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Theorem on the Zeros of Polynomials

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A simple method of finding limits for the absolute values of the zeros of polynomials will be given in this paper. The radius of a circle about the origin in the complex plane will be found from the coefficient of the polynomial such that all the zeros will lie on or within this circle. For certain polynomials a second circle will be found such that the zeros will lie on or outside this circle.

In finding these circles the following well-known theorem of Rouché is used:

Rouché's Theorem. If f(z) and g(z) are analytic on and within a simple closed curve C and $|g(z)| < |f(z)|$ on C, then f(z) and $f(z) + g(z)$ have the same number of zeros within C.

Rouché's Theorem is very useful. By means of it the fundamental theorem of algebra and other important results may be established. The proof of Rouché's Theorem is ordinarily based on residue theory and may be found in most books on the theory of functions of a complex variable. Rouché, however, used series expansions to prove his theorem.

The radius of the circle containing all the zeros may be found from the following:

Theorem 1. The zeros of a polynomial $P(z) = a_o + a_1 z + \cdots + A_n z^n$, $a_n \neq 0$, lie on or within the circle

(a) $|z| = K$, where $K = \dfrac{|a_o| + |a_1| + \cdots + |a_{n-1}|}{|a_n|}$, if $K \geqslant 1$,

or

(b) $|z| = \sqrt[n]{K}$ if $K < 1$.

For the proof let $f(z) = a_n z^n$ and $g(z) = a_o + a_1 z + \cdots + a_{n-1} z^{n-1}$ such that $f(z) + g(z) = P(z)$. Assume first $K \geqslant 1$. Then on the circle $|z| = r$, where $r > K \geqslant 1$, $|f(z)| = |a_n| r^n$, and

$$|g(z)| \leqslant |a_o| + |a_1| r + \cdots + |a_{n-1}| r^{n-1} \leqslant r^{n-1} |a_n| K < |a_n| r^n.$$

The conditions of Rouché's Theorem are satisfied, and since f(z) has n zeros at the origin, P(z) will have n zeros inside the circle $|z| = r$. The radius r may be taken arbitrarily close to K when $K \geqslant 1$; therefore, the zeros lie on or within the circle $|z| = K$ if $K \geqslant 1$.

Next assume $K < 1$. Then on the circle $|z| = r$, where $\sqrt[n]{K} < r < 1$, $|f(z)| = |a_n| r^n$, and

$$|g(z)| \leqslant |a_o| + |a_1| r + \cdots + |a_{n-1}| r^{n-1} \leqslant |a_n| K < |a_n| r^n.$$

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The use of Rouché's Theorem again gives the n zeros of P(z) inside the circle $|z| = r$. Since r may be taken arbitrarily close to $\sqrt[n]{K}$, the zeros lie on or within the circle $|z| = \sqrt[n]{K}$ if $K < 1$.

The following two remarks may now be added:

1. If some $a_i \neq 0$ for $i = 0,1,\cdots,n-2$ and $K > 1$, then the zeros lie within the circle $|z| = K$.
2. If some $a_i \neq 0$ for $i = 1,2,\cdots,n-1$ and $K < 1$, then the zeros lie within the circle $|z| = \sqrt[n]{K}$.

These remarks may be established by taking $r = K$ and $r = \sqrt[n]{K}$ in the respective parts of the preceding proof.

The radius of the circle containing no zeros of certain polynomials may be found from the following:

Theorem 2. The zeros of a polynomial lie on or outside the circle $|z| = \sqrt[n]{M}$ where $M = \frac{|a_0|}{|a_1| + |a_2| + \cdots + |a_n|}$ if $M > 1$.

Let $f(z) = a_0$ and $g(z) = a_1 z + a_2 z^2 + \cdots + a_n z^n$.

On the circle $|z| = r$, where $1 < r < \sqrt[n]{M}$, $|f(z)| = |a_0|$, and

$$|g(z)| \leqslant |a_1|r + |a_2|r^2 + \cdots + |a_n|r^n \leqslant r^n \frac{|a_0|}{M} < |a_0|.$$

The conditions of Rouché's Theorem are again satisfied and P(z) will have no zeros within this circle since f(z) has none. The radius r may be taken arbitrarily close to $\sqrt[n]{M}$; therefore, the zeros lie on or outside the circle $|z|$ $|n| = \sqrt[n]{M}$ when $M > 1$.

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