Proceedings of the Iowa Academy of Science

Volume 62 | Annual Issue Article 40

1955

A Network Representing Elastic Bodies in Spherical Coordinates

William A. Gross
Iowa State College

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Introduction

Circuits representing elastic bodies described in rectangular and cylindrical coordinates have been described by Kron, and Gross and Soroka. Networks are developed here which represent elastic bodies described in spherical coordinates.

Figure 1 shows a representative block of an elastic body with stresses and face areas indicated. Stress superscripts are so arranged that the first designates the normal to the surface on which the stress acts and the second, the direction of the stress. Partial differentiation is indicated by subscript comma and appropriate varibles. The body force is represented by F^a. Incremental angles are required to be small.

The equilibrium relations arising from the block of Figure 1 include, in addition to the familiar relations,

$$r^2 \sin\Theta \, [\sigma^{rr}_{,r} + \sigma^{r\phi}_{,\phi}/r\sin\Theta + \sigma^{r\Theta}_{,\Theta}/r + (2\sigma^{rr} - \sigma^{\Theta\Theta} - \sigma^{\phi\phi} - \sigma^{r\Theta}\cot\Theta)/r + \rho F^r] \, \Delta r \Delta\phi\Delta\Theta = O, \tag{1}$$

$$r^2 \sin\Theta \, [\sigma^{r\phi}_{,r} + \sigma^{\phi\phi}_{,\phi}/r\sin\Theta + \sigma^{\phi\Theta}_{,\Theta}/r + (3\sigma^{r\phi} + 2\,\sigma^{\phi\Theta}\cot\Theta)/r + \rho F^{\phi}] \, \Delta r \Delta\phi\Delta\Theta = O, \tag{2}$$

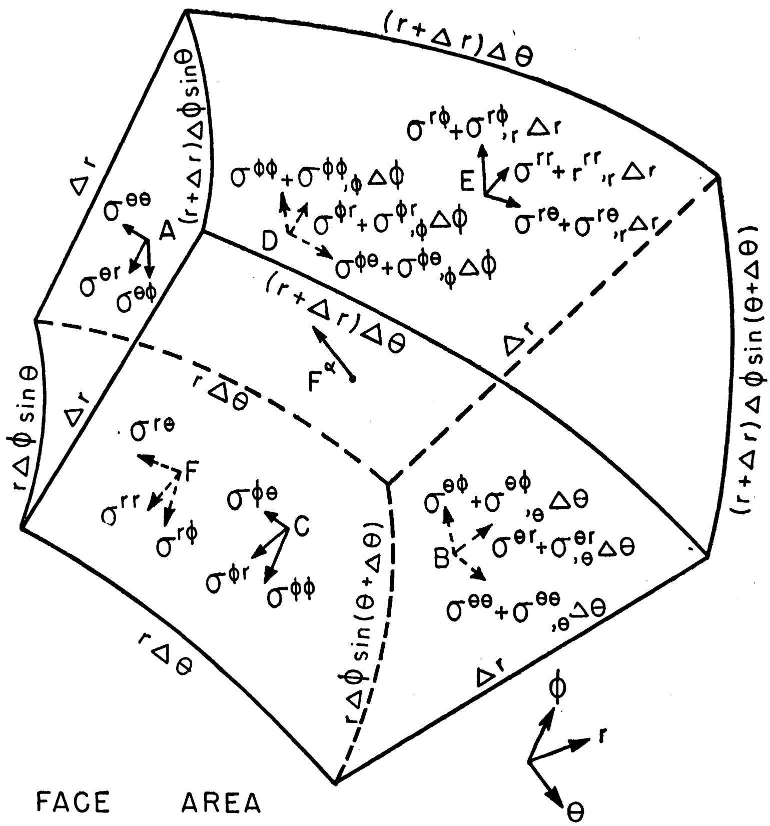
$$r^2 \sin\Theta \, [\sigma^{r\Theta}_{,r} + \sigma^{\phi\Theta}_{,\phi}/r\sin\Theta + \sigma^{\Theta\Theta}_{,\Theta}/r + (\{\sigma^{\Theta\Theta} - \sigma^{\phi\phi}\}\cot\Theta + 3\sigma^{r\Theta})/r + rF^{\Theta}] \, \Delta r \Delta\phi\Delta\Theta = O, \tag{3}$$

about fifteen higher order terms each, which are neglected in the development of the networks. Some of these are of higher order in Δr. Neglecting them leads one to expect reduced accuracy of representation for small r.

Development of the Network

Planning to use the stress-strain relation $\sigma = Ce$, it is convenient to write out the stress in terms of the displacements.

$$\begin{aligned}
\sigma^{rr} &= (\lambda + 2\mu)\, u_{,r} + \lambda\, (v_{,\Theta}/r + 2u/r + w_{,\phi}/r\sin\Theta + v\cot\Theta/r), \\
\sigma^{\phi\phi} &= (\lambda + 2\mu)\, (w_{,\phi}/r\sin\Theta + u/r + v\cot\Theta/r) + \lambda(u_{,r} + u/r + v_{,\Theta}/r), \qquad (4)\\
\sigma^{\Theta\Theta} &= (\lambda+2\mu)\, (v_{,\Theta}/r+u/r) + \lambda(u_{,r}+u/r+v\cot\Theta/r+w_{,\Theta}/r\sin\Theta), \\
\sigma^{r\phi} &= \sigma^{\phi r} = \mu(u_{,\phi}/r\sin\Theta - w/r + w_{,r}), \\
\sigma^{\phi\Theta} &= \sigma^{\Theta\phi} = \mu(v_{,\phi}/r\sin\Theta + w_{,\Theta}/r - w\cot\Theta/r), \\
\sigma^{\Theta r} &= \sigma^{r\Theta} = \mu(u_{,\Theta}/r + v/r + v_{,r}).
\end{aligned}$$



FACE	AREA
A	$(r+\frac{\Delta r}{2})\,\Delta r\,\Delta\phi\,\sin\theta$
B	$(r+\frac{\Delta r}{2})\,\Delta r\,\Delta\phi\,(\sin\theta+\Delta\theta\cos\theta)$
C	$(r+\frac{\Delta r}{2})\,\Delta r\,\Delta\theta$
D	$(r+\frac{\Delta r}{2})\,\Delta r\,\Delta\theta$
E	$(r+\Delta r)^2\,\Delta\theta\,\Delta\phi\,(\sin\theta)$
F	$r^2\,\Delta\theta\,\Delta\phi\,(\sin\theta)$

Fig. 1. Elemental block described in spherical coordinates.

Figure 2 shows, with $\Delta\Theta$ and $\Delta\phi$ expanded for clarity, the complete network which is analogous to the block of Figure 1. The nodes u, v, and w have values equal to the r, Θ and ϕ components of the displacement at similar points in the elastic body. They are shown connected to reference planes. Admittance values, Y, are as given in the accompanying table and apply to the appropriate letters on the figure. Dotted lines represent the paths of the mutual admittances, arrows representing the line on which they act. Normal forces may be considered to flow along the heavy lines and shear forces along the light lines. Forces remain constant on lines between junctions. They are functions of the displacement differences between nodes and the appropriate admittances. Components of the body force $\rho r^2 \Delta r \Delta\phi \Delta\Theta F^a$ are applied at the nodes.

The method by which equilibrium is satisfied in the network may be explained with reference to Figure 3 which shows several forces coming together at a u junction. These forces all have the same

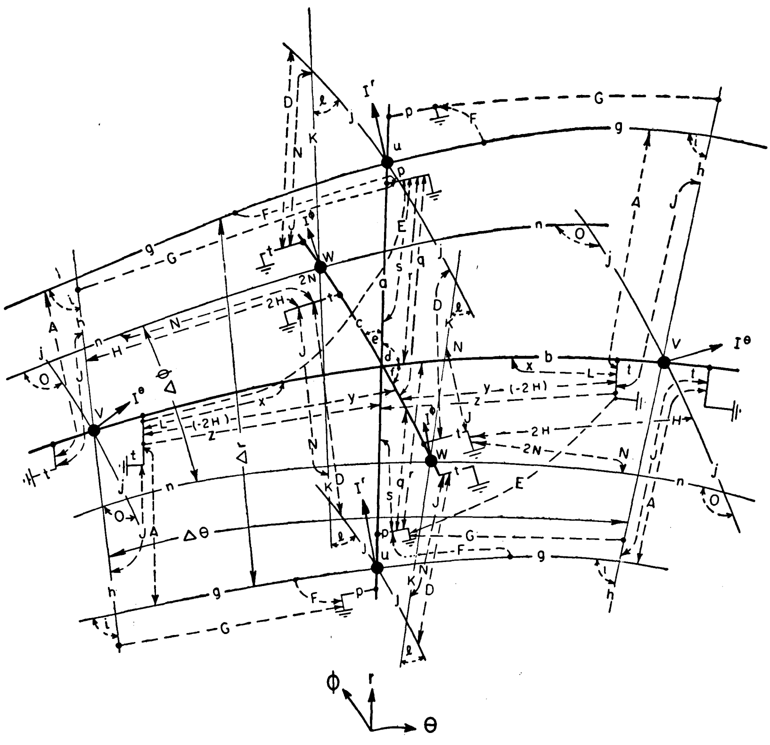


Fig. 2. Equivalent circuit for block of Fig. 1.

line of action although the network analog shows them developed along perpendicular paths.

Referring to Figure 2, we see the force

$$i^1 = Y^a u_r \Delta r + Y^d v,_\theta \Delta\Theta + Y^e w,^\phi \Delta\phi + 2Y^s u + 2Y^z v. \quad (5)$$

Utilization of the admittance values gives

$$i^1 = [r^2(\lambda+2\mu)u,_r + r\lambda v,_\theta + r\lambda w,_\phi/\sin\Theta + 2r\lambda u + r\lambda\cot\Theta v]\Delta\phi\Delta\Theta,$$
$$= r^2\sigma^{rr}\Delta\phi\Delta\Theta.$$

The other forces are similarly developed.

Utilizing a first order approximation, the value of i^2 may be written $i^2 = i^1 + i^1,_r\Delta r$,

$$= r^2\sigma^{rr}\Delta\phi\Delta\Theta + r^2\sigma^{rr}_{,r}\Delta r\Delta\phi\Delta\Theta + 2r\sigma^{rr}\Delta r\Delta\phi\Delta\Theta. \quad (7)$$

The remaining forces of Figure 3 are

$$i^3 = r\sigma^{r\Theta}\Delta\phi\Delta r,$$
$$i^4 = r\sigma^{r\Theta}\Delta\phi\Delta r + r\sigma^{r\Theta}_{,\Theta}\Delta r\Delta\phi\Delta\Theta,$$
$$i^5 = (r/\sin\Theta)\,\sigma^{r\phi}\Delta r\Delta\Theta, \quad (8)$$
$$i^6 = (r/\sin\Theta)\sigma^{r\phi}\Delta r\Delta\Theta + (r/\sin\Theta)\sigma^{r\phi}_{,\phi}\Delta r\Delta\phi\Delta\Theta,$$
$$i^7 + i^8 = r(\sigma^{\Theta\Theta} + \sigma^{\phi\phi} - \sigma^{r\Theta}\cot\Theta)\,\Delta r\Delta\phi\Delta\Theta,$$
$$I^r = \rho r^2\Delta r\Delta\phi\Delta\Theta F^r.$$

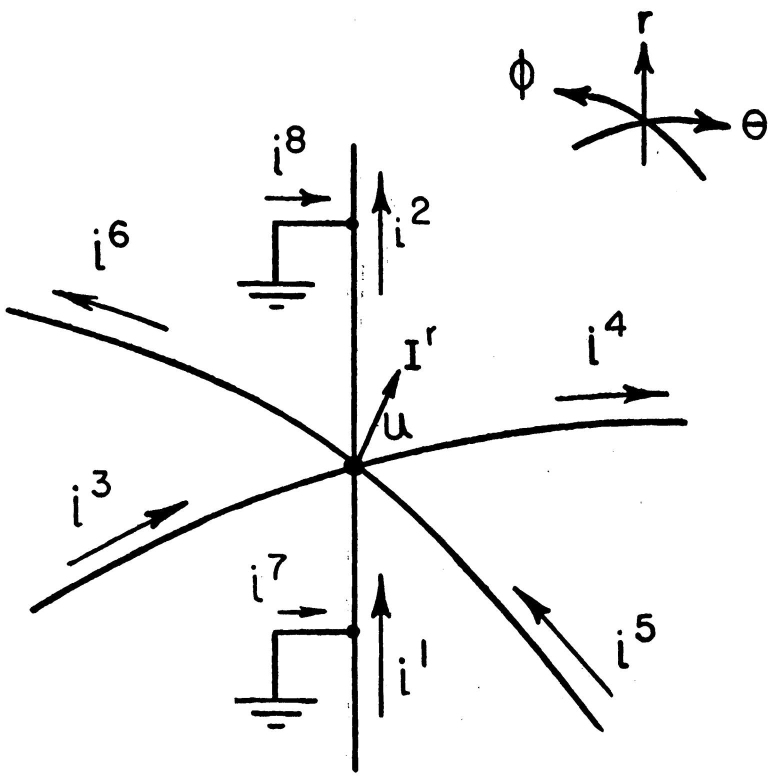


Fig. 3. Representative u junction for spherical coordinate network.

Equilibrium requires that these be no resultant force at this junction. Hence,

$$i^2 - i^1 + i^4 - i^3 + i^6 - i^5 - (i^7 + i^8) + I^r = 0 \qquad (9)$$

or, $$r^2[\sigma^{rr}_{,r} + \sigma^{r\phi}_{,\phi}/r\sin\Theta + \sigma^{r\Theta}_{,\Theta}/r + (2\sigma^{rr} - \sigma^{\Theta\Theta} - \sigma^{\phi\phi} - \sigma^{r\Theta}\cot\Theta)/r + \rho F^r]\,\Delta r\Delta\phi\Delta\Theta = O. \qquad (10)$$

Admittances used in the circuit of Figure 2

element	Y	element	Y
a	$r^2(\lambda+2\mu)\Delta\phi\Delta\Theta/\Delta r$	s	$r\lambda\Delta\phi\Delta\Theta$
b	$(\lambda+2\mu)\Delta r\Delta\phi/\Delta\Theta$	t	$\mu(1+2\cot^2\Theta)$
c	$(\lambda+2\mu)\Delta r\Delta\Theta/\Delta\phi\sin^2\Theta$		$\Delta r\Delta\phi\Delta\Theta/2$
d	$r\lambda\Delta\Theta$	x	$\lambda\cot\Theta\Delta r\Delta\phi/2$
e	$r\lambda\Delta\Theta/\sin\Theta$	y	$(\lambda+2\mu)\cot\Theta$
f	$\lambda\Delta r/\sin\Theta$		$\Delta r\Delta\Theta/2\sin\Theta$
g	$\mu\Delta r\Delta\phi/\Delta\Theta$	z	$r\lambda\cot\Theta\Delta\phi\Delta\Theta/2$
h	$\mu r^2\Delta\phi\Delta\Theta/\Delta r$	A	$-\mu\Delta r\Delta\phi/2$
i	$\mu r\Delta\phi$	D	$-\mu\Delta r\Delta\Theta/2\sin\Theta$
j	$\mu\Delta r\Delta\Theta/\Delta\phi\sin^2\Theta$	E	$(2\lambda+3\mu)\cot\Theta$
k	$\mu r^2\Delta\phi\Delta\Theta/\Delta r$		$\Delta r\Delta\phi\Delta\Theta/2$
l	$\mu r\Delta\Theta/\sin\Theta$	F	$-\mu\cot\Theta\Delta r\Delta\phi/2$
n	$\mu\Delta r\Delta\phi/\Delta\Theta$	G	$-\mu r\cot\Theta\Delta\phi\Delta\Theta$
o	$\mu\Delta r/\sin\Theta$	H	$-\mu\cot\Theta\Delta r\Delta\Theta/2\sin\Theta$
p	$2(\lambda+\mu)\Delta r\Delta\phi\Delta\Theta$	J	$-\mu r\Delta\phi\Delta\Theta/2$
q	$(\lambda+\mu)\Delta r\Delta\Theta/\sin\Theta$	L	$-\mu\cot\Theta\Delta r\Delta\phi$
r	$(\lambda+\mu)\Delta r\Delta\phi$	M	$-\mu\cot\Theta\Delta r\Delta\phi/2$

Eqn. (10) satisfies equilibrium requirements for forces in the r direction. Equilibrium in Θ and ϕ directions is satisfied at v and w nodes. Appropriate ground admittances may be added at nodes to handle transient problems. Thermal stress may be represented by applying proper boundary conditions and body forces.

Compatibility of displacement is automatically satisfied by developing the network on the basis of displacements.

Figure 4 shows a circuit reduced to represent steady state axial symmetry without body forces. If a solid body is represented, a center section like that of Figure 5 should be used. In both cases $\Delta\Theta$ has been expanded for clarity. Boundry alterations may be accomplished to represent irregular shapes.

Solution of the Network

A network of this type may always be solved numerically. In addition, any physical phenomenon which may be described by $X = CY$ may be employed. Electric circuits provide the most obvious application since current may be related to force, voltage to displacement, and the electrical admittance to mechanical admittance.

Application requires that block size be compatible with stress gradients.

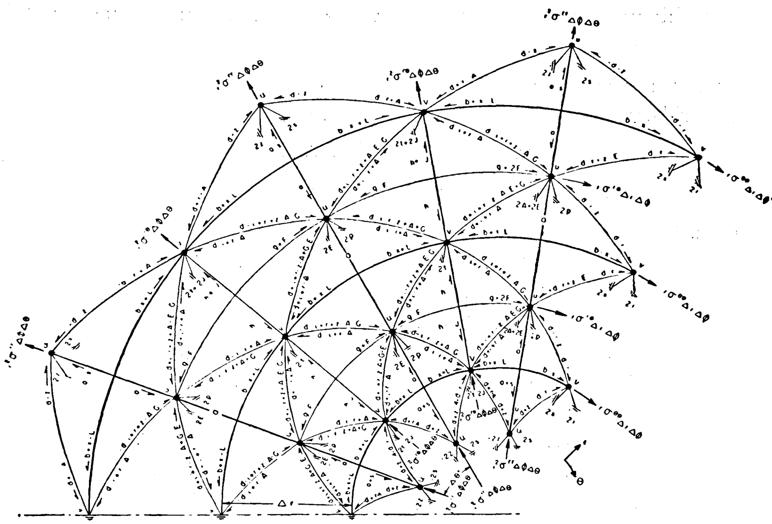


Fig. 4. Equivalent circuit for axially symmetric body described in spherical coordinates, (Θ amplified).

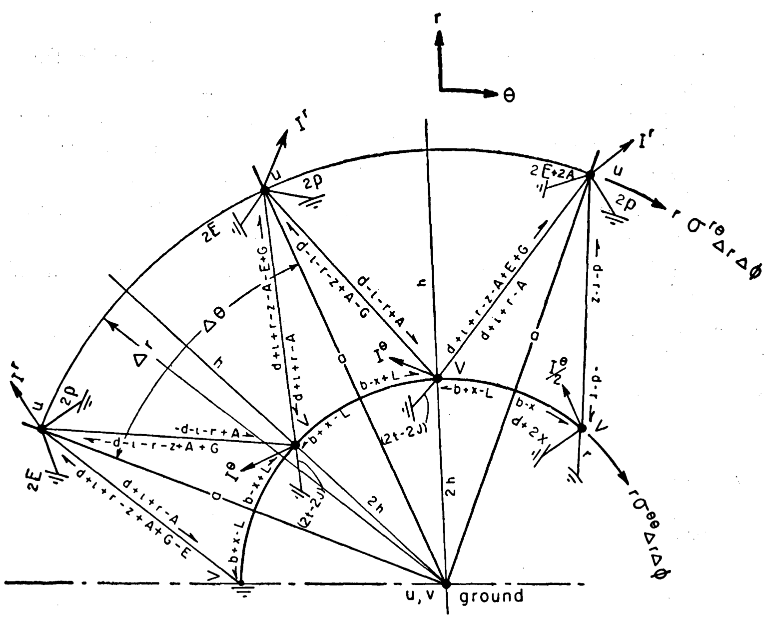


Fig. 5. Equivalent circuit at center of solid axially symmetric body described in spherical coordinates, (Θ amplified).

Literature Cited

1. Kron, G. 1944. Equivalent circuits of the elastic field. Jl. Ap. Mech. 11,3:A-149-61.
2. Gross, W. A., Soroka, W. W. 1952. Equivalent circuit for torsion of a stub shaft. Proc. Eighth Int'l Cong. of Ap. Mech., Istanbul.
3. Gross, W. A., Soroka, W. W. 1954. Network representation of elastic problems in cylindrical coordinates. Proc. Soc. Exp. Stress Anal. 11,2: 45-58.

DEPARTMENT OF THEORETICAL AND APPLIED MECHANICS
IOWA STATE COLLEGE
AMES, IOWA