Proceedings of the Iowa Academy of Science

Volume 60 | Annual Issue Article 60

1953

Generalizations of Continuity

Buchanan Cargal
Iowa State College

Generalizations of Continuity

By Buchanan Cargal

Let X be a Hausdorff space in which each neighborhood is an inexhaustible set (a set of Baire's second category), and let $\{N\}$ denote the system of neighborhoods in X. Thus $N(x)$ is a neighborhood of the element x. Let Y be a regular separable Hausdorff space and let $\{M\}$ denote the system of neighborhoods in Y. If f is a function on X into Y and ξ is a point of X, then f is continuous at ξ if for every $M(f(\xi))$ and for every $N(\xi)$, the set $N(\xi)E[x:f(x)\epsilon M(f(\xi))]$ contains ξ as an interior point. A direct method for generalizing continuity is to weaken the requirements on the set $N(\xi)E[x:f(x)\epsilon M(f(\xi))]$. For example, if $N(\xi)E[x:f(x)\epsilon M(f(\xi))]$ contains an N (not necessarily an $N(\xi)$), then f is neighborly at ξ [1][1]. Other generalizations are obtained by requiring that $N(\xi)E[x:f(\mathrm{x})\epsilon M(f(\xi))]$ contains an inexhaustible set, a set residual on a neighborhood (that is, a set whose complement on the neighborhood is exhaustible, or of Baire's first category), a non-denumerable set, or a non-null dense-in-itself subset. The purpose of this paper is to unify the study of generalized continuity by developing a concept which includes each of the generalizations mentioned above as special cases. It should be observed that properties of $N(\xi)E[x:f(x)\epsilon M(f(\xi))]$ do not characterize functions which are semi-continuous or cliquish [2] at ξ.

Definition I a. Let λ be a set property defined for subsets of X and let S be a subset of X. The point ξ is said to be a *point of* λ *approach by* S if for every $N(\xi)$, $N(\xi)$ S has property λ.[2]

b. The point ξ is said to be a *point of* $k\lambda$ *approach* (*concentrated* λ *approach*) *by* S if there exists an $N(\xi)$ such that for every N contained in $N(\xi)$, NS has property λ.

Lemma I. *For any property* λ *and for any set* S *in* X, *the set* E *of points of* λ *approach by* S *which are not points of* $k\lambda$ *approach by* S *is nowhere dense.*

Proof. If a neighborhood N contains a point of E, then there exists a neighborhood N_1 contained in N such that N_1S does not have property λ. Thus no element of N_1 belongs to E.

Definition II a. Let f be a function on X into Y. The point

[1]Numbers in brackets refer to the list of references.

[2]Definitions I and II are essentially those introduced by Blumberg [3] for the case when the property λ is inexhaustibility

ξ is said to be a *point of* λ *f-approach by* S if for every $M(f(\xi))$ and for every $N(\xi)$, $N(\xi)S\ E[x:f(x)\epsilon M(f(\xi))]$ has property λ.

b. The point ξ is said to be a *point of* $k\lambda$ (*concentrated* λ) *f-approach by* S if for every $M(f(\xi))$, there exists an $N(\xi)$ such that for every N contained in $N(\xi)$, $NSE[x:f(x)\epsilon M(f(\xi))]$ has property λ.

DEFINITION III. The property λ is said to be an ascending set property if for every set S which has property λ and for every subset A of X, it is true that $(S+A)$ has property λ.

LEMMA II. *Let* λ *be an ascending set property, let* $f\ (x)$ *be defined on* X *and let* S *be a subset of* X. *The set* E *of points of* λ *f-approach by* S *which are not points of* $k\lambda$ *f-approach by* S *is an exhaustible set.*

PROOF. Let $\{M^r\}$ be a countable set of neighborhoods which are equivalent to $\{M\}$ and choose $M_j{}^r$ from $\{M^r\}$. By Lemma I, the set E_j of points of λ approach by $SE[x:f(x)\epsilon M_j{}^r]$ which are not points of $k\lambda$ approach by $SE[x:f(x)\epsilon M_j{}^r]$ is nowhere dense. Thus $E_r = \Sigma_j\ E_j$ is exhaustible. We need only to show that $E \subseteq E_r$. Choose ξ in E. Since ξ is not a point of $k\lambda$ f-approach by S, there exists an $M_1\ (f(\xi))$ such that for every $N(\xi)$, there exists an N contained in $N(\xi)$ such that $NSE[x:f(x)\epsilon M_1(f(\xi))]$ does not have property λ. Choose $M_k{}^r(f(\xi))$ from $\{M^r\}$ such that $M_k{}^r(f(\xi)) \subseteq M_1(f(\xi))$. Since λ is ascending, $NSE[x:f(x)\ \epsilon M_k{}^r(f(\xi))]$ does not have property λ. Hence $\xi \epsilon E_r$. Thus E is exhaustible.

COROLLARY 1. *The set of points of inexhaustible f-approach by* X *which are not points of concentrated inexhaustible f-approach by* X *is an exhaustible set* [3].

Let ρ *metrize* Y *and let* $M(f(\xi);\epsilon)$ *denote the set* $E[y:\rho(y,f(\xi)) < \epsilon]$.

THEOREM I. *Let* λ *be an ascending set property and* S *be a set in* X. *If the sequence of functions* $\{f_n(x)\}$ *is defined on* X *and converges uniformly to* $F(x)$, *and if there exists a point* ξ *in* S *such that for every* n, ξ *is a point of* λ f_n*-approach by* S, *then* ξ *is a point of* λ *F-approach by* S.

PROOF. Let $M(F(\xi))$ and $N(\xi))$ be given. We must show that $N(\xi)SE[x:F(x)\epsilon M(F(\xi))]$ has property λ. Choose a positive number ϵ such that $M(F(\xi);\epsilon)\epsilon M(F(\xi))$ and choose n such that for every x in S, $\rho(f_n(x);F(x)) < \epsilon/3$. Now $N(\xi)SE[x:f_n(x)\epsilon M(\xi;\epsilon/3)]$ has property λ. Suppose $\eta\epsilon N(\xi)SE[x:f_n(x)\epsilon M(\xi;\epsilon/3)]$. Then $\rho(f_n(\eta);f_n(\xi)) < \epsilon/3$. Hence $\rho(F(\xi);F(\eta)) \leqslant \rho(F(\xi);f_n(\xi)) + \rho(f_n(\xi);f_n(\eta)) + \rho(f_n(\eta);F(\eta)) < \epsilon$. Thus $\eta\epsilon N(\xi)SE$

$[x:F(x) \epsilon M(F(\xi))]$. Since λ is an ascending set property, $N(\xi) SE[x:F(x) \epsilon M(F(\xi))]$ has property λ.

COROLLARY II. *Let λ be the property that a set contains ξ as an interior point. Then Theorem I yields the well-known result that the uniform limit of a sequence of functions which are continuous at ξ is a function which is continuous at ξ.*

For other particular results, λ may be taken to be the property that a set contain a neighborhood, that a set be infinite, or any other ascending set property.

Let the complement with respect to X of a set S be denoted by $S^{\sim}$ and the closure of S be designated cS.

DEFINITION IV. The property λ is said to be a *locally characterizing set property* if for every set S which has property λ and for every N such that NS has property λ, there exists an N_1 contained in N such that N_1S has property λ and N_1 $(S^{\sim})$ does not have property λ.

For example, the property that a set be residual on a neighborhood is a locally characterizing set property while the property that a set be inexhaustible is not.

LEMMA III. *Let λ be an ascending locally characterizing set property. Let N be a neighborhood and R and S be sets in X such that NRS has property λ. Then there exists an N_1 contained in N such that $N_1R(S^{\sim})$ does not have property λ.*

PROOF. NS has property λ because λ is ascending. But since λ is locally characterizing, there exists an N_1 contained in N such that $N_1(S^{\sim})$ does not have property λ. From the ascending property of λ, it now follows that $N_1R(S^{\sim})$ does not have property λ.

LEMMA IV. *Let λ be an ascending locally characterizing set property. Let f be a function on X onto a subset of Y and let each element x of a set S in X be a point of λ f-approach by S. If ξ is a point of λ f-approach by S, then for every $M(f(\xi))$ and for every $N(\xi)$, there exists an N contained in $N(\xi)$ such that for every x in NS, $f(x) \epsilon M(f(\xi))$.*

PROOF. Let $M(f(\xi))$ and $N(\xi)$ be given. Choose $M_1(f(\xi))$ such that cM_1 $(f(\xi)) \epsilon M(f(\xi))$. Now $N(\xi)SE[x:f(x) \epsilon M_1(f(\xi))]$ has property λ. By Lemma III, there exists an N contained in $N(\xi)$ such that $NSE[x:f(x)$ not ϵ M_1 $(f(\xi))]$ does not have property λ. Suppose η, a point of NS, is such that $f(\eta)$ not ϵ $M(f(\xi))$. Choose $M(f(\eta))$ disjoint from $M_1(f(\xi))$. Now $NSE[x:f(x) \epsilon M(f(\eta))]$ has property λ. Since λ is ascending $NSE[x: f(x)$ not ϵ M_1 $(f(\xi))]$ has property λ, a contradiction. Hence $f(\eta) \epsilon M(f(\xi))$.

COROLLARY III. *Since ρ meterizes the space Y, under the conditions of Lemma IV, for every positive number ϵ, and for every $N(\xi)$, there exists an N contained in $N(\xi)$ such that for every x in NS, $\rho(f(x), f(\xi)) < \epsilon$.*

THEOREM II. *Let λ be an ascending locally characterizing set property. Let $f(x)$ be defined on X and let each element x of a set S in X be a point of λ f-approach by S. If ξ is a point of $k\lambda$ f-approach by S, then $f(x)$ is continuous at ξ with respect to S.*

PROOF. Let $M(f(\xi))$ and $N(\xi)$ be given. Choose $M_1(f(\xi))$ such that $cM_1(f(\xi))\epsilon M(f(\xi))$. Since ξ is a point of $k\lambda$ f-approach by S, there exists an $N_1(\xi)$ in $N(\xi)$ such that for every N in $N_1(\xi)$, $NSE[x: f(x)\epsilon M_1(f(\xi))]$ has property λ. Suppose η, a point of $N_1(\xi)S$, is such that $f(\eta)$ not ϵ $M(f(\xi))$. Choose $M(f(\eta))$ disjoint from $M_1(f(\xi))$. Now $N_1(\xi)SE[x:f(x)\epsilon M(f(\eta))]$ has property λ. By Lemma III, there exists an N_1 in $N_1(\xi)$, such that $N_1SE[x:f(x)$ not ϵ $M(f(\eta))]$ does not have property λ. Since λ is ascending, $N_1SE[x:f(x)\epsilon M_1(f(\xi))]$ does not have property λ, a contradiction. Thus $f(\eta)\epsilon M(f(\xi))$. Hence $f(x)$ is continuous at ξ with respect to S.

Using Lemma II and Theorem II, we may state the following theorem.

THEOREM II′. *If λ is an ascending locally characterizing set property, if $f(x)$ is defined on X, and if each element x of S is a point of λ f-approach by S, then $f(x)$ is continuous over S with respect to S except on an exhaustible set.*

THEOREM III. *Let the sequence of functions $\{f_n(x)\}$ be defined and converge to $F(x)$ on X. Let λ be an ascending locally characterizing set property. Let R be a set residual in X and let it be required that for every n, each x in R is a point of λ f_n-approach by R. Then $F(x)$ is continuous over R with respect to R except on an exhaustible set.*

REMARK. If we say that a set has property λ if it contains a neighborhood and let $R = X$, we obtain Bledsoe's results for neighborly functions [2]. The proof of Theorem III is similar to the proof given by Bledsoe, but we include it here because our theorem is more general.

PROOF. By Theorem II′, $f_n(x)$ is continuous on a residual set R_n with respect to R. Thus for every n, $f_n(x)$ is continuous on the residual set $R' = \Pi_n R_n$ with respect to R. Let $w(x) = \limsup_{\eta \to x} \rho(F(\eta), F(x))$ for η's in R if this limit exists, and let $w(x) = 1$ at all other points of R. Let n and p be positive integers and let $E_p{}^n = \mathrm{E}[x \epsilon R' : w(x) \geqslant 1/p$ and for every $m \geqslant n$, $\rho(f_m(x), F(x)) < 1/7p]$. Suppose there are integers n and p and a neighborhood N such that $E_p{}^n$ is dense on N. Choose ξ in $E_p{}^n N$. Then $\rho(f_n(\xi), F(\xi)) < 1/7p$. Since $f_n(x)$ is continuous at ξ with respect to R, there exists an $N(\xi)$ contained in N such that for every x in $N(\xi)R$, $\rho(f_n(x), f_n(\xi)) < 1/7p$. Since $w(\xi) \geqslant 1/p$, there exists an x_1 in $N(\xi)R$ such that $\rho\ (F(\xi), F(x_1)) \geqslant 6/7p$. Choose $m \geqslant n$ such that $\rho(f_m(x_1), F(x_1)) < 1/7p$. Since x_1 is a point of $\lambda\ f_m$-approach by R, using Corollary III, we can conclude that there exists an N_1 contained in $N(\xi)$ such that for every x in N_1R, $\rho(f_m(x), f_m(x_1)) < 1/7p$. In particular let x_2 be contained in $N_1E_p{}^n$. Then $\rho(f_m(x_2), f_m(x_1)) < 1/7p$. But $\rho(f_m(x_2), f_n(x_2)) < 2/7p$. We have $\rho(F(\xi), F(x_1)) \leqslant \rho(F(\xi), f^n(\xi)) + \rho(f_m(x_1), F(x_1)) + \rho(f_n(\xi), f_n(x_2)) + \rho(f_n(x_2), f_m(x_2)) + \rho(f_m(x_2), f_m(x_1)) < 6/7p$, a contradiction. Thus $E = \Sigma_{n,p} E_p{}^n$ is exhaustible and $F(x)$ is continuous on $(R' - E)$ with respect to R.

References

1. W. W. Bledsoe, *Neighborly Functions,* Proc. Amer. Math. Soc., Vol. 3, No. 1 (Feb. 1952) pp. 114-115.
2. H. P. Thielman, *Types of Functions,* Amer. Math. Monthly, Vol. 60 (1953) pp. 156-161. Also see *Pathological Functions,* Proc. Iowa. Academy of Science, Vol. 59, (1952) pp. 338-343.
3. H. Blumberg, *Arbitrary Point Transformations,* Duke Math. Jour., Vol. 11 (1944) pp. 671-685.

Department of Mathematics
Iowa State College
Ames, Iowa