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INTRODUCTION

Beams with uniform strength subjected only to bending have the same maximum flexural stress at any cross section. Blasius, 1914, Gaede, 1937, and Opatowski, 1945, have included weight of the beam in establishing relations and giving solutions to particular cases. Additional results are provided here.

The bending moment at a section, in terms of the assigned maximum stress is $\sigma I(x)/h$ where $I(x)$ is the moment of inertia of the cross section with respect to its neutral axis, and h is the distance from the neutral axis to the place where the stress is developed.

Restricting the analysis to beams having linear relations between stress and strain, the radius of curvature is $EI(x)/M(x)$, or $\rho = hE/\sigma$. E. is Young's modulus. That the deflection must describe a circular arc for beams of constant height is a convenient property.

Fixed beams with constant height have a middle portion simply supported by cantilever end sections. The length of the center section may vary from zero to the full span width. The slope however will only be continuous if the midsection is one half the total span. Under these conditions we have a practical continuous beam since the cross section must have some area at points of zero moment to withstand shear. The beam would otherwise behave as if its parts were pin connected.

RECTANGULAR BEAMS HAVING CONSTANT HEIGHT

The figure shows the general case of a portion of a beam with specific weight γ, subjected to a uniform load q distributed along the length. The bending moment at an arbitrary section x is:

$$M(x) = M_0 - R_0 x + qx^2/2 + \gamma \int_0^x A(\xi)\ (x-\xi)\, d\xi, \tag{1}$$

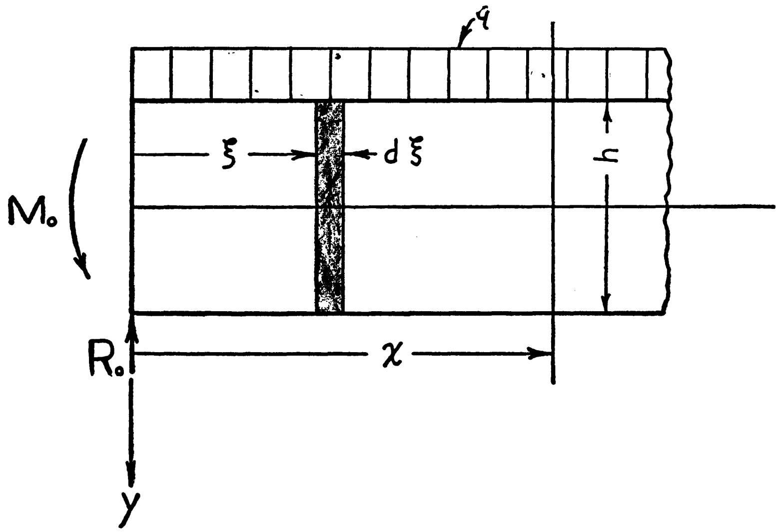
where $A(\xi)$ is the cross-sectional area at section ξ. For uniform strength $M(x) = \sigma S(x)$ where $S(x)$ is the section modulus.

(a) **Cantilever Beam**

For a cantilever beam R_0 and M_0 vanish at the end and Eqn. (1) reduces to an integral equation of convolution type,

$$\sigma h^2 b(x)/6 = qx^2/2 + \gamma h \int_0^x b(\xi)\ (x-\xi)\, d\xi, \tag{2}$$

which can be solved easily by Laplace transformation. The trans-



form of Eqn. (2) is:

$\sigma h^2 b(s)/6 = q/s^3 + \gamma h b(s)/s^2.$ (3)

Hence, $b(x) = q(\cosh ax - 1)/\gamma h,$ (4)

in which $a = \sqrt{6\gamma/\sigma h}.$

The weight of the beam is, by integration,

$W = (q/a)(\sinh aL - aL).$ (5)

(b) **Simply Supported Beam**

In this case M_0 vanishes and $M(x)$ is a negative quantity. Hence the appropriate Laplace transform of Eqn. (1) is

$-\sigma h^2 b(s)/6 = -R_0/s^2 + q/s^3 + \gamma h b(s)/s^2,$ (6)

so that $b(x) = \frac{aR_0}{\gamma h} \sin ax + \frac{q}{\gamma h}(\cos ax - 1).$ (7)

The weight becomes $W = \frac{q}{a}\left(2\tan\frac{aL}{2} - aL\right),$ (8)

and the end reaction, $R_0 = \frac{q}{a}\tan\frac{aL}{2}.$ (9)

Now Eqn. (7) becomes,

$b(x) = \frac{q}{\gamma h}\left[\tan\frac{aL}{2}\sin ax + \cos ax - 1\right].$ (10)

(c) **Fixed Beam**

Considering the preceding remarks about fixed beams, the pertinent shape, weight, and end moment may be obtained by appropriate combinations of results derived for a simple beam and a cantilever beam loaded downward at the end.

By reversing the sign of R_o and in (6) and using (9) with $L=L/2$,

$$b(x) = \frac{q}{\gamma h} [\tan \frac{aL}{4} \sinh ax + \cosh ax - 1] \quad (11)$$

for the cantilever portions. The weight of each of the cantilever sections is $W = \frac{q}{a} [\tan \frac{aL}{4} (\cosh \frac{aL}{4} - 1) + \sinh \frac{aL}{4} - \frac{aL}{4}]$. (12)

By integrating Eqn. (1), the moment at the wall is

$$M_0 = \frac{q}{a^2} [\tan \frac{aL}{4} \sinh \frac{aL}{4} + \cosh \frac{aL}{4} - 1.]. \quad (13)$$

The center section is described in (b) above by substituting L/2 for L.

Rectangular Cantilever Beam Having Constant Width

Opatowski has given the solution to this problem with values available by successive approximations, power series, or reduction to convenient integral forms.

The defining equation is

$$\sigma b h^2(x)/6 = qx^2/2 + \gamma b \int_0^x h(\xi)\,(x-\xi)\,d\xi, \quad (14)$$

Differentiating twice and integrating gives

$$\sqrt{\frac{\sigma}{2\sigma}} \int_0^h \frac{h\,dh}{H} = x + K_2, \quad (15)$$

in which $H = (h^3 + 3qh^2/2\gamma b + K_1^2)^{1/2}$.

To evaluate the integration constants, we note that (16)

$$dx/dh = \sqrt{\sigma/2\gamma}\, h/H,$$

and $\int_0^x h(\xi)\,d\xi = \int_0^h h(dx/dh)\,dh = \sqrt{\sigma/2\gamma} \int_0^h (h^2/H)\,dh.$ (17)

Since $\frac{3}{2} \int_0^h \frac{h^2 + qh/\gamma b}{H} dh = H - K_1,$ (18)

and $\gamma b \int_0^x h(\xi)\,(x-\xi)\,d\xi = \gamma b \int_0^x \int_0^x h(\xi)\,d\xi d\xi,$ (19)

we have $\int_0^x h(\xi)d\xi = \sqrt{\frac{\sigma}{2\gamma}} [\frac{2}{3}(H + K_1) - \frac{q}{\gamma b} \int_0^h \frac{h\,dh}{H}],$ (20)

and $\gamma b \int_0^x h(\xi)\,(x-\xi)\,d\xi = \sigma b h^2/6 - qx^2/2 - (bK_1\sqrt{2\sigma\gamma}/3 + qK_2)\,x.$ (21)

Putting Eqn. (21) into Eqn. (14) shows that the integration constants must vanish, and the solution from Eqn. (15) is

$$X = \sqrt{\frac{\sigma}{2\gamma}} \int_0^h \frac{hdh}{\sqrt{h^3 + 3qh^2/2\gamma b}},$$

$$\text{or } h = \gamma x^2/2\sigma + \sqrt{3\alpha/\sigma b}\, x. \quad (22)$$

The weight of the beam is

$$W = \frac{b\gamma L^2}{6\sigma}\left(L\gamma + 3\sqrt{\frac{3q\sigma}{b}}\right). \quad (23)$$

The defining equations for the simple beam and the fixed beam may be developed similarly. They may be expressed in integral form like Eqn. (15), or developed in terms of Weierstarassian elliptical functions.

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