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A NEW PROOF OF A FORMULA OF KURATOWSKI

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Given a topological space in which the operation of closure of a set, here denoted by 'e', has the properties: e $(A+B) = eA+eB$; A is a subset of eA, $eO = O$, $e^2A = eA$ (where A is any subset of the space and O is the null set), we may define the interior 'i' of a set A by the formula $iA = cecA$ (where cA denotes the complement of A in the space) and write a formula entirely equivalent to one due to C. Kuratowski (Fundamenta Mathematicae, III (1922), p. 183) in the following way:

(1) $$(ei)^2A = eiA.$$

In this notation, eiA is read, 'the closure of the interior of the set A.' Because formula (1) is an operational identity, it is convenient to delete the argument A, and write simply $(ei)^2 = ei$.

The result stated above is an immediate consequence of the following general proposition. Let L be any partially ordered class. We understand by this, that there is a relation represented by '$<$', and read 'precedes', such that if A and B are any two elements of L, distinct or not, then it is determinate whether the relation $A < B$ holds or not. This relation is assumed to be transitive, that is, if $A < B$ and $B < C$, then $A < C$. Let u be an operator defined for all elements of A of L, with values uA in L. An operator u is non-decreasing if $A < B$ implies $uA \leqq uB$. Let v be another operator of the same sort. The symbol uv will be used to represent the operator defined by applying the operator u to the element vA for each element of L.

Theorem. If u *and* v *are non-decreasing operators,* $u \leqq v$, $u^3 = u$, $v^3 = v$, *then* $(uv)^2 = uv$ *and* $(vu)^2 = vu$.

The proof of this theorem is remarkably simple. The operators uv, vu, are both non-decreasing. For $A < B$ implies $vA \leqq vB$, $uA \leqq uB$, and therefore $uvA \leqq uvB$, $vuA \geqq vuB$. It is easily seen that $u^2 \leqq uv \leqq v^2$, and in consequence that $u^3 \leqq uvu$, $vuv \leqq v^3$. The following series of relations establishes the theorem:

$$uv = u^3v \leqq (uvu)v = u(vuv) = (uv)^2 \leqq uv^3 = uv,$$
$$vu = vu^3 \leqq v(uvu = (vuv)u = (vu)^2 \leqq v^3u = vu.$$

To complete the proof of formula (1) we observe that by hypothesis the operator e is non-decreasing and that $e^2 = e$. Therefore, $e^3 = e$. Since $ccA = A$, and $iA = cecA$, it is easy to see that the operator i is non-decreasing and that $i^2 = i$. Finally, the pointsets of a topological space form a partially ordered class L. Formula (1) now follows from the theorem above. Evidently, $(ie)^2 = ie$, for the same reason.

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