Proceedings of the Iowa Academy of Science

Volume 57 | Annual Issue Article 39

1950

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The first appearance of the set of rational numbers of which I am speaking was in James Bernoulli's *Ars Conjectandi.* This work was published in 1713, eight years after the death of its author. In the 2nd chapter, Bernoulli deals with permutations and combinations. He starts with the formula

$$C(n,k) = C(1,k-1) + C(2,k-1) + \ldots + C(n-1,k-1)$$

where naturally

$$c\ (r,s) = 0 \text{ for } r < s;$$

or

$$\frac{n(n-1)\cdots(n-k+1)}{k!} = \sum_{h=1}^{n} \frac{(h-1)(h-2)\cdots(h-k+1)}{(k-1)!}$$

For $k = 3$, for instance,

$$\frac{n(n-1)(n-2)}{3!} = \sum_{h=1}^{n} \frac{(h-1)(h-2)}{2!} =$$

$$\sum \left(\frac{1}{2}n^2 - \frac{3}{2}n + 1\right) = \frac{1}{2}\sum n^2 - \frac{3}{2}\sum n + n$$

Σn was of course known, long before Bernoulli's time, to equal $\frac{n\ (n+1)}{2}$.

$$\therefore \sum n^2 = \frac{n(n-1)(n-2)}{3} + \frac{3n(n+1)}{2} - 2n$$

$$= \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n$$

Likewise, for $k = 4$,

$$\frac{n(n-1)(n-2)(n-3)}{4!} = \sum_{h=1}^{n} \frac{(h-1)(h-2)(h-3)}{3!}$$

$$= \frac{1}{6}\sum n^3 - \sum n^2 + \frac{11}{6}\sum n - n$$

from which follows

$$\sum n^3 = \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2$$

Bernoulli gives thus a table of the values of Σn^c (the notation he used) as far as $c = 10$. Thereupon appears the remarkable generalization:

$$\sum n^c = \frac{n^{c+1}}{c+1} + \frac{n^c}{2} + \frac{c}{2}An^{c-1}$$

$$+ \frac{c(c-1)(c-2)}{2\cdot 3\cdot 4}Bn^{c-3}$$

$$+ \frac{c(c-1)(c-2)(c-3)(c-4)}{2\cdot 3\cdot 4\cdot 5\cdot 6}Cn^{c-5} + \ldots$$

in which $A = 1/6$, $B = -1/30$, $C = 1/42$, $D = -1/30$, $E = 5/66 \ldots$, the first five of the numbers B_2, B_4, $B_6 \ldots$. Bernoulli does not indicate how he obtained this generalization. It seems, however, from the coefficients which appear in the formulas as far as Σn^{10}, that the discovery of the first five numbers should not be too difficult. But to arrive at any general formula, or recurrence relation, seems hopeless by merely following Bernoulli's tracks.

Euler took a different line, and with his tremendous power of computation managed to determine the value of the first 15 non-vanishing values.

Passing on at once to a relatively simple method for computing the numbers, the following has been published in various places and is perhaps the one most frequently used:

Starting with the function $\frac{x}{e^x - 1}$, which is certainly analytic in the vicinity of $x = 0$ if the value of $f(0)$ is defined to be unity, we may write

(1)

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n x^n}{n!} = B_0 + \frac{B_1 x}{1!} + \frac{B_2 x^2}{2!} + \cdots$$

Thus,

$$1 = \left(1 + \frac{x}{2!} + \frac{x^2}{3!} + \cdots\right)\left(B_0 + \frac{B_1}{1!}x + \frac{B_2}{2!}x^2 + \cdots\right)$$

so that $B_0 = 1$, $B_1 = -\frac{1}{2} \ldots$ and in general $(n > 1)$

$$\frac{1}{n!}B_0 + \frac{1}{(n-1)!}\frac{B_1}{1!} + \frac{1}{(n-2)!}\frac{B_2}{2!} + \cdots + \frac{1}{1!}\frac{B_{n-1}}{(n-1)!} = 0$$

If we multiply by $n!$, we get

$$\binom{n}{0}B_0 + \binom{n}{1}B_1 + \binom{n}{2}B_2 + \cdots + \binom{n}{n-1}B_{n-1} = 0$$

or symbolically, $(B + 1)^n - B^n = 0$, replacing B_k by B^k. Returning to

$$f(x) = \frac{x}{e^x - 1}, \quad f(x) + \frac{x}{2} = \frac{x}{2}\left(1 + \frac{2}{e^x - 1}\right)$$

$$= \frac{x}{2}\frac{e^x + 1}{e^x - 1} = \frac{x}{2}\frac{e^{\frac{x}{2}} + e^{-\frac{x}{2}}}{e^{\frac{x}{2}} - e^{-\frac{x}{2}}}$$

(2)

which is an even function. Thus $B_3 = B_5 = \ldots = B_{2k-1} = 0$ and for a few of the others:

for $n = 2, (B + 1)^2 - B^2 = 2B_1 + 1 = 0$
$n = 3, 3B_2 + 3B_1 + 1 = 0 \quad B_2 = 1/6$
$n = 4, 4B_3 + 6B_2 + 4B_1 + 1 = 0 \quad B_3 = 0$ (known)
$n = 5, 5B_4 + 10B_3 + 10B_2 + 5B_1 + 1 = 0 \quad B_4 = -1/30$
$n = 7, 7B_6 + 21B_5 + 36B_4 + 35B_3 + 21B_2 + 7B_1 + 1 = 0$
$B_6 = 1/42$

The case $n = 125$ would seem to be a good problem for one of the super-machines now being constructed. For this would check what is, so far as I know, the last Bernoulli number to have been computed.

The uses of the Bernoulli numbers are almost innumerable. Among the most interesting are the series expansions for tan x, cot x, and csc x. These are found without much difficulty, starting with (2). The results are

$$x \cot x = 1 - \frac{2^2 B_2}{2!} x^2 + \frac{2^4 B_4}{4!} x^4 - + \cdots + (-1)^k \frac{2^{2k} B_{2k}}{(2k)!} x^{2k} + \cdots$$
$$= 1 - \frac{1}{3} x^2 - \frac{1}{45} x^4 - \frac{2}{945} x^6 - \cdots \tag{3}$$

$$\tan x = \sum_{K=1}^{\infty} (-1)^{K-1} \frac{2^{2K}(2^{2K} - 1) B_{2K}}{(2K)!} x^{2K-1}$$
$$= x + \frac{1}{3} x^3 + \frac{2}{15} x^5 + \frac{17}{315} x^7 + \cdots \tag{4}$$

$$x \csc x = \sum_{K=0}^{\infty} (-1)^{K-1} \frac{(2^{2K} - 2) B_{2K}}{(2K)!} x^{2K}$$
$$= 1 + \frac{1}{6} x^2 + \frac{7}{360} x^4 + \frac{31}{15120} x^6 + \cdots \tag{5}$$

We have so far assumed that the B_{2n} in the series for $\frac{x}{e^x - 1}$ were the same B_{2n} which Bernoulli gave as A, B, C . . . in the expansion of Σn^x in powers of n. The proof of this is not very difficult, since we have Euler's expansion of cot x in a series of partial fractions. From this we are led by a direct road to the identity

$$\frac{1}{e^x-1} = \frac{1}{x} - \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2x}{x^2+4n^2\pi^2}$$

Now for $|x| < 2\pi$ each fraction can be expanded in powers of x, giving

$$\frac{2x}{x^2+4n^2\pi^2} = \frac{x}{2n^2\pi^2}\left[1 - \frac{x^2}{4n^2\pi^2} + \frac{x^4}{4^2 n^4 \pi^4} - + \cdots\right] \tag{6}$$

The double series (6) being absolutely convergent within its circle of convergence, can be summed "vertically" in powers of x, so that

$$\frac{1}{e^x-1} = \frac{1}{x} - \frac{1}{2} + \frac{x}{2\pi^2}\sum\frac{1}{n^2} - \frac{x^3}{2^4\pi^4}\sum\frac{1}{n^4}\cdots$$

Thus

$$B_{2r} = (-1)^{r-1}\frac{(2r)!}{2^{2r-1}\pi^{2r}}\sum_{1}^{\infty}\frac{1}{n^{2r}}$$

and we have the connection between the Bernoulli numbers as obtained by their discoverer, and the coefficients in the expansion of $\frac{x}{e^x - 1}$. Incidentally we note that the successive Bernoulli numbers increase greatly as r increases.

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