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ON THE CONVERSE OF A CERTAIN THEOREM IN ANALYTIC GEOMETRY

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The three lines $l_i = a_i x + b_i y + c_i = 0$, $i = 1, 2, 3$, meet in a point if the third order determinant $|a_1 b_2 c_3|$ is zero. This is a necessary and sufficient condition if it is assumed that three parallel lines meet in a point.

This paper is concerned with the answer to the question: How many points can be associated with a given third order determinant which is zero if equations of lines are formed by using the elements of the determinant as the coefficients?

In order to give a systematic treatment we shall agree to proceed as follows:

(a) By transforming the given determinant into another by the so-called "elementary transformations" used in the study of determinants.

(b) The equations of the lines will be formed in the following manner from any *given* determinant. Form a new determinant by multiplying the elements of the first column by x and the elements of the second column by y, leaving the elements of the third column unaltered. The equations of the lines are then found by equating to zero the algebraic sum of the elements of each row.

(c) For the convenience of writing we shall denote the determinant $|a_1 b_2 c_3|$ by the symbol (a, b, c).

It is fairly obvious that a transformation on the elements of a given determinant with whose vanishing we may associate a point in the plane induces a transformation on the coordinates x and y. For example if the point (x_o, y_o) is associated with the determinant $(a, b, c) = 0$, then the point (y_o, x_o) is associated with the determinant $(b, a, c) = 0$. This amounts to saying that the plane has been subjected to the transformation $x = y'$, $y = x'$. This is a reflection in the line $y = x$.

The following tabulation of the transformations may be easily verified assuming that the given determinant is $(a\ b, c) = 0$.

Determinant	Transformation	Remarks
$(-a, b, c) = 0$	$x = x'$, $y = y'$	Reflection in y-axis
$(a, -b, c) = 0$	$x = x'$, $y = -y'$	Reflection in x-axis
$(a, b, -c) = 0$	$x = x'$, $y = -y'$	Reflection in origin
$(b, a, c) = 0$	$x = y'$, $y = x'$	Reflection in line y=x
$(pa, qb, rc) = 0$	$x = px'/r$, $y = qy'/r$	"elongation"
$(c, a, b) = 0$	$x = \frac{y}{x'}$, $y = \frac{1}{x'}$	collineation
$(b, c, a) = 0$	$x = \frac{1}{y'}$, $y = \frac{x}{y'}$	collineation
$(a, c, b) = 0$	$x = \frac{x'}{y'}$, $y = \frac{1}{y'}$	collineation
$(a, b, ah+bk+c) = 0$	$x = x'+h$, $y = y'+k$	translation

If a new determinant is formed from (a,b,c) = 0 by multiplying (a,b,c) by (L,M,N) $\neq$ 0 in the order (L,M,N) (a,b,c) column by row, it is easy to see that the transformation on x and y is the general collineation.

$$x = (L_1x' + M_1y' + N_1)/(L_3x' + M_3y' + N_3),$$
$$y = (L_2x' + M_2y' + N_2)/(L_3x' + M_3y' + N_3),$$

If y_i (i = 1, 2, 3) are constants such that $\sum y_i l_i = 0$, the point associated with the determinant obtained from (a,b,c) by interchanging rows and columns is $(y_1/y_3, y_2/y_3)$. Clearly we may now treat this case in the same manner as the original.

In conclusion, it is seen that by the use of the "elementary transformation" on determinants it is possible to transform the point associated with (a, b, c) = 0 into any other point in the coordinate plane. Obviously the problem may be extended to determinants of any order.

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