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For the purposes of mathematics the general notion of logic 'propositional function' may be used to define a concept which we may wish to regard as a *mathematical function.* The term *function* will be restricted to entities which satisfy the conditions imposed. These conditions may be regarded as axioms. They imply certain elementary properties of functions which we may call theorems, although it seems legitimate to raise the question: Are these propositions in logic or mathematics? The symbolism, and methodology are those of mathematics. The significance is so broad that the subject as a whole may be regarded as at least meta-mathematical if not entirely in the domain of logic. It will be assumed that there is a theory of classes, and certain properties of classes essential to the development of the theory of functions will be stated as need for them arises. (1)

We assume that there exist *things* a, b, c. . . x, y, z represented by lower case Roman letters and *classes* A, B, C, . . . X, Y, Z represented by Roman capitals, an empty class 0, and for each thing x, a class whose only element is x. However, it is convenient to write 'x' both for 'x' as a thing or element and as the class whose only element is 'x'.

Let D and R be any two non-empty classes. The class $D \times R$ of all oriented pairs (x,y) where $x \varepsilon D$ and $y \varepsilon R$ is assumed to exist. Let F be a subset of $D \times R$ with the following properties:

1) *if* $x \varepsilon D$ *there is an element* (x,y) *of* F;
2) *if* $y \varepsilon R$ *there is an element* (x,y) *of* F;

then the class F determine a *function* f on the *domain* D to the range R*). If (x,y) is in F, we write y = fx, reading y *is a value* of f *at the place* x.

The equation

$$f = g$$

is interpreted to mean that the functions f and g are identical, that is, they have the same domain D and range R and are determined by the same subset F of the cross product $D \times R$. It is evident from the definition of equality of functions that the relation is symmetric and transitive.

$$f \neq g$$

means (1) the domains of f and g differ, or (2) if the domains are the same there is either an fx which is not a gx or a gx which is not an fx.

If D is the domain of a function f and R its range, the following conditions are satisfied:

*) This is '*on* D to R' terminology was introduced by E. H. Moore, Introduction to a Form of General Analysis, New Haven (1910). Moore appears to have been the first to propose the formal study of functions of a general variable as a branch of mathematics.

(1) *for every* x ε D *there is an* fx ε R.
(2) *for every* y ε R *there is an* x ε D *such that* y = fx.
(3) *for every* x, *every* fy ε R.

An element y associated with x by a function f, is called an *image* of x, and x is a *counter*-or *inverse*-image of y.

If A is a subset of the domain D of a function f, fA is the class of all elements of D which are images of elements of A, in notation,

$$fA = \sum_{x \, \varepsilon \, A} fx$$

If A = 0, the empty set, then by definition fA = 0.

The function of sets thus defined, with domain *D,* the class of all subsets A of D and range *R,* a class of subsets of R, and a subclass of *R,* the class of all subsets of R, may, without ambiguity, be represented by the symbol f. We shall speak of the *map* fA of A in R. It is convenient to speak of the *map* of x as the *class* fx of all images of x.

If f is given on D to R, A is any subset of D, and *B* is a non-null subset of the fA of A such that each element of *B* is the image of an element of A, then the relation y = fx defines a function g on A to B, called a *reduction* of f. In particular, if B = fA, we may speak of *the* reduction of f determined by A, and refer to the function f on A to B.

The function f on D to R may be regarded as a sub-class F of the class D $\times$ R of oriented pairs (x,y) subject to the conditions; for every x in D there is at least one ,x,y) in F, and also for every y in R there is at least one (x,y) in F. The pairs (x,y) may be called points, and the function f a *point-function.* The function f on *D* to *R* is a set function. If the roles of the classes D and R are interchanged, we obtain the product class R $\times$ D and the class F which defines a function f determines a class F^{-1} of pairs (y,x). The corresponding function f^{-1} is the inverse of the given function f.

If D is a given domain and A is a subset of D, the class cA will denote the subclass of D composed of those elements of D which are not in A. The underlying theory of classes is assumed to provide for the existence of the complementary class cA = D-A. The function 'c' so defined is a *set-function* relative to D and a *point-function* relative to the class *D* of all subclasses of D.

Similarly, with respect to a range R and subsets B of R there is a complement function 'c' on *R* to R. In using the symbol 'c' for the symbol 'c' for the complement function it will be convenient to let the domain be understood. From the complement function c on *D* to *D* we derive a convenient operator on families of sets, also denoted by 'c'. Thus if *F* is any family of subsets of a domain D, cF will mean the class of all sets which are complements of elements of *F*. Since *F* is any family of subsets of D, it is clear that 'c' is in this case a function whose domain is the class *D* of all subsets of *D*, and whose range is also *D*.

The function 'c' in either of the cases above is an example of a 1-1 correspondence. In general, f is a 1-1 correspondence if both f and f^{-1} are single-valued. The identity in which R = D and fx = x, will be denoted by I. Likewise, the set-function which converts a set A into the empty set 0 will be represented by the symbol 0. Thus 0x = 0. 0A = 0.

If f has domain D aand range R, while g has domain R and range S, then fx is an argument of g and gfx is defined. We thus obtain the composite function gf on D to S, called a function of a function.

Theorem. *The set-function* gf *on D to S defined by the point-function* gf *is identical to the composite function of the corresponding set-functions* f and g.*)

*) Proof. Let uA = C where C = gfA. If x ε A then gfx ε C, so that gfa < uA. If y ε c there is x ε A such that gfx = y. Therefore uA < gfa.

In all cases cc = I. If the range R of a function f is a subset of its domain D the iterated function ff is defined with range fD. Thus we may speak of the n-th iterate of f, $f^{(n)}$, with the understanding that the range of f is a subset of the domain of f. The range of $f^{(n)}$ will be the n'th image of the domain of f. If $f^{(n)}x = x$, or $f^{(n)} = I$, $f^{(n)}$ is an operator of order n.

The set-functions fc and cf are easily obtained by composition from the set-function f derived from a point-function f. It will be observed that in fc, the domain and range of c is *D*, while in cf, *R* is the domain and range of c. Thus c, represents two different functions. While we shall have occasion to use the function f* = cfc, which will be called the transform of f, it should be clear that the terminology is based on an analogy with the properties of groups. However, in case f is on D to D, the functions 'c' become identical. This is the case for the topological functions.

If f* = cfc = f, the function f must be a 1-1 correspondence and conversely.

If fA < fcA, then f*A = 0. This follows at once from the Boolean formula

$$fA + fcA = R.$$

If x denotes the class whose only element is x, cx is the class D-x, where D is the domain of a given function fx, and we have the class inclusion, fx < fcx implies f*x = 0.

For all functions f,

$$fA = fA \bullet fcA + f^*A,$$

or

$$(f - f^*)\ A = fA \bullet fcA,$$

where (f—g)A means the intersection of A and cgA.

In all cases where f denotes the set-function derived from a point-function,

$$(f^*)^* = f \quad ^{*)},$$

*) Proof. Let A be any subset of D, fcA is the image of cA in R, cfcA = f*A is the complement in *R* of this image. Then (f*)*A = c(cfcc$\overline{A}$) = c(cfA) = fA.

also

$$cfa < fcA$$

If a function is single-valued and equal to its inverse it is called an *automorphism*. If the range of a function is a subset of its domain, it is a *homomorphism*. If for every set A, $fA < A$, the function is a *retract*. (2).

For all functions f, $A < f^{-1}fA$, $f^{-1}R = D$, $f^{-1}0 = 0$.

Theorem. *If* $A < D$, *then* $f^{-1}cfA < cA$, *that is,* $f^{-1}f^*A < A$; also, $(f^{-1})^*A > A$, $f^{-1}f^*A < A$, $f^*f^{-1}B > B$, $f(f^{-1})^*B < B$, where $B < R = fD$.

In general $(f^{-1})^*$ and $(f^*)^{-1}$ are distinct. We have easily,

$$(f^{-1})^*f^*A < A^* \quad , \quad A^* < (f^*)^{-1}f^*A,$$

where

$$A^* = f^{-1}f^*A.$$

If we let $B^* = f^*A$, then $A^* = f^{-1}B^*$ and in all cases $A^* < A$.

Bibliography

(1) cf. H. Hahn. Reele Funktionen. Erster Teil, pp. 1-4, Leipzig (1932). Also N. Bourbaki, No. 846, Actualites Scientifiques et Industrielles, Paris (1939).
(2) cf. G. T. Whyburn, Analytic Topography, Colloquium Publications of the American Mathematical Society, Vol. XXVII (1942).

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