A Note on Defining an Extension of a Probability Measure on Subsets of Function Space, By Applying One of J. L. Doob's Theorems

W. A. Small
Grinnell College
A Note on Defining an Extension of a Probability Measure on Subsets of Function Space, By Applying One of J. L. Doob's Theorems*

By W. A. Small

The following definitions are based on concepts in references (2) and (4).

Definition 1: Let \( W \) denote the set of all real valued functions, \( w(t) \), of a real variable \( t \).

Definition 2: Let \( t_1, \ldots, t_n \) be a finite set of \( t \) values.
Let \( -\infty \leq a_i < b_i \leq +\infty, \ i = 1, \ldots, n \). Then the set \( N \) of functions:
\[
N = \left\{ \begin{array}{c}
E \ w(t) \in W : a_i < w(t_i) < b_i, i = 1, \ldots, n \end{array} \right\}
\]
is defined to be a neighborhood of any function in the set \( N \).

It is noted that \( W \) is itself a neighborhood of each function in \( W \).

Definition 3: An open set in \( W \) is any union of neighborhoods.

Definition 4: A Borel Field \( F \) of subsets of \( W \) is a class of subsets of \( W \) such that \( W \in F \), and whenever \( A \in F \) and \( B \in F \), then \( A - B \in F \); and whenever each set of the sequence of sets \( A_1, \ldots, A_n, \ldots \), is in \( F \), then so is the union of the sequence.

It is noted that the intersection of the sets in the sequence is also in \( F \).

Definition 5: A probability measure on subsets of function space is a non-negative, completely-additive, complete, set function \( P(A) \) defined on a Borel Field of subsets of \( W \), and such that \( P(W) = 1 \).

Definition 6: \( F_2 \) is the Borel field generated by the open sets.

Definition 7: \( F_0 \) is the Borel field generated by the neighborhoods.

Definition 8: Let \( P \) be any probability measure defined on a Borel field \( F \) of subsets of \( W \); then the three concepts \( W,F,P \), together, are defined to be a Borel Probability Field in Function Space, abbreviated by bpf, and denoted by \( (W,F,P) \).

Definition 9: A bpf \( (W,F_0,P_0) \) is called a Fundamental Borel Probability Field in Function Space, abbreviated fbpf.

*The following note is based on part of a dissertation written under the direction of Professor Dorothy L. Bernstein of the University of Rochester.

Published by UNI ScholarWorks, 1958
Definition 10: If \((W,F,P)\) is any bpf, and if \(A \subseteq W\), then
\[ P^*(A) = \inf_E P(E), \ E \in F, \ E \supseteq A, \]
is defined to be the outer \(P\) measure of the set \(A\).

Definition 11: Let \((W,F,P)\) be a bpf, and let \(W'\) be a subset of \(W\) such that \(P^*(W') = 1\). Let \(\bar{W}\) denote the complement of \(W'\). If \(A\) is any subset of \(W\) such that
\[ A = EW' + H\bar{W}, \]
where \(E\) and \(H\) belong to \(F\), then define the set function \(P'(A)\) by:
\[ P'(A) = P(E). \]

The following theorem is a formalization of statements made by J. L. Doob (2:p.23; 3:p.69) and is also based on a theorem of J. L. Doob's (1,p.109, theorem 1.1):

**Theorem I:** Let \((W,F,P)\) be an arbitrary bpf. A necessary and sufficient condition that there exist an adjunction extension \((W,F',P')\) of \((W,F,P)\) obtained by adjoining \(W'\) to \(F\) is that \(P^*(W') = 1\).

**Proof:** This theorem is proved by showing that \(P'(E)\), as defined in the statement of the theorem is a probability measure (unique up to sets of \(P\) measure zero) such that its domain of definition includes the Borel field \(F' \supseteq F\), and such that \(P'\) reduces to \(P\) on \(F\).

The complete additivity and other probability measure properties of the \(P'\) measure follow from the corresponding properties of \(P\) measure, and from the uniqueness of \(P'\) which itself follows from Doob's theorem (1,p.109, theorem 1.1). The fact that \(F'\) is a Borel field which includes \(F\) follows from the properties of \(F\).

Now following Doob and S. Kakutani (2,p.25) define the set function \(P_2^*\) on the subsets of \(W\) as follows:

**Definition 12:** If \(G\) is any open set in \(W\), and \((W,F_0,P_0)\) is any fbpbf, then let \(P_2^*(G) = \sup_{E_0} P_0(E_0), \ E_0 \in F_0, \ E_0 \subseteq G\).

**Definition 13:** If \(A\) is any subset of \(W\), then let
\[ (A) = \inf_G P_2^*(G), \ G \text{ open, } G \supseteq A \]
It can then be shown that $P_2^*$ is an outer measure, and that the $P_2^*$ measurable sets include the Borel field $F_2$.

Let $P_2$ denote the $P_2^*$ measure of the sets in $F_2$. $P_2$ measure is called *Kakutani measure*.

It is sometimes desirable to know whether, when there exists an adjunction extension $(W,F'_0,P'_0)$ of $(W,F_0,P_0)$, the adjoined set $W'$ belongs to $F_2$ (2,p.29); the reason for this being that it is desirable to use the bpf $(W,F'_2,P'_2)$ in studying probabilities in function space. (2,p.25,26,29). It is clear that an adjunction extension $(W,F'_2,P'_2)$ of $(W,F_2,P_2)$, if it exists, corresponding to the adjunction extension $(W,F'_0,P'_0)$ of $(W,F_0,P_0)$ would serve the same purpose, even though $W'$ might not belong to $F_2$. The condition for the existence of this extension is given in the following theorem, which is Theorem I applied to the bpf $(W,F_2,P_2)$.

**Theorem II**: Suppose $(W,F'_0,P'_0)$ is an adjunction extension of a fbpf obtained by adjoining $W'$ to $F_0$. Then a necessary and sufficient condition that there exist a corresponding adjunction extension of $(W,F_2,P_2)$ obtained by adjoining $W'$ to $F_2$ is that $P_2^*(W') = 1$.

**Proof**: The proof is the same as in Theorem I, except that $(W,F_2,P_2)$ is used instead of the arbitrary bpf.

Without going into the concept of a measurable Borel Probability Field in function space (2,p.26-29), it may be stated that whenever the condition of Theorem II is satisfied for a measurable adjunction extension of a fbpf, then the corresponding adjunction extension of $(W,F_2,P_2)$ is also measurable. This follows from the fact that $F'_0 \subseteq F'_2$.

**Literature Cited**


**DEPARTMENT OF MATHEMATICS**

**GRINNELL COLLEGE**

**GRINNELL, IOWA**