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Fred Robertson
Iowa State College

ROOTS OF QUADRATIC EQUATIONS EXPRESSED IN CONTINUED FRACTIONS

FRED ROBERTSON

The solution of the quadratic equation

$$x^2 - 2bx + c = 0 \tag{1}$$

with c negative is

$$x = b^1 + c/x \tag{2}$$

where $b^1 = 2b$.

If one repeatedly substitutes for x in the right hand member of equation (2) its value as found from the left hand number, one obtains the continued fraction

$$x = b^1 \; \frac{c}{+ b^1 +} \; \frac{c}{b^1 +} \tag{3}$$

The limit of each member of equation (4) is

$$\alpha = b^1 \; \frac{c}{+ b^1 +} \; \frac{c}{b^1 +} \; \frac{c}{b^1 +}$$

where α is a root of equation (1), provided the continued fraction converges. Serret, J. A.[1] and others have proved the theorem.

Theorem I. Any simple recurring continued fraction is a root of a quadratic equation with rational coefficients. The second root β is restricted as follows:

(a) If there is no acyclic part, then $-1 < \beta < 0$.

(b) If the acyclic part consists of a single quotient, then $\beta < -1$ or $\beta > 0$.

(c) If the acyclic part contains at least two quotients, then $\beta > 0$.

The roots of equation (1) may be written

$$x = \frac{+ b + \sqrt{b^2 - c}}{\pm 1} \tag{4}$$

where the radical shall be positive and the plus and minus signs taken successively. Now the variable, x, in equation (4) is of the form $x = \dfrac{E + \sqrt{A}}{D}$ where $E = \pm b$; $A = b^2 - c$; and $D = \pm 1$. (5)

Choose D_{-1} such that

$$-D_{-1}\, D = c$$

1 Serret, J. A. Algèbre Supérieure.

and therefore

$$E^2 + D_{-1} D = A. \tag{6}$$

The variable in equation (5) may be expanded in the form

$$x = a + \frac{D_{-1}}{x_1} \tag{7}$$

where a is the greatest integer, or twice the real part of (5) according as $\sqrt{A}$ is irrational, or either rational or imaginary.

The solution of (7) for x_1 gives by (6)

$$x_1 = \frac{D_{-1} D}{-(aD - E) + \sqrt{A}} = \frac{A - E^2}{-E + \sqrt{A}} = \sqrt{A} + E = x \tag{8}$$

where $a = \frac{2E}{D}$ by definition.

Similarly $x_n = x_{n-1} = \ldots\ldots\ldots\ldots = x$ and the continued fraction is of the form

$$x = a_o \; \frac{b_1}{+ a_1} \frac{b_2}{+ a_2 +} \cdots\cdots\cdots$$

In general

$$x_{n-1} = \frac{E_{n-1} + \sqrt{A}}{D_{n-1}} = a_{n-1} + \frac{D_{-(n-1)}}{x_{n-1}}$$

and

$$x_n = \frac{D_{-n} D_{n-1}}{-(a_{n-1} D_{n-1} - E_{n-1}) + \sqrt{A}}$$

$$= \frac{D_{-n} D_{n-1} [\sqrt{A} + (a_{n-1} D_{n-1} - E_{n-1})]}{A - (a_{n-1} D_{n-1} - E_{n-1})^2} =$$

$$\sqrt{A} + E_{n-1} = x_{n-1}$$

where

$$a_n = \frac{2E_n}{D_n}; \qquad D_{-n} = \frac{A - E_n^2}{D_n} \tag{9}$$

with

$$E_n^2 + D_{-n} D_n = A.$$

The formulas (9) contain the law of formation of the partial quotients of x_n.

In our case $D_n = \pm 1$ and

$$a_n = \pm 2E_n; \qquad D_{-n} = \pm (A - E_n^2). \tag{10}$$

One states the theorem,[2]

Theorem II (a) If α, β are roots of $x^2 - b'x + c = 0$ then the nth convergent of

$$b' \; \frac{-c}{+ b'} \frac{-c}{+ b' +} \cdots\cdots\cdots\cdots,$$

[2] Barnard and Childs; Higher Algebra, p. 393.

equals

$$\frac{\alpha^{n+1}-\beta^{n+1}}{\alpha^n-\beta^n} \text{ or } (1+1/n)\,\alpha \text{ according as } \alpha \neq \beta \text{ or } \alpha=\beta;$$

(b) The continued fraction converges if $(b')^2 \gtreqless 4\,c$ and its value is the numerically greater root of $x^2 - b'x + c = 0$.

One needs a theorem stated by Serret, J. A.,[3]

Theorem III. The periods of the partial quotients of continued fractions which express irrational roots of a quadratic equation with integer coefficients are inverses, i.e., the periodic quotients which give one root are exactly the periodic quotients which give the other root except the order is reversed.

However, in the case considered the continued fraction given by (3) consists of a single period and therefore the representations of the two roots are identical except for algebraic sign.

One could easily derive for the special case the result stated above from equations (9). The proof is valid formally when the roots are rational or complex.

One uses the theorem stated by Perron, O.[4]

Theorem IV. If the elements of the infinite continued fraction

$$b_0 + \frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots\cdots\cdots\cdots$$

are rational numbers then, if for a given index, v the inequalities

$$b_v \gtreqless |a_v|$$
$$b_v \gtreqless |a_v| + 1 \qquad \text{for } a_{v+1} < 0$$

are satisfied, the continued fraction converges to an irrational value, say ξ_0, unless for a given v

$$a_v < 0, \qquad b = |a_v| + 1$$

when the continued fraction is either divergent or has a rational value.

One states the final theorem,

Theorem V. If the variable in the equation

$$x^2 - bx + c = 0$$

is expanded in a continued fraction of the form

$$x = b + \frac{c}{b} + \frac{c}{b} + \cdots\cdots\cdots\cdots$$

then (a) for

$$b \gtreqless |c|$$
$$b \gtreqless |c| + 1 \qquad \text{for } c < 0,$$

the continued fraction converges to an irrational root of the equation unless

3 Serret, J. A. Algèbre Superieure, p. 49.
4 Perron, O. Die Lehre von Den Kettenbrüchen, p. 252.

$$b = |c| + 1 \qquad \text{for } c < 0$$

in which exceptional case the continued fraction diverges or approaches a rational root of the equation.

(b) Otherwise the continued fraction diverges.

(c) In either case the roots of the equation are given by

$$\frac{\pm \frac{b}{2} + \sqrt{c + \left|\frac{b}{2}\right|^2}}{\pm 1}$$

Parts (a) and (b) are contained in the theorems previously stated. In case (c) one finds from equation (3) the real part of the root is $E = \pm \frac{b}{2}$ and from (9) $D_{-n} = \frac{A - E^2}{\pm 1} = + c$ or $A = + c + \left|\frac{b}{2}\right|^2$.

Therefore the roots of the equation in terms of the elements of the repeating continued fraction into which it is expanded are given by

$$x = \frac{\pm \frac{b}{2} + \sqrt{c + \left|\frac{b}{2}\right|^2}}{\pm 1}$$

Example I. Given the continued fraction

$$6 \; \frac{1}{+6+} \frac{1}{6+} \frac{1}{6+} \cdots\cdots\cdots$$

The roots of its associated quadratic equation is by (11) $x = 3 \pm \sqrt{+1+9} = 3 \pm \sqrt{10}$ and the equation is $x^2 - 6x - 1 = 0$.

Example II. Given the continued fraction

$$4 \; \frac{-3}{+4+} \frac{-3}{4+} \frac{-3}{4} \cdots\cdots\cdots$$

The roots of its associated quadratic equation are $x = 2 \pm \sqrt{-3+4} = 3$ or 1 and the equation is $x^2 - 4x + 3 = 0$.

Example III. Given the continued fraction

$$4 \; \frac{-7}{+4+} \frac{-7}{4+} \cdots\cdots\cdots$$

It is divergent but the roots of the associated quadratic equation are $x \doteq 2 \pm \sqrt{-7+4} = 2 \pm i\sqrt{3}$ and the equation is $x^2 - 4x + 7 = 0$.

Department of Mathematics,
Iowa State College,
Ames, Iowa.