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A PROBLEM IN AIR NAVIGATION

H. P. THIELMAN

SUMMARY

The following problem is solved. A pilot is ordered to make a scouting flight around a base. His orders are to fly out from the base along a straight path, make a complete circle whose center is the base and return to the starting point. If a constant wind is blowing and the pilot has fuel for n hours of flight, what is the greatest distance and longest time the pilot can fly away from the base and still have enough fuel left to carry out his orders?

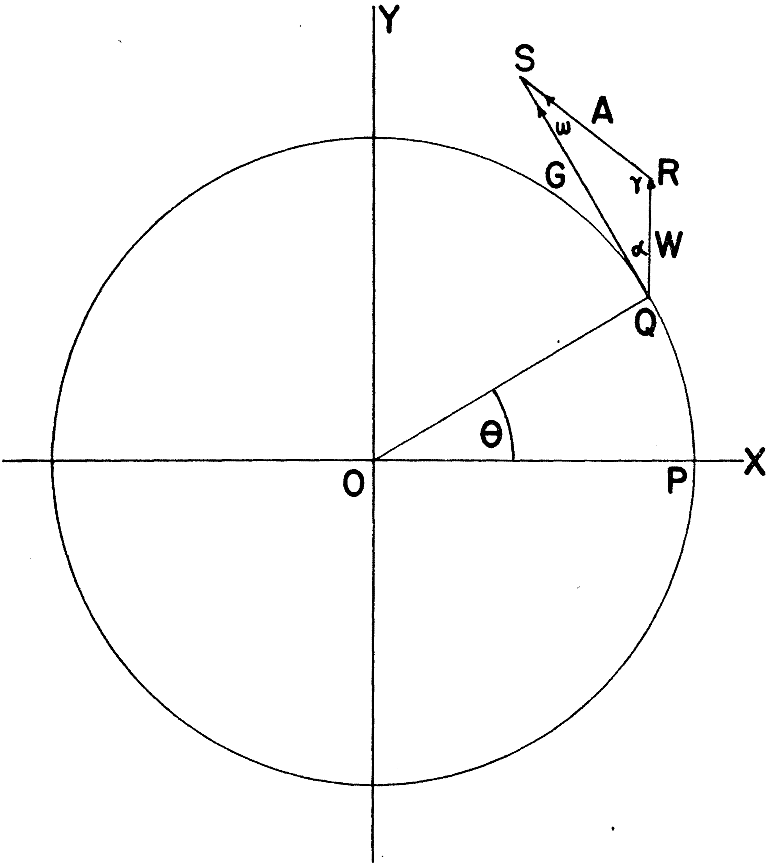
The solution, which involves elliptic integrals, is brought to a usable practical form by a table which reveals at a glance the radius of the circle and the time of flight out for various combinations of wind and air speed.

SOLUTION OF THE PROBLEM

Let the air speed of the airplane be A miles per hour, the magnitude of the velocity of the wind be W miles per hour, and the radius of the largest circle along which the pilot can fly be R. This circle might be called the ''circle of action''. Let the positive direction of the y-axis of a rectangular coordinate system (see figure) coincide with the direction of the wind. From very elementary considerations it is well known that the most economical (as far as fuel consumption is concerned) straight path out and in, under a constant wind, is that path which is at right angles to the direction of the wind. Hence, the pilot will fly out along the x-axis a distance R, complete a circle of radius R with center at the base O, and return along the x-axis.

The time, t_o, going out along the path OP is equal to time back along PO, and is given by $R/\sqrt{A^2-W^2}$, since the ground speed $G = \sqrt{A^2-W^2}$. The mangitude of the velocity of the wind will be assumed to be less than the air speed of the plane.

Let Q be any point in the first quadrant lying on the circle, and let θ be the angle the radius OQ makes with the positive direction of the x-axis. We shall now find the ground speed G of the airplane at any such point Q. We draw the velocity vector triangle QRS, where QR represents the wind, RS the air speed and the direction of the heading of the airplane, and QS the ground speed G and the direction of the true course. By the law of sines, we have



$$G/\sin\gamma = A/\sin\alpha$$

or $$G = A \sin\gamma/\sin\alpha, \tag{1}$$
and $$\sin\omega = W \sin\alpha/A, \tag{2}$$

where α, γ, and ω are the angles in the velocity vector triangle opposite A, G, and W respectively. From the figure we see that $\alpha = \theta$, $\gamma = 180-(\alpha + \omega)$. Hence it follows from equations (1) and (2) that

$$G = \sqrt{A^2-W^2\sin^2\theta} + W \cos\theta.$$

It is easily verified that this formula gives the ground speed for any point along the circle if θ is permitted to vary from O to 2π radians. Hence the total time for a complete passage of the airplane around the circle is given by the integral

$$t = \int ds/G \text{ along the circle, or}$$

$$t = R \int_0^{2\pi} \frac{d\theta}{\sqrt{A^2-W^2\sin^2\theta} + W\cos\theta}.$$

Rationalizing the denominator in the integrand we get

$$t = \frac{R}{A^2-W^2}\int_0^{2\pi}\sqrt{A^2-W^2\sin^2\theta}\,d\theta - \frac{RW}{A^2-W^2}\int_0^{2\pi}\cos\theta\,d\theta.$$

The last integral is zero, and hence

$$t = \frac{4\,RA}{A^2-W^2}\int_0^{\pi/2}\sqrt{1-\left(\frac{W}{A}\right)^2\sin^2\theta}\,d\theta,$$

which can be written in terms of the elliptic integral of the second kind as

$$t = 4\,RAE\,(W/A)/(A^2-W^2).$$

If t_1, t_2, t_3, t_4 stand for the time of flight along the first, second, third, and fourth quadrant sections of the circle respectively we obtain

$$t_1 = t_4 = t/4 - \frac{RW}{A^2-W^2}\int_0^{\pi/2}\cos\theta\,d\theta = t/4 - RW/(A^2-W^2),$$

$$t_2 = t_3 = t/4 + RW/(A^2-W^2).$$

We can now compute the radius R of the circle of action for any given air speed A greater than the magnitude of the velocity of the wind W. Let the total time of flight out along the radius OP, around the circle of radius R, and back to the base along PO be one hour. Then $2t_o + t = 1$, or

$$2R/\sqrt{A^2-W^2} + 4\,RAE\,(W/A)/(A^2-W^2) = 1.$$

Solving this equation for R we obtain

$$R = \frac{\sqrt{A^2-W^2}}{2+4AE(W/A)/\sqrt{A^2-W^2}}. \qquad (3)$$

The longest time t_o, which the pilot may fly out along OP, is given by $\frac{R}{\sqrt{A^2-W^2}}$ or

$$t_o = \frac{1}{2+4AE(W/A)/\sqrt{A^2-W^2}} \tag{4}$$

This may be written in this form

$$t_o = \frac{1}{2+4E(W/A)/\sqrt{1-(W/A)^2}},$$

which shows that the time out depends only on the ratio of the magnitude of the wind to that of the air speed. For n fuel hours R and t_o must be multiplied by n.

A table of values for R and t_o (the time out) for one fuel hour under various wind conditions at different air speeds is attached. These entries were computed by means of formulae (3) and (4). In the tables A and W are expressed in miles per hour, while R and t_o are given in miles and minutes.

TABLE

Radius (R miles) of Circle of Action, Time (t_o minutes) of Flight out for One Fuel Hour in a Wind of W miles per hour and with an Air Speed of A miles per hour.

W	0		10		20	
A	R	t_o	R	t_o	R	t_o
60	7.24	7.24	7.10	7.20	6.68	7.08
80	9.66	7.24	9.56	7.23	9.24	7.15
100	12.07	7.24	11.98	7.23	11.73	7.19
120	14.49	7.24			14.21	7.21
140	16.90	7.24			16.66	7.22
160	19.32	7.24			19.12	7.23
180	21.73	7.24			21.55	7.23
200	24.14	7.24			23.97	7.23

W	30		40		50	
A	R	t_o	R	t_o	R	t_o
60	5.92	6.83	4.40	5.90	2.96	5.35
80	8.69	7.03	7.89	6.83	6.79	6.52
100	11.31	7.11	10.94	7.00	9.87	6.83
120	13.85	7.15	13.35	7.08	12.68	6.97
140	16.36	7.17	15.93	7.12	15.37	7.05
160	18.84	7.19	18.47	7.15	18.31	7.10
180	21.31	7.21	20.98	7.17	20.55	7.13
200	23.77	7.21	23.47	7.18	23.09	7.15

If the ratio of the magnitude of the velocity of the wind is less than one-tenth that of the air speed, the effect of the wind on R and t_o is so small that for practical purposes it may be neglected. For this reason the entries in the tables corresponding to such conditions have been omitted.

The tables show that if the speed of the wind is less than one-third of the air speed of the plane, it is safe to go out for 7 minutes for every fuel hour, and there will be enough fuel left to complete the circle and to return to the base.

If the pilot had been required to fly along a square of side 2R with center at O, the time out would be given by

$$t = \frac{G}{2(3G+2A)}$$

where G is the ground speed out at right angles to the wind. R would then be given by Gt,

or $$R = G^2/2(3G+2A).$$

These formulae permit an easy construction of a table similar to the one above. It follows from these formulae that if the speed of the wind is less than one-third of the air speed of the plane, the pilot can fly out for $10(2-\sqrt{2})$ or 5.86 minutes for every fuel hour, and there will be enough fuel left to complete the flight around the squares and to return to the base.

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