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## Exponential Operators

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EXPONENTIAL OPERATORS

FRED ROBERTSON

1. *Introduction.* The non-commutative operational formula  $Dx = xD + 1$ , where  $D = \frac{d}{dx}$ , is well known. The difficulty of determining the order of the operations  $x$  and  $D$  in a function  $f(x, D)$  retarded the use of operative functions.

Bourlet<sup>1</sup> was the first to discover a way to express an operative function,  $f(x, D)$  such that the operations with  $D$  always precede those with  $x$ . Thus his operators obey the laws of algebra and their operational meaning is unique.

2. *The Bourlet or Generalized Leibnitz Formula.* This formula will be developed from the well known generalized Leibnitz formula of the elementary Calculus,<sup>2</sup> namely,

$$F(x, D) \rightarrow vu = v F(x, D) + \frac{v'}{1!} \frac{F(x, D)}{D} + \frac{v''}{2!} \frac{\delta^2 F(x, D)}{\delta D^2} + \dots \rightarrow u$$

or in the operational form,

$$(2.1) \quad F(x, D) \rightarrow vu = e^{\frac{\delta F(x, D)}{\delta D} \frac{\delta v}{\delta x}} \rightarrow u.$$

By definition

$$[\Psi(x, D) \Phi(x, D)] \rightarrow u = \Psi(x, D) \rightarrow \Phi(x, D) \rightarrow u.$$

When one assumes that  $\Psi(x, D)$  and  $\Phi(x, D)$  are operative functions which possess Taylor's expansions about the origin, then

$$(2.2) \quad \Psi(x, D) \rightarrow \Phi(x, D) \rightarrow u = \Psi(x, D) e^{\frac{\delta \Phi}{\delta D} \frac{\delta}{\delta x}} \rightarrow u.$$

The expansion of the exponential operator; the application of the generalized Leibnitz formula of elementary calculus to the product of  $\Psi(x, D)$  by each term leads to the operational form

$$\begin{aligned} \Psi(x, D) \Phi(x, D) \rightarrow u = & \Psi \Phi + \frac{\delta \Psi}{\delta D} \frac{\delta \Phi}{\delta x} + \frac{1}{2!} \frac{\delta^2 \Psi}{\delta D^2} \frac{\delta^2 \Phi}{\delta x^2} + \dots \rightarrow u \\ & + \frac{\delta \Psi}{\delta D} \Phi + \frac{\delta^2 \Psi}{\delta D^2} \frac{\delta \Phi}{\delta x} + \frac{1}{2!} \frac{\delta^3 \Psi}{\delta D^3} \frac{\delta^2 \Phi}{\delta x^2} + \dots \rightarrow u' \\ & + \frac{1}{2!} \left[ \frac{\delta^2 \Psi}{\delta D^2} \Phi + \frac{\delta^3 \Psi}{\delta D^3} \frac{\delta \Phi}{\delta x} + \frac{1}{2!} \frac{\delta^4 \Psi}{\delta D^4} \frac{\delta^2 \Phi}{\delta x^2} + \dots \right] \rightarrow u'' \end{aligned}$$

<sup>1</sup> Bourlet, C. Sur les opérations en général et les équations différentielles Linéaires d'ordre Infini. Annales d'école Normale Supérieure, Ser. e. Vol. 14 (1897) pp. 133-190.  
<sup>2</sup> Pincherle, S. Funktionenoperationen und Gleichungen, Encyklopädie der Mathematischen Wissenschaften, II, A, 11, p. 769.

The addition by columns gives

$$\Psi(x, D) \Phi(x, D) \rightarrow u = \Psi\Phi + \frac{\delta\Psi}{\delta D} \frac{\delta\Phi}{\delta x} + \frac{1}{2!} \frac{\delta^2\Psi}{\delta D^2} \frac{\delta^2\Phi}{\delta x^2} + \dots \rightarrow u,$$

and the abstraction of the operators gives

$$(2.3) \quad \Psi(x, D) \Phi(x, D) = e^{\frac{\delta\Psi}{\delta D} \frac{\delta\Phi}{\delta x}}$$

The equation (2.3) is the form developed by Bourlet in a different notation and is now called the Generalized Leibnitz expansion formula for operators.

3. *A Fundamental Transfer Theorem.* This theorem, namely,

$$(3.1) \quad F(D) e^{\Phi(x)} = e^{\Phi(x)} F[D + \Phi'(x)]$$

which was originally derived by Robert Murphy in 1837 is readily developed from the generalized Leibnitz formula (2.3).

The substitutions  $\Psi(x, D) = F(D)$  and  $\Phi(x, D) = e^{\Phi(x)}$  in equation (2.3) give

$$(3.2) \quad \begin{aligned} F(D) e^{\Phi(x)} \rightarrow f &= e^{\frac{\delta F}{\delta D} \frac{\delta e^{\Phi}}{\delta x}} \rightarrow f \\ &= F e^{\Phi} + \frac{\delta F}{\delta D} \frac{\delta e^{\Phi}}{\delta x} + \frac{1}{2!} \frac{\delta^2 F}{\delta D^2} \frac{\delta^2 e^{\Phi}}{\delta x^2} + \dots \rightarrow f. \end{aligned}$$

In the right hand member of equation (3.2), the operations with  $D$  precede those with  $x$  but this is not the case in equation (3.1). Therefore one writes

$$\begin{aligned} \frac{\delta e^{\Phi}}{\delta x} &= \Phi' \rightarrow e^{\Phi} \\ \frac{\delta^2 e^{\Phi}}{\delta x^2} &= \frac{\delta}{\delta x} (\Phi' e^{\Phi}) = (\Phi'D + \Phi'') \rightarrow e^{\Phi} = D\Phi' \rightarrow e^{\Phi} \\ &\dots \dots \dots \\ \frac{\delta^n e^{\Phi}}{\delta x^n} &= D^{n-1} \Phi' \rightarrow e^{\Phi}. \end{aligned}$$

Accordingly equation (3.2) becomes

$$(3.3) \quad \begin{aligned} F(D) e^{\Phi} \rightarrow f &= e^{\Phi} [F + \frac{\delta F}{\delta D} \Phi' + \frac{1}{2!} \frac{\delta^2 F}{\delta D^2} \Phi'' + \dots] \rightarrow f \\ &= e^{\Phi} e^{\Phi'} \frac{\delta F}{\delta D} \rightarrow f \\ &= e^{\Phi} F[D + \Phi'] \rightarrow f. \end{aligned}$$

4. *Application to Trigonometric Operators.* The generalized Leibnitz formula introduces trigonometric operators, when  $\Phi(x, D)$  is replaced by  $e^{i\Phi(x, D)}$ .

Thus

$$\Psi(x, D) e^{i\Phi(x, D)} \rightarrow f = e \frac{\delta \Psi}{\delta D} \frac{\delta e^{i\Phi}}{\delta x} \rightarrow f$$

and

$$(4.1) \quad \Psi(x, D) \cos \Phi \rightarrow f = R[e \frac{\delta \Psi}{\delta D} \frac{\delta e^{i\Phi}}{\delta x} \rightarrow f]$$

$$(4.2) \quad \Psi(x, D) \sin \Phi \rightarrow f = I[e \frac{\delta \Psi}{\delta D} \frac{\delta e^{i\Phi}}{\delta x} \rightarrow f].$$

The operations in the right hand members of (4.1 and (4.2) are the real and the imaginary parts respectively of the result of the exponential operating upon the function.

An interesting special case of formulas (4.1 and (4.2) is

$$\cos \Phi(x) D \rightarrow f = R[e^{i\Phi(x)D} \rightarrow f] = R f(x + i \Phi)$$

$$\sin \Phi(x) D \rightarrow f = I[e^{i\Phi(x)D} \rightarrow f] = I f(x + i \Phi),$$

which is obtained by substituting  $\Psi(x, D) = 1$  and  $\Phi = \Phi(x)D$ .

As an example let us compute  $\cos kD \rightarrow x^3$ . Thus

$$\cos kD \rightarrow x^3 = R[e^{ikD} \rightarrow x^3] = R(x + ik)^3 = x^3 - 3xk^2,$$

$$\sin kD \rightarrow x^3 = I[e^{ikD} \rightarrow x^3] = I(x + ik)^3 = 3x^2k - k^3.$$

5. *The Inversion Problem.* When the operational equation

$$\Phi(x, D) \rightarrow f = g$$

is obtained, where  $f$  is the unknown function, then the inverse requires the function  $\Psi(x, D)$  with the property

$$\Psi(x, D) \rightarrow \Phi(x, D) \rightarrow f = 1 \rightarrow f = \Psi(x, D) \rightarrow g.$$

A customary method of procedure to obtain the function  $\Psi(x, D)$  is to solve the equation (2.3) in the form

$$\Psi(x, D) \Phi(x, D) = 1$$

An alternate method of procedure for functions  $\Psi(x)D$  of an exponential type will be explained. Suppose

$$\Phi(x, D) = e^{S(x)D}$$

where  $S(x)$  is a function of a real or complex variable.

The given equation is

$$1 + S(x)D + \frac{S^2(x)}{2!}D^2 + \dots \rightarrow f = g$$

or

$$(5.1) \quad e^{S(x)D} \rightarrow f = g.$$

Then a function  $\Psi(x, D)$  of the form  $e^{T(x)D}$  is desired with the property

$$e^{TD} \rightarrow e^{SD} \rightarrow f = f = e^{TD} \rightarrow g.$$

Now  $e^{SD} \rightarrow f$  is the operational form of the Taylor's expansion of  $f(x + S)$  and  $e^{SD} \rightarrow f(x)$  transforms  $f(x)$  from the domain  $x$  to the domain  $x + S$ . Similarly

$$e^{TD} \rightarrow e^{SD} \rightarrow f = e^{TD} \rightarrow f(x + S) = f[x + T + S(x + T)]$$

The problem is to choose  $T$  in such a way that the result of the second transform, namely,  $f[x + T + S(x + T)]$  is the original function  $f(x)$ . This means that  $T$  is a solution of the equation

$$(5.2) \quad x + T + S(x + T) = x$$

Suppose  $T = \Psi(x)$  is a solution of (5.2). Then

$$e^{\Psi(x)D} \rightarrow e^{S(x)D} \rightarrow f = f = e^{\Psi(x)D} \rightarrow g$$

and the equation (5.1) is formally solved.

Let us apply the theory to two examples.

Example 1.

Solve the equation

$$(5.3) \quad 1 + x^2D + \frac{x^4D^2}{2!} + \dots \rightarrow f = g.$$

Symbolically equation (5.3) is

$$e^{x^2D} \rightarrow f = g.$$

Then

$$e^{TD} \rightarrow e^{x^2D} \rightarrow f = f[x + T + (x + T)^2]$$

and equating the transforms gives

$$(5.4) \quad x + T + (x + T)^2 = x.$$

The solution of (5.4) for  $T$  is

$$T = \frac{-(2x + 1) \pm \sqrt{1 + 4x}}{2} D$$

and the solution of equation (5.3) is

$$f = e^{\frac{-(2x + 1) \pm \sqrt{1 + 4x}}{2} D} \rightarrow g = g\left[x + \frac{-(2x + 1) \pm \sqrt{1 + 4x}}{2}\right].$$

The ambiguous sign means that the transform may be effected in either of two ways.

Example 2.

Solve the equation

$$1 + ixD + \frac{i^2x^2D^2}{2!} + \dots \rightarrow f = g$$

which is

$$(5.5) \quad e^{ixD} \rightarrow f = g$$

in operational form.

Now

$$e^{tD} \rightarrow e^{ixD} \rightarrow f = e^{tD} \rightarrow f[(1+i)x] = f[(1+i)(x+t)].$$

Choose  $T$  satisfying the equation

$$(5.6) \quad (1+i)(x+T) = x$$

Then

$$T = -x \frac{1+i}{2}$$

and the solution of equation (5.5) is

$$f = e^{-x \left(\frac{1+i}{2}\right) D} \rightarrow g = g\left(\frac{1-i}{2} x\right).$$

The preceding examples illustrate the power of this method of solving certain types of differential equations of infinite order.

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