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The operators under consideration will be formed from nine basic first order operators. These will, in their turn, be formed by combining in several ways one of their number, an undefined operator of 'derivation'. Starting with a set A, and the operator d, we obtain:

1. hA = AdA, the concentrated part of A, which is the intersection of A and dA.

2. JA = AcdA, the isolated part of A, that part of A which is not in dA.

3. kA, the kernel of A, which is the maximum dense-in-itself subset of A.

4. sA = AckA, the separated part of A. This is the part of A which is not in the kernel.

5. iA = AcdcA, the interior of A. This is the part of A which is not in the derived set of the compliment of A.

6. bA = AdcA, the border of A, which is the part of A that is not interior.

7. eA = A + dA, the extension or closure of A, the sum of A and its derived set.

8. fA = AcdA + cAdA, the frontier of A, which is the sum of the border of A and of those parts which are both derived points of A and complimentary to A.

When we wish to refer to any one of these operators we will use the letters u and v, and statements involving these letters will be understood to be valid for each of the operators. These operations are performed on elements in a space possessing the following three properties:

1. $d(A + B) = dA + dB$.
2. $d^2A \leq dA$.
3. $d(O) = O$, for O a null element.

Now, the process of taking compliments is not considered one of the basic operations, because of its extreme simplicity. Hence, if we let the operators act upon each other in various ways, and call an operator of second 'order' whenever it contains exactly two of the basic operators, it is clear that all second order operators are of one of the following forms: uv, ucv, cuv, uvc, cuvc, cucv, ucvc, and cucvc. The first two of these forms are of special significance, since any of the others may be obtained from one of them by either complementing the set A before applying the operator, or complementing the resulting set. Any of the nine first order operators may be placed before any other one, and thus 81 operators of the form uv will be obtained. Similarly, there are 81 operators of the form ucv. These 162 operators are the subject of this investigation.

Incidental to his doctoral dissertation, Emmet Carson Stopher showed, in 1937 that many operators in this collection may be 'reduced'; that is, many of them are exactly equal to some other operator of lower order. A rather obvious continuation of his methods reveals that exactly eighty-six are reducible. The proof of the irreductibility of the remaining seventy-six is the object of this paper.

To show that any two operators are unequal in general, we need but produce a space which, if each of the operators is evaluated on it, will yield different values for them. The space will, in fact, divide the collection of seventy-six operators into sub-collections in this way. We may be confident that no operator in any collection is equal to any other in a different collection. If then, we employ succeeding spaces in the same way, the existing collections will become more and more split up. We should suppose that when a collection comes out which contains only one operator, that this one is unequal to all the others. This fact depends, however, upon whether or not we have the right to consider the spaces simultaneously. That this is actually the case will be demonstrated by the following theorem.

Theorem: If $S = S_1 + \dots + S_n$, and if $S_iS_j = O$ for all i and j, and if $dS_i \leq S_j$, and if each space contains a set with corresponding subscript, $S_i \geq A_i$, then $u(A_1 + \dots + A_n) = uA_1 + \dots + uA_n$: and also $cSt\,(A^1 + \dots A_n) = c\underset{1}{S}\,tA^1 + \dots + c\underset{n}{S}\,tA_n$

The proof of this theorem is straightforward, and consists of calculating the values of the expressions in question for each of the first order operators. Then use may be made of the non-intersection of the sets involved, and the result will follow.

Following the procedure outlined above, nine spaces will suffice to show each of the seventy-six operators are independent of all the others.

Space One

The first has but one part, X, with the characteristic that $dX = X$. An operator may have the value X or O on X and X or O on O. Thus our operators fall into four collections containing 12, 12, 8 and 44 operators.

Space Two

This space also has but one part, X, but $dX = O$. It has the effect of further splitting the operators into nine collections.

Space Three

Having exhausted all possible spaces with but one part, space three is the first with two parts, X and Y. $dX = Y$ and $dY = O$. This time, thirteen of the operators become independent, while the remainder are contained in fifteen different collections.

Space Four

$dX = Y$, $dY = Y$. This space leavcs independent fourteen operators, and increases the number of collections to nineteen.

Space Five

$dX = X + Y$, $dY = O$. Ten more operators become independent under this criterion, and the number of collections is reduced to fifteen. Each of these, naturally will now contain a very few operators, and the number of them will reduce substantially with further spaces.

Space Six

$dX = S$, $dY = Y$. Only two operators are reduced in this space, and the number of collections is now fourteen.

Space Seven

$dX = X + Y$, $dY = X + Y$. Three more operators become independent, and the number of collections becomes thirteen. It can easily be seen that any other second order space would be redundant upon one of those already used, or would be incompatible with the postulates. For this reason, the two remaining spaces to be used will be of the third order.

Space Eight

This space contains parts X, Y and Z with the following relationships among them: $dX = Y + Z$, $dY = Z$ and $dZ = O$. It has the effect of eliminating from consideration twenty-two operators, and of reducing the number of collections to three.

SPACE NINE

$dX = X + Y + Z$, $dY = Z$ and $dZ = O$. This space will effect the final separations desired.

We may now use the fundamental theorem, and form a space consisting of spaces one through nine, with the assurance that in this larger space, each operator under consideration will have a value separate from any other.

This result, together with the reductions obtained by Stopher or by his method constitute what might be called a complete classification of all second order operators of basic type formed from the nine operators in question.

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