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## A New Proof of a Formula of Kuratowski

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## A NEW PROOF OF A FORMULA OF KURATOWSKI

E. W. CHITTENDEN

Given a topological space in which the operation of closure of a set, here denoted by 'e', has the properties:  $e(A+B) = eA+eB$ ;  $A$  is a subset of  $eA$ ,  $eO = O$ ,  $e^2A = eA$  (where  $A$  is any subset of the space and  $O$  is the null set), we may define the interior 'i' of a set  $A$  by the formula  $iA = cecA$  (where  $cA$  denotes the complement of  $A$  in the space) and write a formula entirely equivalent to one due to C. Kuratowski (*Fundamenta Mathematicae*, III (1922), p. 183) in the following way:

$$(1) \quad (ei)^2A = eiA.$$

In this notation,  $eiA$  is read, 'the closure of the interior of the set  $A$ .' Because formula (1) is an operational identity, it is convenient to delete the argument  $A$ , and write simply  $(ei)^2 = ei$ .

The result stated above is an immediate consequence of the following general proposition. Let  $L$  be any partially ordered class. We understand by this, that there is a relation represented by ' $<$ ', and read 'precedes', such that if  $A$  and  $B$  are any two elements of  $L$ , distinct or not, then it is determinate whether the relation  $A < B$  holds or not. This relation is assumed to be transitive, that is, if  $A < B$  and  $B < C$ , then  $A < C$ . Let  $u$  be an operator defined for all elements of  $A$  of  $L$ , with values  $uA$  in  $L$ . An operator  $u$  is non-decreasing if  $A < B$  implies  $uA \leqq uB$ . Let  $v$  be another operator of the same sort. The symbol  $uv$  will be used to represent the operator defined by applying the operator  $u$  to the element  $vA$  for each element of  $L$ .

*Theorem.* If  $u$  and  $v$  are non-decreasing operators,  $u \leqq v$ ,  $u^3 = u$ ,  $v^3 = v$ , then  $(uv)^2 = uv$  and  $(vu)^2 = vu$ .

The proof of this theorem is remarkably simple. The operators  $uv$ ,  $vu$ , are both non-decreasing. For  $A < B$  implies  $vA \leqq vB$ ,  $uA \leqq uB$ , and therefore  $uvA \leqq uvB$ ,  $vuA \leqq vuB$ . It is easily seen that  $u^2 \leqq uv \leqq v^2$ , and in consequence that  $u^3 \leqq uvu$ ,  $vuv \leqq v^3$ . The following series of relations establishes the theorem:

$$\begin{aligned} uv &= u^3v \leqq (uvu)v = u(vuv) = (uv)^2 \leqq uv^3 = uv, \\ vu &= vu^3 \leqq v(uvu) = (vuv)u = (vu)^2 \leqq v^3u = vu. \end{aligned}$$

To complete the proof of formula (1) we observe that by hypothesis the operator  $e$  is non-decreasing and that  $e^2 = e$ . Therefore,  $e^3 = e$ . Since  $ccA = A$ , and  $iA = cecA$ , it is easy to see

that the operator  $i$  is non-decreasing and that  $i^2 = i$ . Finally, the pointsets of a topological space form a partially ordered class L. Formula (1) now follows from the theorem above. Evidently,  $(ie)^2 = ie$ , for the same reason.

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