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Real functions of a real variable may be classified according as to whether or not they are continuous, neighborly, cliquish, or discriminative. The purpose of this paper is to investigate these properties, and to discover how they are related.

In this paper X denotes the real continuum E_1 or an interval, open or closed, contained in E_1. The function $f(x)$ is a function from the set X to a subset of E_1. N denotes an arbitrary open interval in E_1, and $N\ (x)$ an arbitrary open interval containing the point x.

Definition 1. The function $f(x)$ is said to be *continuous at the point ξ in X* if for every positive number ϵ, there exists an $N(\xi)$ such that for every x in $N(\xi)\cdot x$, $|f(x)-f(\xi)|<\epsilon$.

The function $f(x)$ is said to be continuous if $f(x)$ is continuous at each point of x.

Definition 2. The function $f(x)$ is said to be *neighborly at the point ξ in X* if for every positive number ϵ and for every $N(\xi)$, there exists an N contained in $N(\xi)\cdot X$ such that for x in N, $|f(x)-f(\xi)|<\epsilon$. [1][1]

The function $f(x)$ is said to be neighborly if $f(x)$ is neighborly at each point of X.

Example 1.

$$f(x) = sin^1/x \text{ for } x \neq O.$$
$$f(O) = O.$$

In this example $f(x)$ is neighborly at every point but discontinuous at $x=O$.

Definition 3. The function $f(x)$ is said to be *cliquish at the point ξ in X* if for every positive number ϵ and for every $N(\xi)$, there exists an N contained in $N(\xi)\cdot X$ such that for every two points x_1 and x_2 in N, $|f(x_1)-f(x_2)|<\epsilon$. [2]

The function $f(x)$ is said to be cliquish if $f(x)$ is cliquish at each point of X.

Example 2.

$$f(x) = 1 \text{ for } x \neq O.$$
$$f(O) = 2.$$

[1]Numbers in brackets refer to the list of references.

In this example $f(x)$ is cliquish but not neighborly at $x=O$.

DEFINITION 4. The function $f(x)$ is said to be *discriminitive at the point* ξ *in* X if for every positive number ϵ and for every $N(\xi)$, there exist an η in the set $E[y: |y-f(\xi)|<\epsilon]$ and an N contained in $N(\xi)\cdot X$ such that for every x in N, $f(x)\neq\eta$.

The function $f(x)$ is said to be discriminative if $f(x)$ is discriminative at each point of X.

EXAMPLE 3.

$$f(x) = 1 \text{ for rational values of } x.$$
$$f(x) = 2 \text{ for irrational values of } x.$$

In this example $f(x)$ is discriminative at every point but is cliquish at no point.

EXAMPLE 4.

Let X be the interval $[O,1]$, and let x be a point of X. Write x in its decimal expansion:

$$x = a_1/10 + a_2/10^2 + a_3/10^3 + \cdots .$$

consider the sequence $a_1, a_3, a_5, \cdots$. If this sequence is not periodic, let $f(x) = \frac{1}{2}$. If the sequence is periodic, and if the first period begins with a_{2n-1}, let $f(x) = a_{2n}/10 + a_{2n+2}/10^2 + a_{2n+4}\ 10^3 + \cdots$ if each term of the sequence $a_{2n}, a_{2n+2} \ldots$ is not equal to 0 or 9. If $a_{2n} = a_{2n+2} = a_{2n+4} = \cdots$ and if a_{2n} is equal to 0 or 9, let $f(x) = \frac{1}{2}$. [3]

In this example $f(x)$ is not discriminitive at any point of $[0,1]$.

It follows from the definitions that if a function $f(x)$ is continuous, $f(x)$ is neighborly; if $f(x)$ is neighborly, $f(x)$ is cliquish; and if $f(x)$ is cliquish, $f(x)$ is discriminitive.

DEFINITION 5.

The function $f(x)$ is said to have the *Darboux property* on X if for every two points x_1 and x_2, $x_1 < x_2$, in X such that $f(x_1) < f(x_2)$ ($f(x_1) > f(x_2)$), and for every η such that $f(x_1) < \eta < f(x_2)$ ($f(x_1) > \eta > f(x_2)$), there exists an x such that $x_1 < x < x_2$ and such that $f(x) = \eta$. [4]

This property is of interest since continuous functions and derivative functions possess the Darboux property. Example 4 shows that the Darboux property does not necessarily even imply discriminativeness. Example 1 illustrates that a neighborly function with the Darboux property is not necessarily continuous. Whether or not there exists a function which is cliquish, has the Darboux property, but is not neighborly, remains an open question. However the following theorem illustrates a level at which Darboux property has an effect.

Theorem 1. *If a bounded function $f(x)$ is discriminative and possesses the Darboux property on X, then $f(x)$ is cliquish on X.*

PROOF. Suppose the bounded discriminative function $f(x)$ with the Darboux property is not cliquish at some point ξ in X. Then there exists a positive number ϵ and an $N(\xi)$ such that for every open interval N_0 contained in $N(\xi)\cdot X$, there exist two points x_0 and x'_0 in N_0 such that $x_0 < x'_0$ and $|f(x_0)-f(x'_0| \geqq \epsilon$. Since $f(x)$ possesses the Darboux property, there exists an x_1 such that $x_0 < x_1 < x'_0$ and $f(x_1) = [f(x_0) + f(x'_0)]/2$. By hypothesis there exists a positive number M such that $|f(x)|<M$. It follows that $M-f(x_1)>\epsilon/2$ and $f(x_1) + M>\epsilon/2$; and since $f(x)$ is discriminative at x_1, there exist a positive number η_1 and an open interval N_1 contained in N_0 such that $M-\eta_1>\epsilon/2$, $\eta_1 + M>\epsilon/2$, and such that for every x in N_1, $f(x) \neq \eta_1$. Now the Darboux property implies that either $f(x)>\eta_1$ or $f(x)<\eta_1$ on N_1. Suppose $f(x)>\eta_1$ on N_1, and since $f(x)$ is not cliquish in $N(\xi)$, there exist two points x'_1 and x''_1 in N_1 *such that* $|f(x'_1)-f(x''_1)| \geqq \epsilon$. As before, there exists an X_2 such that $x'_1<x_2<x''_1$ and $f(x_2) = [f(x'_1) + f(x''_1)]/2$. It follows that $M-f(x_2)>\epsilon/2$ and $f(x_2)-\eta_1>\epsilon/2$; and since $f(x)$ is discriminative at x_2, there exist a positive number η_2 and an N_2 in N_1 such that $M-\eta_2> \epsilon/2$, $\eta_2-\eta_1>\epsilon/2$, and such that for every x in N_2, $f(x) \neq \eta_2$. Continue the process. This construction produces the infinite bounded sequence $\{\eta_n\}$ such that for every two elements η_n and $\eta_{n'}$ of the sequence, $|\eta_m-\eta_{m'}|>\epsilon/2$. This is a contradiction since every infinite bounded sequence of real numbers has at least one limit point. As all the other alternatives lead to this same contradiction, the theorem is proved.

The primary purpose of this paper is to prove the following theorem:

Theorem 2. *If a function $f(x)$ has a derivative $f'(x)$ at each point of X, then $f'(x)$ is neighborly on X.*

We shall first prove a lemma and theorem which lead to a proof of Theorem 2. The proofs depend on the following theorem due to Baire.

Theorem of Baire. A necessary and sufficient condition that a function $f(x)$, defined in a set E which is either perfect or open, is the limit of a sequence of functions all of which are continuous in E, is that $f(x)$ be at most pointwise discontinuous with respect to every perfect set contained in E. [5]

It should be noted that if the domain of definition E of $f(x)$ is an interval, open or closed, and if $f(x)$ is of Baire's class less than two, then Baire's Theorem implies that the set of points of continuity of $f(x)$ is everywhere dense in E.

Lemma 1. *If the function $f(x)$ is defined on the open interval I and if ξ in I is a point at which $f(x)$ is not neighborly, there exist a positive number ϵ and an $N(\xi)$ such that for every continuity point ζ of $f(x)$ in $N(\xi)\cdot I$, $|f(\zeta)-f(\xi)|\geqq\epsilon$.*

Proof. Let ϵ in I be a point at which $f(x)$ is not neighborly. Suppose that for every positive number ϵ and for every $N(\xi)$, there exists a continuity point ζ of $f(x)$ in $N(\xi)\cdot I$ such that $|f(\zeta)-f(\xi)|<\epsilon$. Let the positive number ϵ_1 be such that $|f(\zeta)-f(\xi)|+\epsilon_1<\epsilon$. Since ζ is a continuity point of $f(x)$, there exists an $N(\zeta)$ in $N(\xi)\cdot I$ such that for x in $N(\zeta)$, $|f(x)-f(\zeta)|<\epsilon_1$. Thus for x in $N(\zeta)$, $|f(x)-f(\xi)|\leqq|f(x)-f(\zeta)|+|f(\zeta)-f(\xi)|<\epsilon_1+(\epsilon-\epsilon_1)=\epsilon$. But this contradicts the hypothesis that $f(x)$ is not neighborly at ξ.

Theorem 3. If the function $f(x)$, defined on an open interval I, is of Baire's class less than two and has the Darboux property, then $f(x)$, is neighborly on I.

Proof. Suppose $f(x)$ is not neighborly at the point ξ in I. From Baire's Theorem it follows that the set of points of continuity of $f(x)$ form a set everywhere dense in I. Since $f(x)$ is not neighborly at ξ, Lemma 1 implies that there exist a positive number ϵ and an $N_1(\xi)$ such that for every continuity point ζ of $f(x)$ in $N_1(\xi)\cdot I$, $|f(\zeta)-f(\xi)|>\epsilon$. Choose an $N(\xi)$ contained in $N_1(\xi)\cdot I$ such that the end points of $N(\xi)$ are continuity points of $f(x)$. Denote by R the set of continuity points of $f(x)$ contained in $N(\xi)$. Thus for x in R, $|f(x)-f(\xi)|\geqq\epsilon$.

Let A be the set $\}x:x$ in $N(\xi)$, and $|f(x)-f(\xi)|<\epsilon\}$. Consider the set B of points of A at which the saltus $s_f\ (x)$ relative to A satisfies $s_f(x)>\epsilon/2$. B is not null since ξ is in B. Denote the closure of B by$P$, i.e. $P=B+B'$ where B' is the derived set of B. Note that every point of P is an interior point of $N\ (\xi)$. In order to show that P is a perfect set it is sufficient to show that every point of P is a limit point of P. If x in P is such that $|f(x)-f(\xi)|\geqq\epsilon$, then x is in B' and is therefore a limit point of P. If x in P *is* such that $|f(x)-f(\xi)|<\epsilon$, then x is in B. This follows from the fact that the set of points where $s^f(x)\geqq\epsilon/2$ is closed relative to B. Thus if x is in B for an arbitrary $N(x)$ contained in $N(\xi)$, there exist two points x_1 and x_2 in $N(x)$ such that $|f(x_1)-f(x_2)|\geqq\epsilon/2$. The following possibilities can occur:

(1) $|f(x_1)-f(\xi)|<\epsilon/2,$

(2) $|f(x_2)-f(\xi)|<\epsilon/2,$

(3) $f(\xi)-\epsilon<f(x_1)\leqq f(\xi)-\epsilon/2$
and $f(\xi)+\epsilon/2\leqq f(x_2)<f(\xi)+\epsilon.$

Since R is everywhere dense in $N(\xi)$ and since $f(x)$ satisfies the Darboux property, in case (1) x_1 is an element of P, in case (2) x_2 is an element of P, and in case (3) there exists an x_3 in $N(x)$ with $f(x_3)=f(\xi)$ and therefore x_3 is in P. If it should happen that any one of the points x_1, x_2, *or* x_3 is equal to x, then since $f(x)$ has the Darboux property there exists an x_4 in $N(x)$ with $x_4 \neq x$ and such that $|f(x_4)-f(\xi)|<\epsilon/2$. Hence x_4 is in P. Thus it has been shown that the arbitrary neighborhood $N(x)$ always contains an element of P different from x. Hence P is perfect.

The saltus $s^f(x) \geqq \epsilon/2$ for each point of P, and therefore every point of P is a discontinuity point of $f(x)$ relative to P. By Baire's Theorem $f(x)$ could not be the limit of a sequence of continuous functions. This contradicts the hypothesis that $f(x)$ is of Baire's class less than two, and thus $f(x)$ is neighborly on I.

If a function $f(x)$ has a derivative $f'(x)$ at each point of X, then $f'(x)$ is of Baire's class less than two and has the Darboux property. Thus Theorem 2 follows from Thoerem 3.

The next example shows that if a function $f(x)$ has a derivative $f'(x)$ everywhere on X with the possible exception of a nowhere dense set in X, $f'(x)$ may not be neighborly on its domain of definition.

EXAMPLE 5.

$$f(x) = [(2n+1)-(2n^2+2n+1)\ x]/n(n+1) \text{ for}$$
$$1/(n+1)<x\leqq 1/n;\ n=1,\ \pm 1,\ \pm 3,\cdots.$$
$$f(0) = 0.$$

In this example $f'(0) = 0$, but at all other points where the derivative is defined, $f'(x)<-2$. Thus $f'(x)$ is not neighborly at $x=0$.

Theorem 4. If the function $f(x)$ has a derivative $f'(x)$ everywhere on X with the possible exception of a nowhere dense set S in X, then $f'(x)$ is cliquish on X.

Proof. Let ξ be an arbitrary point of X. Then for every $N(\xi)$, there exists an N contained in $N(\xi)\cdot X$ such that $N \cdot S$ is a null set. By Theorem 3 $f(x)$ is neighborly on N. Thus for every point x_1 in N and for every positive number ϵ, there exists an N_1 in N such that for every point x_2 in N_1, $|f(x_1)-f(x_2)|<\epsilon$. Since N_1 is contained in $N(\xi)$, $f(x)$ is cliquish at ξ.

Definition 6. The function $f(x)$ is said to be *peculiar with respect to the property p on X* if X may be partitioned into two sets X_1 and X_2 such that X_1 and X_2 are each everywherc dense in X, $f(x)$

has the property p at each point of X_1, and $f(x)$ has the property p at no point of X_2. [2]

There are functions peculiar with respect to continuity, neighborliness, and discriminativeness. There are no functions peculiar with respect to cliquishness [2].

For functions which have a derivative at each point of X, the derivative function may be peculiar with respect to continuity as is illustrated by the following well known example.

EXAMPLE 6.

Let the domain of definition of $f(x)$ be the interval $(0,1)$. Order the set $\{x_n\}$ of rational numbers in $(0,1)$.

$$g_n(x,x_n) = (x-x_n)^2 \sin 1/(x-x_n) \text{ for } x \neq x_n.$$
$$g_n(x,x_n) = 0 \text{ for } x = x_n.$$
$$f(x) = \Sigma_n\, g_n(x,x_n)/n^2.$$

In this example $f(x)$ is continuous, and $f'(x)$ is continuous at every irrational number and discontinuous at every rational number in $(0,1)$.

Theorem 3 states that if $f(x)$ has a derivative at each point of X, then $f'(x)$ can not be peculiar with respect to neighborliness.

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