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Sidney D. Nolte
Iowa State University

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An Application of Generalized Means

By SIDNEY D. NOLTE

Abstract. The generalized mean $M(x,y)$ is defined to be $\Psi^{-1}[p\Psi(x) + \alpha\Psi(y)]$ where $p, \alpha > 0, p + \alpha = 1$ and $\Psi(x)$ is monotone and continuous.

This mean is applied to the second difference.

$$\Delta^2(f; x, h) = f(x+h) + (f-h) - 2f(x)$$

to form a generalized second difference

$$\Delta^2\Psi(f; x, h) = M\Psi[f(x+h), f(x-h)] - f(x)..$$

A study is made of functions whose generalized second differences satisfy certain conditions. Maxima of classes of generalized quasi-smooth functions are examined.

It is the purpose of this note to apply the generalized mean to the study of second differences. A generalized second difference will be defined and certain properties of the second difference will be examined under this generalization.

MEANS DEFINED

A generalized mean is defined to be a single valued function $M(x,y)$ of two variables x and y ($\alpha \leqq x, y \leqq \beta$) if $M(x,y)$ satisfied the postulates:

- (i) Strictly monotonic: This means that if $x < x'$, then $M(x,y) < M(x',y)$ and likewise for y .
- (ii) Continuous:
- (iii) Bisymmetric: This means that $M[M(x_1,x_2), M(y_1,y_2)] = M[M(x_1,y_1), M(x_2,y_2)]$
- (iv) Reflexive: $M(x,x) = x$
- (v) Symmetric: $M(x,y) = M(y,x)$.

It follows immediately from postulates (i) and (iv) that any $M(x,y)$ will have the property that if $x < y$, then $x < M(x,y) < y$.

Aczel [1] has proved that postulates (i) through (v) are necessary and sufficient conditions for the existence of a strictly monotone, continuous function $\Psi(x)$ ($\alpha \leqq x \leqq \beta$) by which $M(x,y)$ has the form

$$(1) \quad M(x,y) = \Psi^{-1} \left[\frac{\Psi(x) + \Psi(y)}{2} \right]$$

Further, a necessary and sufficient condition for the function $M(x,y)$ to satisfy postulates (i) through (iv) is that there exists a strictly monotone, continuous function $\Psi(x)$ ($\alpha \leqq x \leqq \beta$) and a pair of positive numbers p, q such that $p + q = 1$ and by which $M(x,y)$ has the form

$$(2) \quad M(x,y) = \Psi^{-1}[p\Psi(x) + q\Psi(y)].$$

The function $\Psi(x)$ is said to generate the mean $M_\Psi(x, y)$. It is easy to see that $\Psi(x)$ is not unique. That is, if the mean M_Ψ is generated by the function $\Psi(x)$, it is also generated by the function $\Phi(x) = A\Psi(x) + B$ (A, B constants).

The well known arithmetic, geometric and harmonic means are means satisfying (i) through (v) and can be generated by the functions x , $\log x$, $1/x$ respectively. Non-symmetric means are referred to as weighted means. The weighted arithmetic mean is written $M_x(x,y) = px + qy$, the weighted geometric mean is written $M_{10gx}(x,y) = x^p y^q$ and the weighted harmonic mean is written $M_{1/x}(x,y) = \frac{xy}{py + qx}$, where $p + q = 1$, and $p, q > 0$ in all these cases.

MEANS APPLIED TO THE SECOND DIFFERENCE

Consider the second difference of a function f

$$(3) \quad \Delta^2(f;x,h) = f(x + h) + f(x - h) - 2f(x).$$

Then

$$(4) \quad \frac{\Delta^2(f;x,h)}{2} = \frac{f(x + h) - f(x - h)}{2} - f(x).$$

This equation is an expression of the difference between $f(x)$ and the arithmetic mean of $f(x + h)$ and $f(x - h)$. This might suggest a generalized form of the second difference by use of other means.

Therefore the generalized second difference is defined by the expression

$$(5) \quad \Delta^2_\Psi(f;x,h) = M_\Psi[f(x + h), f(x - h)] - f(x)$$

where the domain of f is an interval $[\alpha, \beta]$ and the range of f is a subset of the domain of Ψ .

Let $\wedge(\alpha, \beta)$, M be the class of all continuous functions $f(x)$ ($\alpha \leq x \leq \beta$) which satisfy the condition

$$|f(x + h) + f(x - h) - 2f(x)| \leq 2Mh, M, h > 0.$$

Such functions are called quasi-smooth. Timan [2] has shown that

$$(6) \quad \omega^*(h) = \sup_{f \in \wedge(\alpha, \beta)M} \{ \omega(f,h) \} = \sup_{f \in \wedge(\alpha, \beta)M} \left\{ \sup_{|x_1 - x_2| \leq h} |f(x_1) - f(x_2)| \right\}$$

possesses the property

$$(7) \quad \omega^*(h) = \frac{M}{\ln 2} \left(h \ln \frac{1}{h} \right) + o(h).$$

Since $g(x) = f(x) + Ax + B$ is also quasi-smooth if $f(x)$ is, and since $f(x) \in \wedge(\alpha, \beta)M$ implies that $\frac{1}{M}f(x) \in \wedge(\alpha, \beta)1$, a normalization leads one to consider the class $\wedge^*(-1, 1)$ where $\wedge^*(-1, 1)$ is the class of functions $\wedge(-1, 1)$ which takes on the value zero at $+1$ and -1 .

In the course of the proof of the above relation, Timan shows that if

$$K = \sup_{f \in \wedge^*(-1,1)} \left\{ \max_{-1 \leq x \leq 1} |f(x)| \right\},$$

then

$$K \leq 4/3.$$

This result can be generalized by considering the class $\wedge^*_p(-1,1)$ of all continuous $f(x)$ ($-1 \leq x \leq 1$) such that if $(1) = f(-1) = 0$ and

$$(8) \quad |pf(x+h) + qf(x-h) - f(x)| \leq h, \quad h > 0.$$

Theorem: Let $K = \sup_{f \in \wedge^*_p(-1,1)} \left\{ \max_{-1 \leq x \leq 1} |f(x)| \right\}$

$$\text{Then } K \leq -\frac{4}{3} \left\{ \max[p,q] + 1/2 \right\} = \frac{4}{3} \left[1 + \left| p - \frac{1}{2} \right| \right]$$

Proof: Take $f(x) \in \wedge^*_p(-1,1)$ and let $\max_{-1 \leq x \leq 1} f(x) = f(x_0) = K - \epsilon$,

$\epsilon > 0$. Let $x_1 \leq x_2 \leq \dots \leq x_n$ be the points in $[-1,1]$ at which $f(x) = L$, $0 < L < K - \epsilon$. Then there exists two points x_i, x_{i+1} such that $x_i \leq x_0 \leq x_{i+1}$. Now consider the function

$$\Psi(x) = \frac{2}{x_{i+1} - x_i} \left\{ f \left[\frac{x_{i+1} - x_i}{2} x + \frac{x_{i+1} + x_i}{2} \right] - L \right\}.$$

Then it is easy to show that $\Psi(1) = \Psi(-1) = 0$ and that for $h > 0$,

$$|p\Psi(x+h) + q\Psi(x-h) - \Psi(x)| \leq h \quad (-1 \leq x \leq 1).$$

Therefore $\Psi(x) \in \wedge^*_p(-1,1)$. Hence $\max_{-1 \leq x \leq 1} \Psi(x) = \frac{2}{x_{i+1} - x_i}$

$$\left\{ \max_{-1 \leq x \leq 1} f(x) - L \right\} = \frac{2}{x_{i+1} - x_i} \left\{ K - \epsilon - L \right\} \leq K.$$

from which it follows that

$$(9) \quad K \leq \frac{2(L + \epsilon)}{2 - (x_{i+1} - x_i)}$$

Now let $x = 0, h = 1$, to obtain

$$(10) \quad |pf(1) + qf(-1) - f(0)| \leq 1,$$

set $x = 1/2, h = 1$ to obtain

$$(11) \quad |pf(1) + qf(0) - f(1/2)| \leq 1/2,$$

and set $x = -1/2, h = 1$ to obtain

$$(12) \quad |pf(0) + qf(-1) - f(-1/2)| \leq 1/2$$

Then from (10), (11) and (12) it follows that

$$(13) \quad |f(0)| \leq 1, |f(1/2)| \leq q + 1/2, |f(-1/2)| \leq p + 1/2.$$

Hence if in (15), $L = \max [p,q] + 1/2$ then $x_{i+1} - x_i \leq 1/2$.

Therefore $K \leq \frac{2\{\max [p,q] + \frac{1}{2} + \epsilon\}}{2 - 1/2} = \frac{4}{3} \left\{ \max [p,q] + \frac{1}{2} + \epsilon \right\}$

and it follows that $K \leq \frac{4}{3} \left\{ \max [p,q] + \frac{1}{2} \right\}$.

CONVEX FUNCTIONS

A function $f(x)$ is said to be convex if

(14)
$$\frac{f(x_1) + f(x_2)}{2} \geq \frac{f(x_1 + x_2)}{2}$$

for every x_1, x_2 in the domain of f . This leads one to consider "generalized convex" functions. A generalized convex function will be defined as one which satisfies

(15)
$$M_{\Psi} [f(x_1), f(x_2)] \geq f [M_{\Phi} (x_1, x_2)],$$

for $x_1 > x_2, x_1, x_2 \in [\alpha, \beta]$.

In particular, if M_{Ψ} is the weighted arithmetic mean and if M_{Φ} is the arithmetic mean, (15) becomes

(16)
$$pf(x_1) + qf(x_2) \geq f\left(\frac{x_1+x_2}{2}\right), p, q > 0, p + q = 1, x_1 > x_2.$$

The case where $p = q = \frac{1}{2}$ reduces to ordinary convex functions.

It is easy to see that equality holds in (14) if and only if $f(x) = Ax + B$, and equality holds in (16) if $f(x) = C$. This leads to the question of whether there exists non-constant solutions to the functional equation

(17)
$$pf(x) + qf(y) = f\left(\frac{x+y}{2}\right) = 0, p + q = 1, p, q > 0, x > y.$$

Let $f(x)$ be a non-constant solution of (17) for $a \leq x \leq b$, and let $x_{0\epsilon} \in (a,b)$. Then for a positive h sufficiently small, $x_0 - 2h$ and $x_0 + 2h$ are contained in the interval (a,b) . Also, $g(x) = f(x) - f(x_0)$ is also a solution of (17). Hence

(18)
$$\begin{aligned} pg(x_0 + 2h) + qg(x_0 - 2h) &= 0 \\ pg(x_0 + 2h) - g(x_0 + h) &= 0 \\ qg(x_0 - 2h) - g(x_0 - h) &= 0 \\ pg(x_0 + h) + qg(x_0 - h) &= 0 \end{aligned}$$

Since there exists a non-trivial solution of this system of equations, the determinant of the coefficients $\Delta = pq(q - p) = 0$. Therefore if (17) has a non-constant solution then $p = q = 1/2$. If $p \neq q$ then (17) has only a constant solution.

Now since there are no non-constant solutions of (17) in the case $p \neq q$, what types of functions are such that the inequality holds? This question is answered by the following theorem.

Theorem: If $f(x)$ is continuous for $\alpha \leq x \leq \beta$ and if 16 holds then $f(x)$ is monotone.

Proof: Let $a, b \in (\alpha, \beta)$, $a < b$. Then subdivide the interval (a, b) by the points $a + h, a + 2h, \dots, a + nh = b$ and let h be such that $a - h, b + h \in [\alpha, \beta]$. Then

$$(19) \quad \left\{ \begin{array}{l} pf(a + h) + qf(a - h) \geq f(a) \\ pf(a + 2h) + qf(a) \geq f(a + h) \\ pf(a + 3h) + qf(a + h) \geq f(a + 2h) \\ \cdot \\ \cdot \\ \cdot \\ pf(b + h) + qf(b - h) \geq f(b) \end{array} \right.$$

addition of these inequalities yields

$$qf(a-h) + pf(a) + pf(b) + pf(b+h) \geq f(a) + f(b)$$

This inequality is equivalent to

$$q[f(a-h)-f(a)] + p[f(b+h)-f(b)] \geq (p-q) [f(a)-f(b)].$$

By continuity of f , for any $\epsilon > 0$, h can be made sufficiently small so that $\epsilon \geq (p-q) [f(a)-f(b)]$. Therefore $(p-q)[f(a)-f(b)] \leq 0$ and $f(x)$ is monotone.

Corollary: If M_Ψ is any weighted mean generated by Ψ , $p \neq q$, and if f is such that

$$M_\Psi [f(x + h), f(x - h)] \geq f(x)$$

then if $\Psi f(x)$ is continuous it will be a monotone function.

Proof: Since $\Psi^{-1}[p\Psi f(x + h) + q\Psi f(x - h)] \leq f(x)$ and since $\Psi(t)$ is monotone, we have $p\Psi f(x + h) + q\Psi f(x - h) \geq \Psi f(x)$

Application of the previous theorem yields the desired result.

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DEPARTMENT OF MATHEMATICS
 IOWA STATE UNIVERSITY
 AMES, IOWA