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Lebesgue's density theorem states that at almost every point of a measurable set S in E_n, the metric density of S exists and is 1 and at almost every point of the complement of S, the density of S exists and is 0. This theorem was first proven for E_1 by Lebesgue using his theory of integration. It was later proven by Denjoy [1], Lusin [2], and Sierpiński [3] for E_1 without the use of integration. The theorem was first proven for E_n by de la Vallée Poussin.

In the light of the density theorem one might say that the "usual" values of metric density are 0 and 1. It is easy to give examples where the density does not exist. The purpose of this note is to show that the density in E_1 may have any value between 0 and 1.

Let $\{I_n\}$ be a sequence of intervals in E_1. The sequence is said to converge to the real number x, if (i) x is a member of I_n for each n and (ii) $\{m(I_n)\}$, where m denotes Lebesgue measure, is a null sequence.

Let ϕ be a real valued function defined on a subclass I of the class of all intervals in E_1. If $\{I_k\}$ is a sequence of intervals from I, then lim sup ϕ (I_k) and lim inf ϕ (I_k) will denote the right most and left most limit points of the sequence $\{\phi\ (I_k)\}$. Denote by I (x) the class consisting of all sequences of intervals from I which converge to x. Then $\limsup_{I \to x} \phi\ (I)$ is defined to be sup $\{\limsup \phi\ (I_k)$: $\{I_k\} \epsilon\ I\ (x)\}$, and $\liminf_{I \to x} \phi\ (I)$ is defined to be inf $\{\liminf \phi\ (I_k)$: $\{I_k\} \epsilon\ I\ (x)\}$.

The relative measure of a measurable set S in an interval I, denoted by $\rho(S{:}I)$, will mean the ratio of the measure of $S \frown I$ to the measure of I. Some obvious properties of $\rho(S{:}I)$ are the following:

(1) For a given measurable set S, $\rho(S{:}I)$ is defined for all bounded intervals in E_1.

(2) $0 \leq \rho(S{:}I) \leq 1$.

(3) If $m(S \frown I) = 0$, $\rho(S{:}I) = 0$.

(4) If $S \supseteq I$, $\rho(S{:}I) = 1$.

(5) If –S denotes the complement of S, then $\rho(-S{:}I) = 1 - \rho(S{:}I)$.

Statement (5) follows from the fact that

$m(S \frown I) + m(-S \frown I) = m(I)$.

Now let S be any measurable set in E_1 and let x be any real number. Then the upper metric density of S at x, denoted by $\overline{D}_x(S)$, is defined to be $\limsup_{I \to x} \rho(S:I)$, and the lower metric density, $\underline{D}_x(S)$ is defined to be $\liminf_{I \to x} \rho(S:I)$.

If $\overline{D}_x(S) = \underline{D}_x(S)$, then the common value is called the metric density of S at x and is denoted by $D_x(S)$. By the definition of $\overline{D}_x(S)$ and $\underline{D}_x(S)$ and (2) it follows that

(6) $0 \leq \underline{D}_x(S) \leq \overline{D}_x(S) \leq 1$.

If $D_x(S)$ exists, then

(7) $D_x(S) = \lim \rho(S:I_k)$

where $I_k \to x$.

From (7) and statement (5) it follows that if $D_x(S)$ exists then

(8) $D_x(-S) = 1 - D_x(S)$.

Example 1. Let $S = \{x: 0 \leq x < 1\}$. Then if $0<x<1$, $D_x(S) =1$, and if $x > 1$ or $x < 0$, then $D_x(S) = 0$. For the points 0 and 1, $D_x(S)$ fails to exist.

It is obvious that $D_x(S) = 1$ for points of S different from zero, and $D_x(S) = 0$ for points of $-S$ different from 1. Let $I_k = [-\frac{1}{k}, 0]$. Then $I_k \to 0$ and $\rho(S:I_k) = 0$. Hence $\underline{D}_0(S) = 0$. If $I_k = [0, \frac{1}{k}]$, $I_k \to 0$ and $\rho(S:I_k) = 1$. Therefore $\overline{D}_0(S) = 1$, and it follows that $D_0(S)$ does not exist.

In example 1, where $D_x(S) = 0$, there is some interval, I, about x such that $m(S \frown I) = 0$. An interesting example due essentially to Goffman [4], is the following in which $D_0(S) = 0$ but for each open interval, I, containing 0, $m(S \frown I) > 0$.

Example 2. For each positive integer n, let

$$A_n = \left\{ x: \frac{1}{n} < x < \frac{1}{n} + \frac{1}{2^n} \right\}.$$

Then if $S = \bigcup_{n=1}^{\infty} A_n$, $D_0(S) = 0$. For, if I is any interval about 0, and if $J_n = [0, \frac{1}{n-1}]$ and $K_n = [0, \frac{1}{n} + \frac{1}{2^n}]$, there is an n such that

$$\frac{m(S \frown J_n)}{m(J_n)} \leq \frac{m(S \frown I)}{m(I)} \leq \frac{m(S \frown K_n).}{m(K_n)}$$

Therefore

$$\overline{D}_0\ (S) \leq \lim_{K_n \to 0} \frac{m(S \frown K_n)}{m(K_n)} = \lim_{n \to \infty} \sum_{k=n}^{\infty} \frac{1}{2^k} \Big/ \frac{1}{n} + \frac{1}{2^n} = 0.$$

The following example gives a set whose density exists at 0 and has any given value between 0 and 1.

Example 3.

Let $0 < \lambda < 1$ be given. For each positive integer n, let

$A_n = L_n \smile \overline{L_n}$ and $B_n = M_n \smile \overline{M}_n$
where

$$L_n = \left\{ x: \frac{1}{n+1} < x < \frac{1}{n+1} + \frac{\lambda}{n(n+1)} \right\}$$

$$\overline{L}_n = \left\{ x: -\frac{1}{n+1} - \frac{\lambda}{n(n+1)} < x < -\frac{1}{n+1} \right\}$$

$$M_n = \left\{ x: \frac{1}{n+1} + \frac{\lambda}{n(n+1)} < x < \frac{1}{n} \right\}$$

$$\overline{M}_n = \left\{ x: -\frac{1}{n} < x < -\frac{1}{n+1} - \frac{\lambda}{n(n+1)} \right\}.$$

Then let $A = \bigcup_{n=1}^{\infty} A_n$ and $B = \bigcup_{n=1}^{\infty} B_n$. Then $D_0\ (A) = \lambda$.

Proof.

Let I be any interval containing 0 whose length is less than 1/2. Denote the left and right end points of I by h and k respectively. Since $m(I) < 1$, there exists an integer N such that

$$(9) \quad \frac{1}{N+1} \leq m(I) \leq \frac{1}{N}.$$

Also there exist integers p and q such that

$$(10) \quad \frac{1}{p} \leq k \leq \frac{1}{p-1}$$

$$(11) \quad -\frac{1}{q-1} \leq h \leq -\frac{1}{q}.$$

Let $I_R = (\frac{1}{p}, k)$ and $I_L = (h, -\frac{1}{q})$. Then

$$m(I_R) = k - \frac{1}{p} \leq \frac{1}{p(p-1)} \text{ and } m(I_L) \leq \frac{1}{q(q-1)}.$$

From inequalities (9) and (10)

$$\frac{1}{p} \leq k \leq m(I) \leq \frac{1}{N}$$

and $p \geq N$. Thus $\frac{1}{p(p-1)} \leq \frac{1}{N(N-1)}$. It follows from inequalities (9) and (11) in a similar manner that $\frac{1}{q(q-1)} \leq \frac{1}{N(N-1)}$.
Therefore, if $E = I_R \cup I_L$,

$$m(E) \leq \frac{2}{N(N-1)}.$$

Again using inequality (9) it follows that

$$(12) \quad \frac{m(E)}{m(I)} \leq \frac{2(N+1)}{N(N-1)}.$$

Let $H = (A \cap I) \cup (B \cap I) - E$. Then H consists of disjoint open intervals L_n, M_n; $n = p, p+1, \ldots$ and $\bar{L}_m$, $\bar{M}_m$; $m = q, q+1, \ldots$ The interval I may be written as

$$(13) \quad I = H \cup E \cup D,$$

where D is a countable set consisting of the point 0 and the end points of the disjoint intervals in H. Since H, E, and D are disjoint,

$$(14) \quad m(I) = m(H) + m(E).$$

Now,

$$(15) \quad m(H) = \sum_{n=p}^{\infty} \frac{1}{n(n+1)} + \sum_{n=q}^{\infty} \frac{1}{n(n+1)}$$
$$= \frac{1}{p} + \frac{1}{q}$$

and

$$(16) \quad m(A \cap H) = \sum_{n=p}^{\infty} m(L_n) + \sum_{n=q}^{\infty} m(\bar{L}_n)$$
$$= \sum_{n=p}^{\infty} \frac{\lambda}{n(n+1)} + \sum_{m=q}^{\infty} \frac{\lambda}{m(m+1)}$$
$$= \lambda\left(\frac{1}{p} + \frac{1}{q}\right).$$

Therefore $\rho(A:H) = \lambda$.

From equation (14) it follows that

$$(17) \quad \frac{m(A \cap H)}{m(I)} \leq \rho(A:H) = \lambda.$$

It is also true that

$$(18) \quad \frac{m(A \cap H)}{m(I)} = \lambda m(H)/m(I),$$

but division of equation (14) by m(I) and rearrangement of terms gives

$$\frac{m(H)}{m(I)} = 1 - \frac{m(E)}{m(I)} .$$

It then follows from inequality (12) that

$$(19) \quad \frac{m(H)}{m(I)} \geq 1 - \frac{2(N+1)}{N(N-1)}.$$

Combining inequalities (17), (18), and (19) gives that

$$(20) \quad \lambda - \frac{2\lambda(N+1)}{N(N-1)} \leq \frac{m(A\frown H)}{m(I)} \leq \lambda.$$

From equation (13)

$$(21) \quad m(A\frown I) = m(A\frown H) + m(A\frown E).$$

Therefore

$$\frac{m(A\frown I)}{m(I)} = \frac{m(A\frown H)}{m(I)} + \frac{m(A\frown E)}{m(I)}$$

$$\leq \lambda + \frac{2(N+1)}{N(N-1)},$$

and

$$\frac{m(A\frown I)}{m(I)} \geq \frac{m(A\frown H)}{m(I)}$$

$$\geq \lambda - \frac{2\lambda(N+1)}{N(N-1)}.$$

Thus

$$(22) \quad \lambda - \frac{2\lambda(N+1)}{N(N-1)} \leq \rho(A:I) \leq \lambda + \frac{2(N+1)}{N(N-1)}.$$

For any sequence $I_r \longrightarrow 0$, the sequence N_r of integers associated with I_r must approach ∞. Therefore by (22), $D_0(A) = \lambda$.

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