

# Proceedings of the Iowa Academy of Science

---

Volume 43 | Annual Issue

Article 75

---

1936

## A Criterion for Prime Numbers

Carl W. Strom

*American Consulate General, Vancouver*

Copyright ©1936 Iowa Academy of Science, Inc.

Follow this and additional works at: <https://scholarworks.uni.edu/pias>

---

### Recommended Citation

Strom, Carl W. (1936) "A Criterion for Prime Numbers," *Proceedings of the Iowa Academy of Science*, 43(1), 263-265.

Available at: <https://scholarworks.uni.edu/pias/vol43/iss1/75>

This Research is brought to you for free and open access by the Iowa Academy of Science at UNI ScholarWorks. It has been accepted for inclusion in Proceedings of the Iowa Academy of Science by an authorized editor of UNI ScholarWorks. For more information, please contact [scholarworks@uni.edu](mailto:scholarworks@uni.edu).

A CRITERION FOR PRIME NUMBERS

CARL W. STROM

Let  $f_n$  denote the sum of the divisors of the positive integer  $n$  and  $f(n-a)$ ,  $a < n$ , similarly denote the sum of the divisors of the positive integer  $(n-a)$ . Then,

$$f_n = \sum_k (-1)^{k+1} \left[ f\left(n - \frac{3k^2 - k}{2}\right) + f\left(n - \frac{3k^2 + k}{2}\right) \right], \quad k = 1, 2, 3, \dots,$$

in which the sum on the right is to be extended to include all positive values of  $n - \frac{3k^2 - k}{2}$  and  $n - \frac{3k^2 + k}{2}$ , and  $f(n-n)$ , if it occurs is defined to be equal to  $n$ .

A necessary and sufficient condition that  $n$  be prime is that the sum on the right be equal to  $(n+1)$ .

A.

Consider the infinite product,  $S = (1-x)(1-x^2)(1-x^3)\dots(1-x^n)\dots$

Euler<sup>1</sup> obtained the following relations:

$$S = (1-x) - A_1 x^2, \quad A_1 = (1-x) + x(1-x)(1-x^2) + x^2(1-x)(1-x^2)(1-x^3) + \dots,$$

$$A_1 = (1-x^3) - A_2 x^5, \quad A_2 = (1-x^2) + x^2(1-x^2)(1-x^3) + x^4(1-x^2)(1-x^3)(1-x^4) + \dots,$$

It is easily shown by mathematical induction that

$$A_{n-1} = 1 - x^{2n+1} - A_n x^{3n+2},$$

$$A_n = (1-x^n) + x^n(1-x^n)(1-x^{n+1}) + x^{2n}(1-x^n)(1-x^{n+1})(1-x^{n+2}) + \dots$$

It follows by another mathematical induction that

$$S = 1 - x - x^2 + x^5 + x^7 + \dots + (-1)^n \left[ x^{-\frac{3n^2-n}{2}} + x^{\frac{3n^2+n}{2}} \right] + \dots$$

B.

Let  $z = x f_1 + x^2 f_2 + x^3 f_3 + \dots + x^n f_n + \dots$

This series is convergent in the interval  $0 \leq x < \frac{1}{2}$  by Cauchy's radical test, for

$$f_n \leq \sum_{r=1}^n r = \frac{n^2 + n}{2}, \quad \lim_{n \rightarrow \infty} \sqrt[n]{\frac{n^2 + n}{2}} = 1.$$

<sup>1</sup> Euleri, Opera Omnia, Comm. 175, 541.

<sup>2</sup> See Euleri, Opera Omnia, Comm. 175, 243, and 244. In these papers Euler obtained the law for the formation of the sums of the divisors of the positive integers and produced evidence, largely inductive, in its support. In the following, Euler's result is generalized and a proof of it is given that meets the requirements of modern function theory.

Therefore,

$$\lim_{n \rightarrow \infty} \sqrt[n]{x^n \int^n} \leq x < \frac{1}{2}, 0 \leq x < \frac{1}{2}.$$

The series of functions,

$$\frac{x}{1-x} + \frac{2x^2}{1-x^2} + \frac{3x^3}{1-x^3} + \dots + \frac{nx^n}{1-x^n} + \dots,$$

is uniformly convergent in the interval  $0 \leq x < \frac{1}{2}$  for the terms are respectively less than the corresponding terms of the convergent series,  $\sum_{n=1}^{\infty} \frac{n}{2^{n-1}}$ . The series of functions may be expanded into a multiple series as follows:

$$\begin{array}{cccccccc} x + x^2 + x^3 + x^4 & + x^5 + x^6 + x^7 + x^8 + x^9 + x^{10} & + & \dots & & & & \\ + x^2 & + 2x^4 & + 2x^6 & + 2x^8 & + 2x^{10} & + & \dots & \\ + 3x^3 & & + 3x^6 & & + 3x^9 & + & \dots & \\ + 4x^4 & & & + 4x^8 & & + & \dots & \\ + 5x^5 & & & & + 5x^{10} & + & \dots & \\ & & & & & & & + \dots \end{array}$$

In this form n is multiplied by the sum of all those powers of x whose indices are multiples of n.

Since the series of the sums of the rows is uniformly convergent, the multiple series may be summed by columns for values of x lying within the interval of uniform convergence as follows:

$$x + x^2(1+2) + x^3(1+3) + x^4(1+2+4) + x^5(1+5) + \dots$$

In this form each power of x is multiplied by the sum of all the positive integers of which the index of the power is an integral multiple.

Hence, the given series of functions equals

$$x \int 1 + x^2 \int 2 + x^3 \int 3 + \dots + x^n \int n + \dots = z.$$

Since the series of functions is uniformly convergent, it may be integrated term for term for values of x lying within the interval of convergence. Therefore,

$$\begin{aligned} \int -\frac{z \, dx}{x} &= \log(1-x) + \log(1-x^2) + \log(1-x^3) + \dots + \log(1-x^n) + \dots, \\ &= \log(1-x)(1-x^2)(1-x^3)\dots(1-x^n)\dots \\ &= \log\left(1-x-x^2+x^5+x^7+\dots(-1)^n \left[ x^{\frac{3n^2-n}{2}} + x^{\frac{3n^2+n}{2}} \right] + \dots\right), \text{ by A.} \end{aligned}$$

Differentiating:

$$z = \frac{-x \frac{d}{dx} \left( 1-x-x^2+x^5+x^7+\dots+(-1)^n \left[ x^{\frac{3n^2-n}{2}} + x^{\frac{3n^2+n}{2}} \right] + \dots \right)}{1-x-x^2+x^5+x^7+\dots+(-1)^n \left[ x^{\frac{3n^2-n}{2}} + x^{\frac{3n^2+n}{2}} \right] + \dots}$$

<sup>3</sup> This is not the full interval of convergence for this series or for those that follow, but it suffices for the purposes of this argument.

Since the series in the numerator is a power series, it may be differentiated term for term and the new series will be convergent for values of  $x$  lying within the interval of convergence of the original series. Then,

$$z = \frac{x + 2x^2 - 5x^5 - 7x^7 + \dots + (-1)^{n+1} \left[ \frac{3n^2-n}{2} x^{\frac{3n^2-n}{2}} + \frac{3n^2+n}{2} x^{\frac{3n^2+n}{2}} \right] + \dots}{1 - x - x^2 + x^5 + x^7 + \dots + (-1)^n \left[ x^{\frac{3n^2-n}{2}} + x^{\frac{3n^2+n}{2}} \right] + \dots}$$

Hence,

$$\begin{aligned} & [x \int 1 + x^2 \int 2 + x^3 \int 3 + \dots + x^n \int n + \dots] \cdot \\ & \left[ 1 - x - x^2 + x^5 + x^7 + \dots + (-1)^n \left( x^{\frac{3n^2-n}{2}} + x^{\frac{3n^2+n}{2}} \right) + \dots \right] = \\ & \left[ x + 2x^2 - 5x^5 - 7x^7 + \dots + (-1)^{n+1} \left( \frac{3n^2-n}{2} x^{\frac{3n^2-n}{2}} + \frac{3n^2+n}{2} x^{\frac{3n^2+n}{2}} \right) + \dots \right] \end{aligned}$$

The series on the left hand side are both absolutely convergent,  $0 \leq x < \frac{1}{2}$ , and may therefore be multiplied according to the ordinary rule for the multiplication of series. Since the expression is an identity in  $x$ ,  $0 \leq x < \frac{1}{2}$ , multiplying and equating coefficients of like powers of  $x$  in the two members:

$$\begin{aligned} \int 1 &= 1 \\ \int 2 &= \int 1 + 2 \\ \int 3 &= \int 2 + \int 1, \end{aligned}$$

$$\int n = \sum_k (-1)^{k+1} \left[ \int \left( n - \frac{3k^2-k}{2} \right) + \int \left( n - \frac{3k^2+k}{2} \right) \right], \quad k = 1, 2, 3, \dots,$$

in which the sum on the right is to be extended to include all positive values of  $n - \frac{3k^2-k}{2}$  and  $n - \frac{3k^2+k}{2}$  and  $\int(n-n)$ , if it occurs, is defined to be equal to  $n$ .

Since  $\int n = (n + 1)$  is a necessary and sufficient condition that  $n$  be prime, the proposition follows.

THE AMERICAN CONSULATE GENERAL,  
VANCOUVER, B. C.