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INTRODUCTION

Morse and Leighton (Singular quadratic functionals. Transactions of the American Mathematical Society, volume 40 (1936) pp. 252-286) gave a systematic approach to the problem of minimizing a singular quadratic functional for one dependent variable considering integrands of the type

$$f(x, y, y') = r(x)y'^2 + 2q(x)y\,y' + p(x)y^2$$

where r, q, and p are single-valued continuous functions of the real variable x on the interval (0, d)[2] and r is positive. They defined first conjugate point of the singular point $x = 0$. They defined minimum limit of a functional and determined conditions under which [o, b] would afford such a limit to a functional among several classes of comparison curves.

In this paper we extend to n dependent variables the definition of a conjugate point and the analogue of the Jacobi necessary condition. Criteria will be presented for locating conjugate points.

The repeated suffix notation for summation and the notation $f(x, y, y')$ for $f(x, y_1, \ldots, y_n, y'_1, \ldots, y'_n)$ will be used. Summations will in general be from 1 to n.

1. *The functional.* Let

(1.1) $2\Omega(x,y,y') = r_{ij}\, y'_i\, y'_j + 2q_{ij}\, y'_i\, y_j + p_{ij}\, y_i\, y_j$ $(i, j = 1, \ldots, n)$ where r_{ij}, q_{ij}, and p_{ij} are functions of class C^1 of the real variable x on (0,d) and $r_{ij}\,\pi_i\,\pi_j > 0$ for $0 < x \leqq b$ and for every set $(\pi) \neq 0$. The constant b is fixed but arbitrary on the interval (0, d). We consider the functional

$$(1.2) \qquad J(y)\Big|_e^b = \int_e^b 2\,\Omega\,(x, y, y')\,dx \qquad (0 < e < b < d).$$

We call $y_i(x)$ and the curve $y_i = y_i(x)$ C — admissible on [0, b] if

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[1]The results of this paper were included in a doctoral thesis written under the direction of Professor A. E. Pitcher.

[2]In future discussions intervals will be designated as follows:
- [a, b] means the interval $a \leqq x \leqq b$,
- (a, b] means the interval $a < x \leqq b$,
- (a, b) means the interval $a < x < b$,
- [a, b) means the interval $a \leqq x < b$.

1. $y_i(x)$ are continuous on the closed interval $[0, b]$ and $y_i(0) = y_i(b) = 0$;

2. $y_i(x)$ are absolutely continuous on each closed subinterval of $(0, b]$.

It will be observed that the segment $[0, b]$ of the x — axis is C — admissible and that on this segment $J = 0$. If

$$\liminf_{x=0} \int_x^b 2\,\Omega\,(x, y, y')\,dx \geqq 0 \qquad x > 0$$

holds for each curve $y_i = y_i(x)$ of a given class we say that $[o, b]$ affords a minimum limit to J among curves of the given class.

We seek an analogue to the Jacobi necessary condition. The Euler equations, and also the Jacobi equations, take the form

$$(1.3) \qquad \frac{d}{dx}(r_{ij}\,y'_j + q_{ij}\,y_j) - (q_{ji}\,y'_j + p_{ij}\,y_j) = 0 \qquad (i, j = 1, \ldots, n).$$

The determinant of the coefficient of y'' of this system of linear homogeneous differential equations is $|r_{ij}|$. To every solution $y_i(x)$ of (1.3) we set

$$(1.4) \qquad \zeta_i^y = r_{ij}\,y'_j + q_{ij}\,y_j$$

It is a well known fact that if (η_1, ζ_1) and (η_2, ζ_2) are two solutions of the Jacobi equation then

$$(1.5) \qquad \eta_{1i}(x)\,\zeta_{2i}(x) - \zeta_{1i}(x)\,\eta_{2i}(x) = \text{constant}.$$

If this constant is zero, we call the two solutions conjugate. Form the determinant

$$(1.6) \qquad D(x, c) = |\,v_{ij}\,(x, c)\,|$$

of which the columns are solutions of the Jacobi equations for a constant c and which satisfy the initial conditions

$$(1.7) \qquad v_{ij}(c, c) = 0, v_{ijx}(c,c) = \delta_{ij} \qquad (i, j = 1, \ldots, n)$$

where δ_{ij} is the Kronecker delta. It will be observed that the columns of $D(x,c)$ are mutually conjugate solutions.

A system of n linearly independent mutually conjugate solutions will be called a conjugate base. The set of all solutions dependent on the solutions of a conjugate base will be called a conjugate family. We shall refer to the determinant of a conjugate base. By this shall be meant the determinant

$$(1.8) \qquad D(x) = |\,\eta_{ij}(x)\,|$$

for which the columns are solutions from a given base. If $D(x)$ vanishes to the rth order at $x = a$ then $x = a$ will be called a focal point of the rth order of the given family. If a focal point $x = c$ of a conjugate family F is of order n, then the focal points of F other than $x = c$ are the conjugate points of $x = c$.

A very useful separation theorem for future use may be stated as follows: *Theorem 1.1.* The number of focal points of any conjugate family on a given interval (open or closed at either end) differs from that of any other conjugate family by at most n.

2. *Definition of conjugate points.* The conjugate point for a singular quadratic functional will be defined so that the known classical results for the non-singular case are included in the derived results. Let $0 < a < b$. Let $x_1(a) \leqq x_2(a) \leqq \ldots \leqq x_q(a)$ be the first q conjugate points of $x = a$ that follow a, if these conjugate points exist. Conjugate points of order r will be counted as r conjugate points. It is known that the qth conjugate point of $x = a$ following $x = a$, if it exists, advances or regresses continuously with $x = a$. We therefore define the qth conjugate point of $x = 0$ as the limit of $x_q(a)$ as a tends to zero. Let

$$x_q = \lim_{a = 0} + x_q(a). \tag{2.1}$$

If $x_1(a)$ exists for no value of a on $(0, b)$, $x = 0$ will be said to have no conjugate point on $[0, d)$. It should be noted that x_1 may coincide with $x = 0$. Necessary and sufficient condition for this to happen will be found later.

3. *An analogue of the Jacobi necessary condition.* We recall our definition of C — admissible curves and now prove the following theorem.

Theorem 3.1. If $[0, b]$ affords a minimum limit to J among C-admissible curves, there can be no point conjugate to $x = 0$ on the interval $[0, b)$.

Proof: Let $h > 0$. If $x = 0$ has a conjugate point on $[0, b)$ then $x = h$ will have a conjugate point on (h, b) if h is sufficiently small. Consider the class of C-admissible curves on $[0, b]$ which follow the segment of the x-axis from $x = 0$ to $x = h$. By hypothesis $[0, b]$ affords a minimum limit to J for this class of curves. But the part of the curves from $x = 0$ to $x = h$ contributes nothing to J and $[h, b]$ affords a minimum among absolutely continuous curves joining points $x = h$ and $x = b$ on the x-axis. Hence a point conjugate to $x = h$ cannot exist on (h, b) and the theorem follows at once.

4. *Theorems on conjugate points.* We state the following theorem.

Theorem 4.1. If $x = 0$ has no conjugate point on $(0, b)$ there is a conjugate family with no focal points on $(0, b)$.

Proof: Let the determinant

$$D(x, b) = | y_{ij}(x, b) | \qquad (i, j = 1, \ldots, n) \tag{4.1}$$

where $y_{ij}(b, b) = 0$ and $y_{ijx}(b, b) = -\delta_{ij}$ be the determinant of a conjugate base of family F. It is asserted that family F is a family satisfying the conditions of the theorem. For, let $0 < h < b$, h being otherwise arbitrary. Suppose $D(x, b)$ vanishes at some point x_0, $h < x_0 < b$. But $x = h$ has no conjugate point on $(h, b]$ while $D(x, b)$ vanishes at least $(n + 1)$ times in that interval. This contradicts Theorem 1.1. The proof of the theorem is complete.

Morse and Leighton have shown for $n = 1$ that if $x = 0$ is not its own conjugate point there exists a solution $w(x) \not\equiv 0$ of the Euler equations such that the first conjugate point of $x = 0$, if it exists, is the first position zero of $w(x)$. This result can be generalized. To shorten the proof it will be convenient to first state some results that will be used.

We shall say that conjugate systems $u_{ij}(x)$ and $v_{ij}(x)$ form a *double* conjugate system if

$$(4.2)\qquad \begin{aligned} u_{ij}(x)\,\zeta^u{}_{ik}(x) - u_{ik}(x)\,\zeta^u{}_{ij}(x) &= 0 \\ v_{ij}(x)\,\zeta^v{}_{ik}(x) - v_{ik}(x)\,\zeta^v{}_{ij}(x) &= 0 \\ u_{ij}(x)\,\zeta^v{}_{ik}(x) - v_{ik}(x)\,\zeta^u{}_{ij}(x) &= \delta_{jk}. \end{aligned}$$

Such systems exist as is shown in Hadamard (page 344, Lecons Sur Le Calcul Des Variations. Tome Premier. Librairie Scientifique. A. Herman et Fils, 1910). Let

$$(4.3)\qquad A = \begin{pmatrix} u & v \\ \zeta^u & \zeta^v \end{pmatrix}$$

be a 2n by 2n matrix where the elements are n by n blocks formed as indicated. Let

$$(4.4)\qquad V = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

be a 2n by 2n matrix in the same block arrangement. Then $V^{-1} = -V$.

By the results of Morse and Pitcher (On certain invariants of closed extremals. Proceedings of the National Academy of Sciences, vol. 20 (1934) pp. 282-288) it is known that for a double conjugate system

$$(4.5)\qquad A^T V A = V$$

where A^T denotes the transpose of matrix A. The following lemma will be used in the proof of the next theorem.

Lemma 4.1 The matrix $\begin{pmatrix} \zeta^{vT}{}_{ik} & -v^T{}_{ik} \\ -\zeta^{uT}{}_{ik} & u^T{}_{ik} \end{pmatrix}$ is the inverse of matrix

$$A = \begin{pmatrix} u_{ij} & v_{ij} \\ \zeta^u{}_{ij} & \zeta^v{}_{ij} \end{pmatrix}$$

Proof: From (4.5) it follows that

(4.6) $$A^{-1} = V^{-1} A^T V = \begin{pmatrix} \zeta^{vT} & -v^T \\ -\zeta^{uT} & u^T \end{pmatrix}$$

This completes the proof of the lemma.

We now proceed to prove an important theorem.

Theorem 4.2. If $x = 0$ is not its own conjugate point ($x_1 > 0$ if it exists) there exists a conjugate system y_{ij} of extremals such that

$$| y_{ij}(x) | = 0 \qquad 0 < x \leqq b$$

if and only if $x = x_q$, the multiplicity of the zeros of $| y_{ij}(x) |$ and of the conjugate points of zero being the same.

Proof: This theorem reduces to Theorem 4.1 if $x = 0$ has no conjugate point on $[0, b)$. Accordingly the proof is limited to the case that x_1 exists and $x_1 > 0$.

Choose a double conjugate system $u_{ij}(x)$, $v_{ij}(x)$ as given in (4.2). Then at $x = a$ form equations

(4.7) $$u_{ij}(a) A_{jk} + v_{ij}(a) B_{jk} = 0$$
$$\zeta^u{}_{ij}(a) A_{jk} + \zeta^v{}_{ij}(a) B_{jk} = \delta_{ik}.$$

Recalling Lemma 4.1 it can be seen that equations (4.7) have unique solutions

(4.8) $$A_{jk}(a) = -v_{kj}(a), \; B_{jk}(a) = u_{kj}(a).$$

Set

(4.9) $$\rho(x) = \sqrt{(u_{ik}(x))^2 + v_{jk}(x))^2} \neq 0$$

where summation is on both j and k from 1 to n,

$$a_{jk}(x) = \frac{-v_{kj}(x)}{\rho(x)}, \; b_{jk}(x) = \frac{u_{kj}(x)}{\rho(x)} .$$

Then

(4.10) $$\Sigma a^2{}_{jk}(x) + \Sigma b^2{}_{jk}(x) = 1.$$

As we are dealing with compact sets we may form convergent subsequences of the sets a_{jk} and b_{jk}. There exists a sequence h_q converging monotonically to zero such that each sequence $a_{ij}(h_q)$, $b_{ij}(h_q)$ is convergent.

Set

(4.11) $$a_{ijq} = a_{ij}(h_q)$$
$$b_{ijq} = b_{ij}(h_q)$$

Let $\lim_{q\to\infty} a_{ijq} = \alpha_{ij}$ and $\lim_{q\to\infty} b_{ijq} = \beta_{ij}$. Then

(4.12) $$\Sigma \alpha^2{}_{ij} + \Sigma \beta^2{}_{ij} = 1 \qquad (i, j = 1, \ldots, n).$$

Set

(4.13) $$y_{ikq}(x) = u_{ij}(x) a_{jkq} + v_{ij}(x) b_{jkq}$$
$$\zeta_{ikq}(x) = \zeta^u{}_{ij}(x) a_{jkq} + \zeta^v{}_{ij}(x) b_{jkq}$$
$$y_{ik}(x) = u_{ij}(x) \alpha_{jk} + v_{ij}(x) \beta_{jk}$$
$$\zeta_{ik}(x) = \zeta^u{}_{ij}(x) \alpha_{jk} + \zeta^v{}_{ij}(x) \beta_{jk}.$$

Then

$$y_{ikq} \rightarrow y_{ik}, \zeta_{ikq} \rightarrow \zeta_{ik}$$

and y_{ik} has the desired properties if $x_1 > 0$.[3] For, let $0 < x_0 < x_1$. Consider problem q as

$$(4.14) \qquad J_q(y) = y_{ijq}(x_0)\, \zeta_{ikq}(x_0)\, c_j\, c_k + \int_{x_0}^{b_1} 2\,\Omega\, dx$$

$$y_i(x_0) = y_{ijq}(x_0)\, c_j,\ y_i(b_1) = 0.$$

The focal points for this problem are the conjugate points of h_q and the zeros of $|\, y_{ijq}(x)\,|$.

Suppose $x_0 < b_1 < x_1$. Then $J_q(y) > 0$ for all y satisfying conditions (4.14). Hence, by continuity consideration

$$(4.15) \qquad J(y) = y_{ij}(x_0)\, \zeta_{ik}(x_0)\, c_j\, c_k + \int_{x_0}^{b_1} 2\,\Omega\, dx$$

is non-negative subject to

$$(4.15)' \qquad y_i(x_0) = y_{ij}(x_0)\, c_j,\ y_i(b_1) = 0.$$

Equations (4.15) and (4.15)′ will be referred to as problem O. Hence, there is no zero of $|\, y_{ij}(x)\,|$ on $x_0 < x < b_1$. Since b_1 is arbitrary there is no zero on $x_0 < x < x_1$. As a matter of fact x_0 is also arbitrary so there is no zero on $0 < x < x_1$. More generally if $x_r < b_1 < x_{r+1}$ and $0 < x_0 < x_1$ the Morse index [The Calculus of Variations in the Large, American Mathematical Society Colloquium Publications, vol. XVIII, Chapter III] of problem q is r provided q is chosen so large that $x_r(h_q) < b_1$. Hence the index of problem 0 is also r, as can be seen by the use of the Morse index form. The zeros of $|\, y_{ij}(x)\,|$ on $x_0 < x < b_1$ are therefore r in number counting multiplicity. Again x_0 is arbitrary. So there are r zeros of $|\, y_{ij}(x)\,|$ on $0 < x < b_1$.

This result is obtainable without the use of the Morse index form. For example, by choosing $x_r < x_0 < b_1 < x_{r+1}$ if $x_r < x_{r+1}$, it is seen as above $|\, y_{ij}(x)\,|$ has no zeros on $x_r < x < x_{r+1}$. If m is the greatest integer less than r such that $x_m < x_r$ and if $x_r < x_{r+1}$, then by choosing

$$x_m < x_0 < x_r < b < x_{r+1}$$

it is seen that $|\, y_{ij}(x)\,|$ has an $r - m$ fold zero at x_r.

The case where $r_{ij} = 0$ if $i \neq j$, $q_{ij} = q_{ji}$, and $q'_{ij} = p_{ij}$ if $i \neq j$ will be called the *separated case*. If $x = 0$ is not its own conjugate point the determinant of Theorem 4.2 takes a diagonal form as expressed by the following theorem which will be stated without proof.

Theorem 4.3. If $x = 0$ is not its own conjugate point in the separat-

[3] The proof from this point on was suggested by Prof. M. R. Hestenes.

ed case the determinant of the conjugate base which vanishes at the conjugate points of $x = 0$ will be of the form

$$(4.16) \qquad | y_{ij}(x) |$$

where $y_{ij} = 0$ if $i \neq j$.

Whether or not $x = 0$ is its own conjugate point can be determined by investigating the number of conjugate points that exist for an arbitrary value $x = k > 0$ in the interval $(0, k)$.

Theorem 4.4. Let k be an arbitrary constant $0 < k < d$ and F^k be the conjugate family whose base determinant $D(x, k) = | y_{ij}(x, k)|$ where $y_{ij}(k, k) = 0$ and $\zeta_{ij}(k, k) = \delta_{ij}$. Then $x = 0$ is its own conjugate point if and only if $x = k$ has an infinite number of conjugate points on $(0, k)$.

Proof: Let $0 < h < k$. Form the conjugate family F^h whose base determinant is $D_1(x, h) = | u_{ij}(x, h) |$ where $u_{ij}(h, h) = 0$ and $\zeta_{ij}(h, h) = \delta_{ij}$. By hypothesis $D(x, k)$ vanishes infinitely often in the interval $(0, k)$. Hence, by separation theorem (1.1) $D_1(x, h)$ must vanish infinitely often in that interval and $x = 0$ is its own conjugate point.

Conversely. If $x = 0$ is its own conjugate point then corresponding to $e > 0$ there is an h with $0 < h < e$ such that $x_1(h) < e$. Then $D_1(x, h)$ vanishes at least $n + 1$ times on $[h, e]$ so that $D(x, k)$ vanishes on that interval by virtue of Theorem 1.1. Since e is arbitrary, $D(x, k)$ vanishes infinitely often on $(0, k)$.

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