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The Neumann problem requires the determination of a function $\emptyset$ which satisfies Laplace's equation in a domain V bounded by a closed surface B and is such that its normal derivative takes on assigned values f on B. It is assumed that f is piecewise continuous function of position on B and satisfies the condition $\int f\, dB = 0$.

The method of the present paper was first developed in connection with the Dirichlet problem,[1] which is based on the idea of the hypercircle in function space introduced by W. Prager and J. L. Synge.[2] The present work differs from that of Prager and Synge in that they were interested in obtaining bounds in the mean square sense for elastostatic boundary value problems, whereas we have as our present objective the determination of bounds at a point for the derivatives of the solution of the Neumann problem.

In the present paper, it is assumed that the hypercircle has already been found. This in itself is a rather difficult task but has been carried out for the Neumann problem by Synge in a previous paper.[3] However, once the solution has been located on a hypercircle the remainder of the work is relatively simple. There are certain weaknesses in the method, namely, that the method does not apply to a point on the boundary except in very special cases,[4] and that as the point at which bounds are being sought approaches the boundary the bounds become progressively weaker. Hence, we shall restrict the present work to apply only to points interior to the domain of definition of the problem.

We may consider the problem in Euclidean N-space E_N which enables us to treat simultaneously the two most interesting cases, $N = 2, 3$. Let V denote an open domain in E_N, bounded by a closed surface B. We shall have occasion to make use of an N-dimensional sphere with center at a general point P contained in V and having radius a. The interior of any such sphere will be denoted by v and its bounding surface by b. The unit normal n (with components n_i, $i = 1, 2, \ldots, N$) will always be directed away from P on either B or b.

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1 C. G. Maple, The Dirichlet problem: bounds at a point for the solution and its derivatives, Quart. App. Math., 8, 213-228, (1950).

2 W. Prager and J. L. Synge, Approximations in elasticity based on the concept of function space, Quart. App. Math., 5, 241-269 (1947).

3 J. L. Synge, The method of the hypercircle in function space for boundary value problems, Proc. Roy. Soc. (London) A *191*, 447-467 (1947).

4 As an example of one such special case, see my paper on the Dirichlet problem.

The summation convention holds for repeated Latin suffixes and the range of such indices is always 1, 2, . . . , N. The coordinate system will be rectangular cartesian with origin at P and differentiation with respect to a coordinate x_i is indicated by a comma ($u,_i = \frac{\delta u}{\delta x_i}$). Integration will be denoted by a single integral sign and the range indicated by the element of integration unless otherwise indicated.

We introduce the idea of function space F and distinguish vectors if F from those in E_N by writing those in F in heavy type. A vector **S** in F is defined to be any vector field in $V + B$. The components p_i of this field are assumed to have continuous first derivatives.

The scalar product of two vectors **S** and **S′** will be denoted by **S**. **S′** and defined by

$$(1) \qquad \mathbf{S}.\mathbf{S}' = \int p_i p_i' \, dV.$$

If $\mathbf{S} = \mathbf{S}'$, we get the metric for the space:

$$(2) \qquad \mathbf{S}^2 = \mathbf{S}.\mathbf{S} = \int p_i p_i \, dV.$$

The assumption that the solution vector **S** lies on a hypercircle Γ is expressed by writing

$$(3) \qquad \mathbf{S} = \mathbf{C} + R\mathbf{J},$$

where **C** is a known vector representing the center of Γ and R, a known positive number, the radius of Γ. **J** is an arbitrary vector except for the restrictions

$$(4) \qquad \mathbf{J}.\mathbf{J} = 1, \mathbf{J}.\mathbf{I}\alpha = 0, \ (\alpha = 1, 2, \ldots, m),$$

where the vectors $\mathbf{I}\alpha$ form a set of known vectors satisfying the conditions of orthonormality $\mathbf{I}\alpha.\mathbf{I}\beta = \delta\alpha\beta$.

Let **G** be any vector in F and consider the scalar product of **S**. **G**, where **S** is the solution vector. We determine maximum and minimum values of **S**. **G** as **S** ranges over the hypercircle. We may represent **G** as the sum of its projections onto each of the unit vectors $\mathbf{I}\alpha$ and the subspace of the hypercircle; that is,

$$(5) \qquad \mathbf{G} = M\mathbf{J}_o + \sum_{\alpha=1}^{m} N_\alpha \mathbf{I}_\alpha,$$

where $M\mathbf{J}_o$ ($\mathbf{J}_o.\mathbf{J}_o = 1$) is the vector projection of **G** onto the plane of the hypercircle and

$$N_\alpha = \mathbf{I}_\alpha \cdot \mathbf{G}, \tag{6}$$

$$M^2 = \mathbf{G}^2 - \sum_{\alpha=1}^{m} (\mathbf{I}_\alpha \cdot \mathbf{G})^2, \quad M > 0.$$

Now due to the fact that **S** is on the hypercircle, we have

$$\mathbf{S}\cdot\mathbf{G} - \mathbf{C}\cdot\mathbf{G} = R\,\mathbf{J}\cdot\mathbf{G} \tag{7}$$

$$= R\,\mathbf{J}\cdot(M\mathbf{J}_0 + \sum_{\alpha=1}^{m} N_\alpha \mathbf{I}_\alpha)$$

$$= RM\,\mathbf{J}\cdot\mathbf{J}_0 .$$

Hence

$$|\mathbf{S}\cdot\mathbf{G} - \mathbf{C}\cdot\mathbf{G}| \leqslant RM. \tag{8}$$

We are interested in obtaining bounds at a point P for the derivative $\emptyset,_p$, where $\emptyset,_p$ represents the derivative in the direction x_p of the solution function $\emptyset$. For this purpose, we define a function

$$G^{(p)} = x_p r^{-N}, \tag{9}$$

where $r^2 = x_i x_i$ is the square of the distance measured from the point P. The partial derivatives of this function given by (10) below form a vector field in $V + B$ which is made to correspond to a vector $\mathbf{G}^{(p)}$ in function space.

$$G^{(p)}_{,i} = \delta_{ip} r^{-N} - N x_i x_p r^{-N-2}, \quad r \geqslant a, \tag{10}$$

$$= 0 \qquad\qquad , \quad r < a.$$

In defining the vector field as given by (10), we have cut out its singularity in V so that the scalar product $\mathbf{S}.\,\mathbf{G}^{(p)}$ has meaning. Using Green's theorem, this scalar product may be written

$$(11) \quad \mathbf{S}\cdot\mathbf{G}^{(p)} = \int \phi_{,i} G^{(p)}_{,i}\, dV = \int_{V-v} \phi_{,i} G^{(p)}_{,i}\, dV$$

$$= \int \phi_{,i} G^{(p)} n_i\, dB - \int \phi_{,i} G^{(p)} n_i\, db$$

$$- \int \phi_{,ii} G^{(p)} dV.$$

The last integral in (11) is zero due to the fact that $\emptyset$ is harmonic in V and the integral over the surface B is calculable since $\emptyset,_i n_i = f$ is known on B. The integral over b may be expressed in terms of the value of $\emptyset,_p$ at the point P by use of a mean value theorem for harmonic functions.[5] Thus by (9) and Green's theorem, we have

$$(12) \quad \int \phi_{,i} G^{(p)} n_i\, db = a^{-N} \int \phi_{,i} x_p n_i\, db$$

$$= a^{-N} \int \phi_{,p}\, dV = L_N\, \phi_{,p}(P),$$

where

$$(13) \qquad L_N = \frac{\pi^{N/2}}{\Gamma\left(\frac{N}{2}+1\right)}$$

We note that L_N is independent of the radius a and that $L_2 = \pi$ and $L_3 = 4\pi/3$. Hence, the scalar product $\mathbf{S}.\,\mathbf{G}^{(p)}$ may be expressed in terms of a calculable integral and the value of $\emptyset,_p$ at the point P:

$$(14) \quad \mathbf{S}\cdot\mathbf{G}^{(p)} = \int \phi_{,i} G^{(p)} n_i\, dB - L_N\, \phi_{,p}(P).$$

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[5] Cf. O. D. Kellogg, Potential Theory (Berlin, 1929) p. 224.

This may now be combined with the inequality (8) to give the results which are summarized in the following theorem.

THEOREM. *Let* $\emptyset$ *be a function which is harmonic in* V *and such that its normal derivative assumes assigned values on the boundary* B *of* V. *Let* P *be any interior point of* V. *Then bounds for* $\emptyset,_p$ (P) are given by the inequality

$$(15) \quad \left| \int f G^{(p)} dB - \mathbf{C} \cdot \mathbf{G}^{(p)} + L_N \emptyset_{,p}(P) \right| \leq R M^{(p)}$$

where **C** *and* R *represent the center and radius respectively of the hypercircle and*

$$(16) \quad G^{(p)} = x_p r^{-N}, \quad L_N = \frac{\pi^{N/2}}{\Gamma\left(\frac{N}{2}+1\right)},$$

$$\left(M^{(p)}\right)^2 = \left(\mathbf{G}^{(p)}\right)^2 - \sum_{\alpha=1}^{m} \left(\mathbf{I}_\alpha \cdot \mathbf{G}^{(p)}\right)^2 .$$

It may also be desirable to have bounds for a second derivative $\emptyset,_{pq}$ at a point *P*. In this case, it is convenient to define a function

$$(17) \quad G^{(pq)} = x_p x_q r^{-N-2}$$

and obtain a vector field in *V* by differentiation of this function in each of the coordinate directions. We define this vector field in *V* as follows:

$$(18) \quad G^{(pq)}_{\cdot i} = \delta_{pi} x_q r^{-N-2} + \delta_{iq} x_p r^{-N-2}$$
$$- (N+2) x_p x_q x_i r^{-N-4}, \quad r \geq a$$
$$= 0 \qquad , \quad r < a$$

and make the vector $\mathbf{G}^{(pq)}$ in F correspond to (18). By means of a procedure similar to that used in the preceding case, it is now possible to express the scalar product $\mathbf{S} . \mathbf{G}^{(pq)}$ (**S** is the solution vector) in terms of calculable integrals and the value of $\emptyset,_{pq}$ at the point P. Explicitly,

$$(19) \quad \mathbf{S} \cdot \mathbf{G}^{(pq)} = \int \phi_{,i} G^{(pq)}_{,i} \, dV$$

$$= \int \phi_{,i} G^{(pq)} n_i \, dB - \int \phi_{,i} G^{(pq)} n_i \, db.$$

Here we have made use of Green's theorem and the harmonic character of $\emptyset$ to transform the above volume integral into integrals over the bounding surfaces. Now the integral over B is calculable; so we turn our attention to the remaining integral, which depends on the radius a. Substitution of (17) into this integral and use of Green's theorem gives

$$(20) \quad \int \phi_{,i} G^{(pq)} n_i \, db = a^{-N-2} \int \phi_{,i} x_p x_q n_i \, db$$

$$= a^{-N-2} \int (\phi_{,ii} x_p x_q + \phi_{,i} \delta_{pi} x_q + \phi_{,i} \delta_{qi} x_p) \, dv$$

$$= a^{-N-2} \int (\phi_{,p} x_q + \phi_{,q} x_p) \, dv.$$

Let us set

$$(21) \quad J = \int \phi_{,p} x_q \, dv$$

and differentiate J with respect to a. We get

$$(22) \quad \frac{dJ}{da} = \int \phi_{,p} x_q \, db = a \int \phi_{,p} n_q \, db$$

$$= a \int \phi_{,pq} \, dv = a^{N+1} L_N \, \phi_{,pq}(P),$$

since $\emptyset$ is a harmonic function. Integration of this last form gives

$$(23) \quad J = \frac{a^{N+2}}{N+2} L_N \, \phi_{,pq}(P),$$

where the constant of integration is zero (cf. equa. (21)). Using these results, the scalar product (19) may be written

$$\mathbf{S}\cdot\mathbf{G}^{(pq)} = \int \phi_{,i}\, G^{(pq)} n_i \, dB - \frac{2}{N+2} L_N \, \phi_{,pq}(P). \tag{24}$$

When we substitute this expression in (8), we get the following theorem:

THEOREM. *Let* Ø *be a function which is harmonic in* V *and such that its normal derivative assumes assigned values on the boundary* B *of* V. *Let* P *be any interior point of* V. *Then bounds for* $\phi_{,pq}$ (P) *are given by*

$$\left| \int f\, G^{(pq)} dB - \mathbf{C}\cdot\mathbf{G}^{(pq)} - \frac{2}{N+2} L_N \, \phi_{,pq}(P) \right| \leq R M^{(pq)}, \tag{25}$$

where **C** *and* R *represent the center and radius respectively of the hypercircle and* $G^{(pq)}$ *is defined by* (17) *and* $M^{(pq)}$ *by* (6).

It may be noted that the Green's vectors given by (10) and (18) in the two preceding cases are defined in such a way that only the particular derivatives of Ø for which bounds are sought enter into the scalar product of the solution vectors and the Green's vectors. This property does not persist for derivatives of higher order than the second when the function which leads to the vector field in V is defined in a manner analogous to (9) and (17). However, it is possible to select a function from which a vector field may be obtained by differentiation in such a way as to preserve this property. In the case of a third order derivative, say $\phi_{,pqs}$, one such function may be obtained by differentiation of the fundamental solution of Laplace's equation once with respect to each of the variables x_p, x_q and x_s. This of course leads to a much more complicated expression for the vector field in V and consequently to a more elaborate procedure in the evaluation of the scalar product of the solution vector and the Green's vector corresponding to this field.

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