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A Modern Approach to Elementary Analysis

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Recall, on the one hand, that mathematics has grown up in the past one hundred fifty years. With the acceptance of non-euclidean geometrics, of algebraic structures beginning with groups, and of symbolic logic, the very idea of what mathematics is had to change. It took a while for mathematicians to digest the ideas of men like Lobachevski, Galois, and Boole. The Bourbaki notion of mathematics as a 'storehouse of structures' is new in the last few decades.

Recall, on the other hand, that the mathematics of the high school and the first two years of college is classical mathematics. Some of it is a diluted version of Euclid. Some of it is as modern as Euler.

Reacting to this instance of cultural lag, many good and thoughtful men are speaking out; and, as you would expect, they are reaching different conclusions.

The conservative is saying: "You cannot learn to run before you learn to walk. Modern mathematics is builtt upon classical mathematics. Without a thorough study of our mathematical heritage, a modern student will lack the perspective to distinguish fads from basic developments."

The radical is saying: "You cannot wait to relive the dead past. Let pupils begin early to make contact with the vigorous mathematics of their own day. Anyway, boolean algebra is simpler than classical algebra. Why wait?"

Sometimes one suspects that people advocate what they know. A mathematician may see no problem in teaching young people the branch of mathematics that he himself knows. He finds, in his own special field, beauty, clarity and great utility. He sees no obstacle to teaching children what his graduate students ("with all their lack of perception") manage to learn; so he writes a book, trying to address a new, and younger, audience by modifying slightly the form of presentation he has practiced with graduate students.

In sharp contrast, a college or high school teacher, who is ignorant of this mathematician's special field of study, may find his book dull, useless, and, most of all, incomprehensible. "Of course," he says, "I could learn it if I felt the effort worth my while. But this material would be completely beyond my pupils."

Let us consider the question: Is it feasible to introduce high school and college freshmen to portions of modern mathematics? I do not think I need to persuade you that the question is important, and timely; but permit me to share a quotation from the thirtieth Josiah Willard Gibbs Lecture, delivered at Rochester, New York, on December 27, 1956. Professor Marshall H. Stone of the University of Chicago discusses, under the title Mathematics and the Future of Science, the need to keep open the lines of communication among the pure and applied mathematicians and all men of science. He says, in conclusion, "What we must realize above all is that the mathematical education of the past has to a disturbing extent failed to lay the groundwork for the kind of intercommunication among mathematics, the various sciences, and engineering, which we now see to be necessary. By and large mathematical instruction has been little touched, except at the graduate level, by the mathematical advances of this century. Until it is, such improvements as are made in it will be mainly of a technical or pedagogical nature. The most serious obstacle to a modernization of the mathematical curriculum is the utilitarian spirit which pervades secondary education and governs the manner in which scientists teach the use of mathematics in the various fields where it is applied. Because students have been taught in high school to understand mathematics more in its practical aspects than in its technical and logical fullness, they arrive in college with their mathematical abilities blunted instead of sharpened and strengthened as they should have been. They are further encouraged to take a utilitarian view of mathematics by the way they see it handled—and, at times, mishandled—in nearly every scientific or engineering course they may elect. The consequence is that the attempt to teach calculus properly or to introduce the elements of modern algebra into the curriculum is often resented by students and criticised by other departments. Despite this difficulty some progress has been made. Much more, however, is essential before American mathematics can be considered to be in a sound and healthy state." He finishes with a quotation from Francis Bacon, "if you will have a tree bear more fruit than it used to do, it is not anything you can do to the boughs, but it is the stirring of the earth and putting of new mould about the roots that must work it."*

I shall limit my remarks to elementary analysis, that is, to the study of the elementary functions. This excludes much that is important to the beginners in mathematics, and natural and social science. It ignores, for example the realm of discrete mathematics that leads into probability and statistics.

Moreover I am considering only those students for whom mathematics is special, as opposed to general, education. This is mathematics for teachers, prospective mathematicians, scientists and

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engineers. Judging from the trend of events, this may still be mathematics for large numbers of young Americans.

Our experience, to date, at Iowa State Teachers College is with eighth and ninth graders at our Campus school, and with college freshmen. We begin with a *set of elements;* we discuss *sentences* with *variables* whose *domain* is the set of elements.

Consider, for example, the set of natural numbers from 1 to 9 inclusive. We write $\{1, 2, 3, \ldots 9\}$. Consider the sentences

$$x < 5,$$
$$x + y = y + x, \text{ and}$$
$$3x = 7.$$

Each *condition* has a *solution set;* this is the set of roots of the condition. $x < 5$ has the solution set $\{1, 2, 3, 4, 5\}$; similarly, the solution set of $x + y = y + x$ is $\{1, 2, 3, \ldots 9\}$, and of $3x = 7$ is $\{\ \}$, the empty set. Each solution set has a *graph* on the *line of numbers.* From the beginning, we emphasize the graph as a teaching and learning aid.

With a college freshmen we pass rapidly to the set of real numbers, and to subsets of the real numbers that satisfy a condition. To each point on the line of numbers corresponds a number that is its *coordinate.* To each point on the line of numbers corresponds a *directed distance,* or *vector,* from the origin.

We introduce the cartesian product, and subsets of the cartesian product defined by conditions on two variables. Consider the solution set of each of the following conditions:

$$y < 3x^2 - 2x + 1,$$
$$y - x = 5 \text{ and } x^2 + y^2 = 36,$$
$$y = 2 \sin 3x.$$

Each solution set is a *relation* in the cartesian product. The second and third relations are *functions*; that is, the set $\{ (x, y) \}$ of ordered number pairs is such that, for two elements (x_1, y_1) and (x_2, y_2), $x_1 = x_2 \rightarrow y_1 = y_2$; that is, the second element of a number-pair is unique.

We have found it possible to avoid many of the confusions that plague the beginner as he learns the classical language of mathematics. For example, variables do not vary. They are symbols that may be replaced by any one of a definite set of symbols. Replacement instances of sentences become statements that are either true or false. The word *unknown* refers to a variable; when you seek the set of values of a variable that satisfy a condition, you may wish to speak of the unknown rather than the variable. The word *parameter* refers to a variable. A condition contains one or more variables; for example, $y = mx + b$ contains the variables x, y, m, and b. When you interpret $y = 3x + 2$ as a set of points, $y = mx + b$ is a set of sets of points. A Parameter is a variable that you use to select one set of elements from a set of sets of elements. The language of families of graphs comes easily and naturally.

Our ninth graders pick up the new language more easily than our college freshmen; our college freshmen pick up the new language more easily than we do.

We are encouraged by our experience to date. We beileve we are teaching our pupils sounder mathematics, and preparing them better for further work in mathematics. Some of our failures to communicate ideas to our pupils have already been explained as we located blind spots in our thinking. We catch ourselves using words as we learned to use them; indulging in contradictions that bother the beginner.

In closing, I wish to raise a theoretical question. Perhaps one can explain the growing up of mathematics as a product of efforts to probe the foundations. Mathematicians have tried to ask the right questions in the right terms. As a pay-off, they have discovered unity, and identity of structure, where previously there appeared to be lack of unity and common structure. Meantime psychologists have discovered that the whole is greater than the sum of its parts. The word structure is as important to the psychologist as to the modern mathematician. This leads me to my question. Is mathematics teachable just in case it is good mathematics? I suspect that the answer is yes.

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