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In almost all books on College Algebra, the Pascal Triangle is placed in such a position as to show the binomial coefficients in a horizontal line. This position is appropriate when the only use of the trangle is to help students remember these coefficients; but Pascal had a wider purpose in mind.

Pascal's Construction. Pascal first sets up a lattice-work consisting of equal squares, assigning to each square a definite positive integer which he determines as follows.

1. The number in each square of the 1st (top) row, and also in each square of the 1st (left-hand) column is to be unity.

2. To each other square is assigned an integer determined by the recurrence relation which he states thus: "The number in any other square is equal to the sum of the two numbers immediately to the left, and immediately above it."

It will be convenient to use modern notation to interpret Pascal's definitions and theorems. Let us then introduce a coordinate system in which the X-axis is the upper boundary of the lattice-work, and the Y-axis, directed downward, is the left-hand boundary. The unit of the coordinate system is to be the side of a square. Thus the equation of the *lower* boundary of the squares in the first row will be $y=1$; and similarly for the 2nd, . . . kth row the equations of the lower boundary of the squares will be $y=2$, $y=3$, . . . $y=k$. And for the columns of squares the equations of the *right-hand* boundaries will be $x=1$, $x=2$, . . . $x=k$. Thus the coordinates (x, y) determine the lower right-hand vertex of an arbitrary square.

We can now apply Pascal's Definitions to determine the number (positive integer) which is to be assigned to each square. The 1st row, and the 1st column, will be occupied by 1's by the first Definition. The second Definition now enables us to fill up the squares successively, thus: for the 2nd row, the 1st "cell" (Pascal's term) is of course 1, while the 2nd, 3rd, . . . are to be formed by adding unity to the preceding number in the row. But this is precisely the way in which the set of positive integers is obtained, starting from 1. In symbols, $\sum_{x=1}^{x} 1 = x$. The 3rd row will now be assigned the numbers

1, 3, 6, 10, . . .

which we recognize (as did Pascal) to be the triangular numbers of the ancient Greeks. They are seen to be the numbers given by $\sum_{1}^{x} x = x(x+1)/2$. If we now let f(x,y) equal the "cell" corresponding to the coordinates (x,y), we have thus far found:

1. $f(x,1) = 1$ 2. $f(x,2) = x$ 3. $f(x,3) = x(x+1)/2$.

By successive finite summation we find the following:

$$f(x,4) = \sum_{1}^{x} \frac{x(x+1)}{2} = \frac{x(x+1)\ (x+2)}{2\ .\ 3}$$

and in general

$$f(x,y) = \frac{x(x+1)(x+2)\ldots(x+y-2)}{(y-1)!} = \frac{(x+y-2)!}{(x-1)!\ (y-1)!} \qquad (1$$

We at once notice two facts:

1. $f(x,y) = (y,x)$, that is, the cells are symmetrical to the line $y = x$. As would be expected, Pascal notes this fact.

2. The numbers f(x,y) are all binomial coefficients.

Returning to Pascal, he now draws a diagonal line passing through the lower left and upper right vertices of a set of squares. This line he calls a "base." Thus a single base passes through the squares (1,6), (2,5), (3,4), (4,3), (5,2), and (6,1). The corresponding numbers, which Pascal calls "cells on the same base," are seen to be 1, 5, 10, 10, 5, 1. These are indeed the binomial coefficients for exponent 5. And the equation of this base is $X+Y=7$, so that (1) is confirmed in this case. Moreover, the equation of any line parallel to a base is $X + Y = \text{const}$. It follows that $x+y-2$, the numerator of (1), is constant along any base, and the cells on any base are the binomial coefficients for a single exponent.

We shall now state a few of Pascal's many theorems derived from the arithmetical triangle.

Theorem 1. Any cell is equal to the sum of all the cells of the preceding row up to and including the cell immediately above the one specified.

In symbols,

$$f(x,y) = \sum_{x=1}^{x} f(x, y-1).$$

The proof by mathematical induction, the method Pascal uses, is not difficult. (Of course, the summation may be made by using a single column as well as a single row.)

The importance of this Theorem may be judged from the fact that James Bernoulli made it the foundation upon which he developed his great work, Ars Conjectandi. His discovery of the Bernoulli numbers was based upon this Theorem.

Theorem 2. The sum of the cells of any base is equal to double the sum of the cells of the preceding base.

This theorem is of course well known.

Theorem 3. The sum of the cells of any base is greater by one than the sum of the cells of all the preceding bases.

For, $1 + 2 + 4 \ldots + 2^{n-1} = 2^n - 1$.

Theorem 4. Of two adjacent cells on the same base, the upper is to the lower as the number of cells from the upper to the end of the base, is to the number of cells from the lower end of the base to the lower cell(both inclusive.)

In symbols, $f(x + 1, y - 1)/f(x,y) = (y - 1)/x$.

With the aid of symbols, the proof is short and painless, even for an average student. Pascal's proof is by mathematical induction. The statement of the next theorem requires a new definition. Pascal calls "exponent of the triangle" the number $x + y - 1$, that is, the number 1 greater than the exponent of the binomial which corresponds to a given base.

Theorem 5. In an arithmetical triangle the sum of the cells of any row is to the last cell in the row as the "exponent of the triangle" is to the y-coordinate of the row itself.

In symbols,

$$\sum_{x=1}^{x} f(x,y) : f(x,y) = (x + y - 1) : y.$$

This is also very easily proved by writing out the values of the symbols on the left, noting that the summation gives $f(x,y + 1)$.

Theorem 6. Any cell, added to the preceding ones in the same column, is to the sum similarly formed from the cells in its row as the number of cells in the column is to the number of cells in the row.

Here also the proof using symbols is very easy.

Theorem 7. In an arithmetical triangle two successive cells on the bisecting line being chosen, the higher-valued is to 4 times the lower-valued as the "exponent of the base" of the latter is to a number greater by unity.

We recall that the "exponent of the base" means $x + y - 1$, or since in this case $y = x$, this number is $2x - 1$. The two cells involved are $f(x,x)$ and $f(x + 1, x + 1)$. Thus the first ratio mentioned is,

$$f(x + 1, x + 1) : 4f(x,x)$$

while the second ratio is $(2x - 1) : 2x$.

The proof of this theorem also is very easy.

Pascal applies these and other theorems to the solution of many problems relating to combinations, to probability, and to games of chance.

The final theorem which I shall mention is one which Pascal could not include in this collection, since he did not consider using negative exponents in the Binomial Theorem. But it makes an interesting addition to the uses ordinarily made of Pascal's Arithmetical Triangle.

Theormen 8. The numbers in the kth column (or kth row) are the coefficients in the expansion of $(1-u)^{-k}$, $k = 1, 2, 3, \ldots$ The proof would make a good problem for the better students in a class of College Algebra.

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