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Waring's Problem, Modulo p , and the Representation Symbol

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Abstract. The representation symbol $[a,b,c]$ is the statement that an integer of n -ic type a is congruent to the sum of an integer of n -ic type b and an integer of n -ic type c . The symbol is extended to include any definite number of elements. New properties, together with a list of symbols involving the n -ic types of specific integers, are derived for use in studying Waring's problem, modulo p , for a particular exponent n . Let $T_p(n)$ be the least number such that every integer is congruent to the sum of $T_p(n)$ or fewer n -ic residues. Then for primes of the form $22k + 1$, $k > 3$, $2 \leq T_p(11) \leq 4$.

Waring's problem, modulo p , is the determination, for given integers n and p , of a least s depending on n and p such that for every integer c , the equation

$$c = \sum_{i=1}^s x_i^n + pq$$

will have a solution where the x_i are integers. The integer s shall be denoted by $T_p(n)$.

Throughout this discussion the letter g will be used to designate a fixed primitive root for a given prime p . An integer g is a primitive root of a prime p if and only if $p-1$ is the least positive integer t such that

$$g^t \equiv 1 \pmod{p}.$$

If c is an integer not divisible by p , then there exists an integer t such that $c \equiv g^t \pmod{p}$ where $0 \leq t < p-1$.

For any given integer n , an integer a is called the n -ic type, modulo p , of the integer c , if $t = nq + a$, where $0 \leq a < n$. If $a = 0$, then c is called an n -ic residue since $c \equiv (g^q)^n \pmod{p}$.

Waring's problem, modulo p , can now be stated as the determination of the least number $T_p(n)$ such that any integer is congruent to the sum of $T_p(n)$ or fewer n -ic residues.

The letter I denotes a variable integer. In a single expression its value need not be the same if it occurs twice. u_k denotes the n -ic type of the integer k . Thus $k \equiv g^{nI+u_k} \pmod{p}$.

The representation symbol, $[a,b,c]$, as defined by Torline (1955)¹, is the statement that an integer of n -ic type a is representable as the sum of an integer of n -ic type b and an integer of n -ic type c .

This may be written:

$$[a,b,c] \Leftrightarrow g^{nI+a} \equiv g^{nI+b} + g^{nI+c} \pmod{p}.$$

The extended representation symbol, $[a_1; a_2, a_3, \dots, a_s]$, is the statement

$$g^{nI+a_1} \equiv g^{nI+a_2} + g^{nI+a_3} + \dots + g^{nI+a_s} \pmod{p}.$$

To justify these definitions it can be shown that all integers of the same n -ic type are representable as the same number of n -ic residues.

The following properties were derived by Torline:

Property 1. If $[a,b,c]$ holds and if integers of n -ic types b and c can be written as the sum of t_1 and t_2 n -ic residues respectively, then integers of n -ic type a are representable as the sum of $t_1 + t_2$ n -ic residues.

Property 2. Since -1 is an n -ic residue for primes of the form $2nk + 1$, permutations of the elements in the representation symbol do not change the validity of the symbol for these primes. Thus,

$$[a,b,c] \iff [b,c,a] \iff [c,a,b].$$

Property 3. $[a,b,c] \iff [a+r,b+r,c+r]$ since by multiplying the congruence

$$g^{nI+a} \equiv g^{nI+b} + g^{nI+c} \pmod{p}$$

by g^r , we get the congruence

$$g^{nI+a+r} \equiv g^{nI+b+r} + g^{nI+c+r} \pmod{p}.$$

The following theorem utilizes the symbol which has more than three elements.

Theorem: If $[a,b,c]$ holds, then $[2a;2b,u_2+b+c,2c]$ and $[3a;3b,u_3+2b+c,u_3+b+2c,3c]$ hold.

Proof: If $[a,b,c]$ holds, then $g^{nI+a} \equiv g^{nI+b} + g^{nI+c} \pmod{p}$.

By squaring this congruence we have

$$g^{nI+2a} \equiv g^{nI+2b} + 2g^{nI+b}g^{nI+c} + g^{nI+2c} \pmod{p},$$

and cubing the same congruence we have

$$g^{nI+3a} \equiv g^{nI+3b} + 3g^{nI+2b}g^{nI+c} + 3g^{nI+b}g^{nI+2c} + g^{nI+3c} \pmod{p}.$$

Interpreting these congruences as representation symbols, recalling that

$$2 \equiv g^{nI+u_2} \pmod{p} \text{ and } 3 \equiv g^{nI+u_3} \pmod{p},$$

the desired symbols are obtained.

In particular, if $[a,0,0]$ and if $u_2 = 0$, then $[2a;0,0,0]$.

Also, if $[a,0,0]$ and if $u_2 = 0$, then $[3a;0,0,0,0]$.

The simple relation $16 = 15 + 1$ or $2^4 = 5 \cdot 3 + 1$ gives the congruence

$$g^{nI+4u_2} \equiv g^{nI+u_5+u_3} + 1 \pmod{p},$$

which is equivalent to the statement A: $[4u_2,u_5+u_3,0]$. Similarly, the following valid symbols, as well as other useful symbols not included here, are obtained.

$$B: [u_3,u_2,0]$$

$$C: [2u_3,3u_2,0]$$

$$D: [2u_5,2u_3,2u_4]$$

$$E: [u_5,u_3,u_2,]$$

The following theorem, dealing with Waring's problem, modulo p , with $n = 11$, makes use of these ideas.

Theorem 2. If p is a prime of the form $22k + 1$ where $k > 3$, then $2 \leq T_p(11) \leq 4$.

For a fixed prime p of the form $22k + 1$, the proof was divided into three cases.

Case I: $u_2 = 0, u_3 = 0$; in which case an argument involving sequences of integers was used together with some of the above theory.

Case II: $u_2 = 0, u_3 = a \neq 0$; in which all possible values for u_5 were considered; that is, $u_5 = 0, a, 2a, \dots, 10a$.

Case III: $u_2 = a \neq 0$; in which case all possible values for u_5 were considered for each possible value for u_3 .

The case when $u_2 = 0, u_3 = a \neq 0$, and $u_5 = 7a$ will serve as an example of the technique used in cases II and III. By substituting these values in the representation symbols A, B, C, D, and E as given above, the following valid symbols, together with their derived symbols, are obtained.

A: $[0, 8a, 0] \iff [8a, 0, 0]$ by property 2
 $\implies [5a; 0, 0, 0]$ by theorem 1

B: $[a, 0, 0]$

C: $[2a, 0, 0] \implies [4a; 0, 0, 0]$ by theorem 1

D: $[3a, 2a, 0] \iff [a, 0, 9a] \iff [0, 10a, 8a]$ by properties 2 and 3

E: $[7a, a, 0] \iff [6a, 0, 10a]$ by property 3

From the symbols containing italicized elements it may be concluded that integers of n -ic types $a, 2a$, and $8a$ are representable as the sum of two n -ic residues. Also integers of n -ic types $3a, 4a, 5a, 7a, 9a$, and $10a$ are representable as the sum of three n -ic residues, and integers of n -ic type $6a$ are representable as the sum of four n -ic residues. Thus integers of all n -ic types are representable as the sum of not more than four n -ic residues for this particular case. Tables were set up using this basic device for all possible combinations of u_2, u_3 , and u_5 . In addition, in several cases representation symbols involving the n -ic types of the integers 7, 23, and 67 were used in the proof of the theorem.

Literature Cited

1. Torline, Sister Mary Ferdinand, C.S.J. "Waring's Problem, Modulo p ," Unpublished Ph.D. dissertation, Department of Mathematics, Saint Louis University, 1955.

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