 - m^2$, $\quad \frac{1}{2}(z-y^2) = 2lm$,
(6) $\frac{1}{2}(z+y^2) = 2lm$, $\quad \frac{1}{2}(z-y^2) = l^2 - m^2$.

(4) becomes (3) on changing the sign of z, (5) so reduces on changing the sign of m, and (6) reduces to (4) on interchanging l, m. Hence it is only necessary to consider (3) and to allow l, m to be interchanged, or z, m to become negative. From (3),

(7) $y^2 = l^2 + 2lm - m^2$, $z = l^2 - 2lm - m^2$, hence l is odd. Also $(l+m)^2 - y^2 = 2m^2$, and the G.C.D. of $l+m+y$, $l+m-y$ is 2, hence either $l+m+y = 2f^2$, $l+m-y = 4g^2$ or $l+m+y = 4g^2$, $l+m-y = 2f^2$, where f, g, are co-prime positive integers and f is odd. Therefore

(8) $l + m = f^2 + 2g^2$, $y = \pm(f^2 - 2g^2)$, $m = 2fg$.

From (2),

(9) $l = r^2 - s^2$, $m^2 = 2rs$, $x = r^2 + s^2$, where r, s are co-prime positive integers, not both odd. From (7), (8), (9),

(10) $z = 2(r^2 - s^2)^2 - (f^2 + 2g^2)^2$,

(11) $fg = rs$, $f^2 + 2g^2 - 2fg = r^2 - s^2$. The same equations (11) are obtained from (4), (5), (6), except that (5) leads to negative values of f and r.

Eliminate r from (11) and express the result as a quadratic in f.

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Thus $f^2(g^2 - s^2) + 2fgs^2 - s^4 - 2s^2g^2 = 0$, $f(g^2 - s^2) = -gs^2 + \sqrt{2g^4 - s^4}$. Hence

(12) $2g^4 - s^4 = t^2$

(13) $rg - fs + gs = \pm t$. If $t = 0$, then from (12) $g = s = 0$, hence $m = 0$, $x = y = z = 1$, contrary to hypothesis; hence t may be taken > 0. Let h be the G.C.D. of g, s, and write

(14) $g = hx_1$, $s = hy_1$, $t = h^2z_1$, then

(15) $2x_1{}^4 - y_1{}^4 = z_1{}^2$, where x_1, y_1, z_1 are odd and co-prime in pairs. Also $x = r^2 + s^2 = (f - g)^2 + g^2 + 2s^2 > g^2 \geqq x_1 > 0$.

Unless $x_1 = y_1 = z_1 = 1$, the process can be repeated; also this solution must ultimately be reached since there cannot be an infinite sequence of decreasing x's.

To reverse the process, suppose first that $x_1 = y_1 = z_1 = 1$. Then from (14) and (11) we have $g = s$, $f = r$, $2rs = 3s^2$. Now r is prime to s, hence $s = 2$, $r = 3$, $x = 13$, $y = 1$, $z = 239$. Next suppose that $x_1 \pm y_1$; then from (14), (11) and (13), $fx_1 = ry_1$, $fy_1 - rx_1 = h(x_1y_1 \pm z_1)$, hence $r(y_1{}^2 - x_1{}^2) = hx_1(x_1y_1 \pm z_1)$. Choose the least positive integral h for which $h(x_1y_1 \pm z_1) / (y_1{}^2 - x_1{}^2)$ is an integer λ, then for each determination of h and λ, x_1, y_1, h, λ are co-prime in pairs. Then $r = \lambda x_1$, $f = \lambda y_1$, and x, y, z are given by (9), (8), (10). It readily follows that z is prime to x and y.

The next three solutions of (1) in order of magnitude are $x = 1525$, $y = 1343$, $z = 2750257$; $x = 2165017$, $y = 2372159$, $z = 3503833734241$; $x = 42422452969$, $y = 9788425919$, $z = 2543305831910011724639$.

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