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Pathological Functions[1]

By H. P. Thielman

1. Introduction

Definition 1.1. Let f be a function whose domain of definition X is a neighborhood space. Let P be a point property of the function f. The function f is said to be *peculiar* or *pathological with respect to the property* P if there exists a partition of X into two subsets X_1 and X_2 each everywhere dense in X and such that the property P holds at every point of X_1 and fails to hold at every point of X_2. (Since the term *pathological function* has been used in a wider sense before even in mathematical literature, the term *peculiar* is to be preferred in the above definition.)

For example, a function f whose domain of definition is the interval (a,b), whose points of continuity are everywhere dense in (a,b) and whose points of discontinuity are everywhere dense in (a,b), is peculiar with respect to continuity, and also with respect to discontinuity.

It is well known that there exist functions which are peculiar according to the above definition with respect to such properties as continuity, differentiability and neighborliness. The last concept was defined in a recent article by Woodrow W. Bledsoe[2] for real functions of one real variable. We shall state the definition of this concept for functions whose domains of definition and whose ranges are neighborhood spaces.

The object of this paper is to give some examples of functions which are peculiar with respect to continuity and neighborliness, and also to show that there exist point properties (of functions) with respect to which no function can be peculiar. One such point property, namely that of *cliquishness* of a *function* as defined here is a natural generalization of continuity and neighborliness, but yields results which are quite different from those which hold for continuous or neighborly functions. This property of cliquishness reduces to the property of neighborly' as defined by Bledsoe[3] for the case when the domain of definition of the function is an interval (a,b) and the range of the function is a metric space.

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[1]Selected as the most meritorious paper presented to the Mathematics Section of the Iowa Academy of Science in 1952.

[2]Bledsoe, W. W., Proc. Amer. Math. Soc., Vol 3, pp. 114, 115 (1952).

[3]Bledsoe, *loc. cit.* p. 115.

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2. Neighborly Functions

Definition 2.1. Let f be a function whose domain of definition is a neighborhood space X with neighborhoods N, and whose range is a subset of a neighborhood space with neighborhoods M. The function f is said to be *neighborly at a point* c of its domain of definition if for every neighborhood N_c of the point c, and for every neighborhood $M_{f(c)}$ of $f(c)$ there exists a neighborhood N contained in N_c (but not necessarily containing c) such that for every element x of N it is true that $f(x)$ is an element of $M_{f(c)}$. A function which is neighborly at each point of its domain of definition is said to be *neighborly*. A function is said to be *non-neighborly at a point* if it is not neighborly at that point.

It is obvious that if a function is continuous at a point then it is neighborly at that point.

Example 2.1. The function $f(x)$ given by the equation

$$f(x) = \sin \frac{1}{x} \qquad \text{if } x \neq 0,$$
$$= 0 \qquad \text{if } x = 0,$$

is discontinuous at $x = 0$, but it is neighborly at that point.

In analogy with the standard definition of pointwise continuous functions we make the definitions.

Definition 2.2. A function f is said to be *pointwise neighborly* in X if the set of points x where $f(x)$ is non-neighborly is everywhere dense in X but not closed relative to X. A function f is said to be *pointwise non-neighborly* in X if the set of points where $f(x)$ is neighborly is everywhere dense in X but not closed relative to X.

Example 2.2. The function $f(x)$ defined in the open interval $(0,1)$ and given by

$$f(x) = 0 \text{ if } x \text{ is irrational,}$$
$$= 1/q \text{ at } p/q$$
where p and q are relatively prime positive integers, is pointwise continuous, pointwise discontinuous, pointwise neighborly, and pointwise non-neighborly in $(0,1)$. It is continuous and neighborly at each irrational point in $(0,1)$. It is thus peculiar with respect to continuity, and also with respect to neighborliness.

For the function of this last example the points of continuity constitute a set of measure one, while the points of discontinuity form an exerywhere dense set of measure zero. In the next example these characteristics of the corresponding sets are reversed, that is, the set of point of continuity will be of measure zero, while the set of points of discontinuity will be a set of measure one. It is well known, and easily proved, that the set of points of discontinuity of every pointwise discontinuous function is an exhaustible set. (An exhaustible set, or a set of Baire's first category, is a set which can be expressed as the sum of at most denumerably many nowhere dense sets.) Hence in order to construct the required function we will have to construct an exhaustible set of measure one on which the required function is discontinuous. The complement of an exhaustible set in the real continuum is a residual set which is everywhere dense. If the required function is thus so constructed as to be continuous on this residual set then our goal will have been attained.

Example 2.3. By the well known method used for the construction of the famous Cantor ternary set, we construct a sequence of sets $N_1, N_2, N_3, \ldots$, such that each set is nowhere dense in the closed interval $[0,1]$. The complement of the set N_n consists of the points of a denumerable number of non-overlapping, non-abutting open intervals. We construct the required sets N_n so that the sum of the lengths of the complementary intervals is $1/2^n$. Then the set

$$S = \Sigma_{i=1}^{\infty} N_i$$

is an exhaustible set whose measure is one. Its complement is everywhere dense in $[0,1]$, and has measure zero. We now define a sequence of functions as follows:

$$f_n(x) = \frac{1}{2^n} \quad \text{if } x \text{ is an element of } N_n,$$
$$= 0 \quad \text{if } x \text{ is not an element of } N_n.$$

The function

$$F(x) = \Sigma_{n=1}^{\infty} f_n(x) \qquad (0 \leqslant x \leqslant 1)$$

is the required function. That $F(x)$ is discontinuous at each point of S is obvious from the fact that if x belongs to S, then x is in at least one of the sets $N_n (n = 1, 2, 3, \cdots)$. Suppose x is an element of N_k. Then $F(x) > 1/2^k$. But in every neighborhood of x there are points of the complement of S where $F(x) = 0$. Next, $F(x)$ is continuous at each point of the complement of S. Let c be a point of complement of s, and let a positive number ϵ be given. Since the series $\Sigma f_n(x)$ is uniformly convergent in $[0,1]$, there exists a positive interger m such that the remainder $\Sigma_{m+1}^{\infty} f_n(x)$ is less than ϵ for all x in $[0,1]$. We can find an interval (α, β) containing c such that (α, β) contains no point of the finite set of nowhere dense sets $N_1, N_2, \ldots, N_m$. Then for every x of (α, β), x is either an element of the complement of S, or x belongs only to those nowhere dense sets N_n for which $n > m$. In either case $|F(c) - F(x)| < \epsilon$, and $F(x)$ is thus seen to be continuous at c.

3. Cliquish Functions.

Definition 3.1. Let the domain of definition of a function f be a neighborhood space with neighborhoods N, and let the range of the function be a subset of a metric space with metric ρ. The function f is said to be *cliquish at a point c* of the closure of the domain of definition if for every positive number ϵ, and for every neighborhood N_c of c there exists a neighborhood N contained in N_c (but not necessarily containing c) such that for every two element x_1 and x_2 of N it is true that

$$\rho\,[f(x_1), f(x_2)] < \epsilon\ .$$

It is obvious that if a function is continuous, or neighborly at a point then it is cliquish at that point. The function $f(x)$ given by

$$f(x) = \sin 1/x \qquad \text{if } x \neq 0$$
$$= 2 \qquad \text{if } x = 0$$

is not neighborly at $x = 0$, but it is cliquish at that point.

A function which is cliquish at every point of its domain of definition is said to be cliquish.

The functions of Examples 2.2, and 2.3 are only pointwise continuous and pointwise neighborly but they are cliquish. As a matter of fact, we shall see that every function which is pointwise non-neighborly is cliquish.

A function is said to be *non-cliquish* at a point if it is not cliquish at that point. The real function which is zero at each rational point, and one at each irrational point, is non-cliquish at every point.

Definition 3.2. A function is said to be *pointwise cliquish* in X if the set of points where the function is non-cliquish is everywhere dense in X but not closed relative to X. A function is said to be *pointwise non-cliquish* in X if the set of points where the function is cliquish is everywhere dense in X but not closed relative to X.

A function which was both pointwise cliquish and pointwise non-cliquish in its domain of definition would be peculiar with respect to cliquishness. We shall now prove the *interesting result that there exists no function which is peculiar with respect to cliquishness.*

THEOREM 3.1 *Let f be a function with domain of definition X. If f is cliquish at each point of a set which is everywhere dense in X, then f is cliquish in X.*

Note: This means that there exists no function which is pointwise cliquish and also pointwise non-cliquish in a given set).

Proof: Let f be the function given in the statement of the theorem. Then there exists a set C everywhere dense in X, and such that for each point c of C the function f is cliquish at c. Let x be a given point of X, and let N_x be an arbitrarily given neighborhood of x. In N_x there exists at least one point c of C. Let a positive number ϵ be given, and let N_c be a neighborhood of c such that N_c is contained in N_x. Since f is cliquish at c, there exists a neighborhood N contained in N_c (and hence contained in N_x) such that for every x_1 and x_2 of N it is true that

$$\rho\,[f(x_1)\ f(x_2)] < \epsilon.$$

Since N is contained in N_x and since N_x was an arbitrarily given neighborhood of x, f is cliquish at x. But x was an arbitrarily given point of X. Therefore f is cliquish at every point of X. In other words, f is cliquish.

As a direct consequence of this we have the result to which we alluded above that every pointwise non-neighborly function is cliquish. In particular, every pointwise discontinuous function is cliquish.

THEOREM 3.2. *The set of points at which a pointwise cliquish function is cliquish is nowhere dense.*

Proof: Suppose that the set C where f is cliquish were not nowhere dense in the domain of definition X of f. Then there would exist at least one neighborhood N such that C would be everywhere dense in N. Then f would be point-wise non-cliquish in N. Hence by Theorem 3.1, f would be cliquish at every point of N. This contradicts the hypothesis that the set of points where f is non-cliquish is everywhere dense in X.

It might be of interest to note that a type of converse of this theorem holds. That is, for every set S which is nowhere dense in an interval (a,b) there exist pointwise cliquish functions which are cliquish on S and non-cliquish at each point of the everywhere dense complement of the closure of S in (a,b). An example of such funtions can be constructed in the following way. The complement of the closure of S in (a,b) is an everywhere dense set which consists of the points of a denumerable number of non-overlapping open intervals. On each of these intervals the function is defined as follows: Let (α, β) be such an interval. Then

$$f(x) = (x\text{-}\alpha) \text{ if } \alpha < x \leqslant \frac{\alpha + \beta}{2}, \text{ and } x \text{ is rational,}$$

$$= -(x\text{-}\beta \text{ if } \frac{\alpha + \beta}{2} < x < \beta, \text{ and } x \text{ is rational,}$$

$$= 0 \qquad \text{if } \alpha < x < \beta \text{ and } x \text{ is irrational.}$$

For each point x of the closure of S, $f(x) = 0$. This function is easily seen to be continuous, and hence cliquish, at each point of S. On every open interval (α, β) of the everywhere dense complement of S in (a, b), $f(x)$ is totally discontinuous and non cliquish.

THEOREM 3.3. *The limit $F(x)$ of a sequence of cliquish functions can be non-cliquish at every point of its domain of definition.*

Proof: Let $F(x) = 1$ if x is a rational number in the open interval $(0, 1)$; let $F(x) = 0$ if x is an irrational number in $(0, 1)$. This function is non-cliquish at each point of $(0, 1)$. It is, however, the limit of the sequence $\{F_n(x)\}$, where

$$F_n(x) = \Sigma_{q=2}^{q=n} f_q(x),$$

and where $f_q(x) = 1$, if $x = p/q$, $p < q$, p and q are relatively prime positive integers, while $f_q(x) = 0$ if $x \neq p/q$. Each $F_n(x)$ is obviously cliquish at each point in $(0, 1)$.

The three theorems stated illustrate how different the point property of cliquishness of a function is from the properties of continuity and neighborliness. The corresponding statements for continuous and neighborly functions are false. Thus for example, it is not true that every function, which is continuous on a set dense in the domain of definition of the function, is continuous, nor is every function, which is neighborly on a set dense in the domain of definition of the function, neighborly. As opposed to Theorem 3.2 we have for continuous and neighborly functions the result that the set of points at which a pointwise continuous function is continuous may be everywhere dense. An analogous result holds for neighborly functions. Instead of Theorem 3.3 we have the important result that the limit of a sequence of continuous (neighborly) functions is at most pointwise discontinuous (non-neighborly).

The next two theorems and their proofs are analogues of similar theorems and proofs in the theories of pointwise discontinuous and neighborly functions.

THEOREM 3.4. *The points of discontinuity of every cliquish function form a set of Baire's first category.*

THEOREM 3.5. *Every cliquish function is at most pointwise discontinuous.*

The proofs of these theorems are left as exercises for the reader.

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