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An Application of Generalized Means

By Sidney D. Nolte

Abstract. The generalized mean \( M(x,y) \) is defined to be
\[
\Psi^{-1}[p\Psi(x) + a\Psi(y)]
\]
where \( p, a > 0, p + a = 1 \) and \( \Psi(x) \) is monotone and continuous.

This mean is applied to the second difference.
\[
\Delta^2(f; x, h) = f(x+h) + (f+h) - 2f(x)
\]
to form a generalized second difference
\[
\Delta^2\Psi(f; x, h) = M\Psi[f(x+h), f(x-h)] - f(x).
\]

A study is made of functions whose generalized second differences satisfy certain conditions. Maxima of classes of generalized quasi-smooth functions are examined.

It is the purpose of this note to apply the generalized mean to the study of second differences. A generalized second difference will be defined and certain properties of the second difference will be examined under this generalization.

Means Defined

A generalized mean is defined to be a single valued function \( M(x,y) \) of two variables \( x \) and \( y \) \((\alpha \leq x, y \leq \beta)\) if \( M(x,y) \) satisfied the postulates:
(i) Strictly monotonic: This means that if \( x < x' \), then
\[
M(x,y) < M(x',y)
\]
and likewise for \( y \).

(ii) Continuous:

(iii) Bisymmetric: This means that
\[
M[M(x_1,x_2), M(y_1,y_2)] = M[M(x_1,y_1), M(x_2,y_2)]
\]
(iv) Reflexive: \( M(x,x) = x \)

(v) Symmetric: \( M(x,y) = M(y,x) \).

It follows immediately from postulates (i) and (iv) that any \( M(x,y) \) will have the property that if \( x < y \), then \( x < M(x,y) < y \).

Aczel [1] has proved that postulates (i) through (v) are necessary and sufficient conditions for the existence of a strictly monotone, continuous function \( \Psi(x) \) \((\alpha \leq x \leq \beta)\) by which \( M(x,y) \) has the form
\[
M(x,y) = \Psi^{-1} \left[ \frac{\Psi(x) + \Psi(y)}{2} \right]
\]

Further, a necessary and sufficient condition for the function \( M(x,y) \) to satisfy postulates (i) through (iv) is that there exists a strictly monotone, continuous function \( \Psi(x) \) \((\alpha \leq x \leq \beta)\) and a pair of positive numbers \( p, q \) such that \( p + q = 1 \) and by which \( M(x,y) \) has the form
\[
M(x,y) = \Psi^{-1}[p\Psi(x) + q\Psi(y)].
\]

The function \( \Psi(x) \) is said to generate the mean \( M_\Psi \) \((x, y)\). It is easy to see that \( \Psi(x) \) is not unique. That is, if the mean \( M_\Psi \) is generated by the function \( \Psi(x) \), it is also generated by the function \( \Phi(x) = A\Psi(x) + B \) (A,B constants).
The well known arithmetic, geometric and harmonic means are means satisfying (i) through (v) and can be generated by the functions $x, \log x, 1/x$ respectively. Non-symmetric means are referred to as weighted means. The weighted arithmetic mean is written $M_x(x,y) = px + qy$, the weighted geometric mean is written $M_{\log x}(x,y) = x^py^q$ and the weighted harmonic mean is written $M_{1/x}(x,y) = \frac{xy}{pq + qx}$, where $p + q = 1$, and $p,q > 0$ in all these cases.

**Means Applied to the Second Difference**

Consider the second difference of a function $f$

$$\Delta^2(f;x,h) = f(x + h) + f(x - h) - 2f(x).$$

Then

$$\frac{\Delta^2(f;x,h)}{2} = \frac{f(x + h) - f(x - h)}{2} - f(x).$$

This equation is an expression of the difference between $f(x)$ and the arithmetic mean of $f(x + h)$ and $f(x - h)$. This might suggest a generalized form of the second difference by use of other means.

Therefore the generalized second difference is defined by the expression

$$\Delta^2(f;x,h) = M_{\Psi} [f(x + h), f(x - h) - f(x)]$$

where the domain of $f$ is an interval $[\alpha, \beta]$ and the range of $f$ is a subset of the domain of $\Psi$.

Let $\wedge (\alpha, \beta)$, $M$ be the class of all continuous functions $f(x)$ ($\alpha \leq x \leq \beta$) which satisfy the condition

$$|f(x + h) + f(x - h) - 2f(x)| \leq 2Mh, \quad M, h > 0.$$  

Such functions are called quasi-smooth. Timan [2] has shown that

$$\omega^*(h) = \sup_{f \in \wedge (\alpha, \beta) M} \{ \omega(f,h) \} = \sup_{f \in \wedge (\alpha, \beta) M} \left\{ \sup_{x_1,x_2 \in [x_1,x_2] \leq h} |f(x_1) - f(x_2)| \right\}$$

possesses the property

$$\omega^*(h) = \frac{M}{1n2} (h \ln \frac{1}{h}) + O(h).$$

Since $g(x) = f(x) + Ax + B$ is also quasi-smooth if $f(x)$ is, and since $f(x) \epsilon \wedge (\alpha, \beta) M$ implies that $\frac{1}{M}f(x) \epsilon \wedge (\alpha, \beta) 1$, a normalization leads one to consider the class $\wedge^*(-1,1)$ where $\wedge^*(-1,1)$ is the class of functions $\wedge (-1,1)$ which takes on the value zero at $+1$ and $-1$. 

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In the course of the proof of the above relation, Timan shows that if

\[ K = \sup_{f \in \mathcal{F}} \left\{ \max_{-1 \leq x \leq 1} |f(x)| \right\}, \]

then

\[ K \leq 4/3. \]

This result can be generalized by considering the class \( \mathcal{F} \) of all continuous \( f(x) \) \((-1 \leq x \leq 1) \) such that if (1) \( f(1) = -f(-1) = 0 \) and

\[ |pf(x + h) + qf(x - h) - f(x)| \leq h, \quad h > 0. \]

**Theorem:** Let \( K = \sup_{f \in \mathcal{F}} \left\{ \max_{-1 \leq x \leq 1} |f(x)| \right\} \) and if \( f(-1) = 0 \) then

\[ K \leq -\frac{4}{3} \left\{ \max[p,q] + 1/2 \right\} = \frac{4}{3} \left[ 1 + \left| p - \frac{1}{2} \right| \right] \]

**Proof:** Take \( f(x) \in \mathcal{F} \) and let \( \max f(x) = f(x_0) = K < l \leq x \leq 1 \). Let \( x_1 < x_2 < \ldots < x_n \) be the points in \([-1,1]\) at which \( f(x) = L, \) \( 0 < L < K - \epsilon \). Then there exits two points \( x_i, x_{i+1} \) such that \( x_i \leq x \leq x_{i+1} \). Now consider the function

\[ \Psi(x) = \frac{2}{x_{i+1}-x_i} \left\{ f \left[ \frac{x_i \cdot x_{i+1}}{2} - x \right] + \frac{x_{i+1} - x}{2} \right\} - L. \]

Then it is easy to show that \( \Psi(1) = \Psi(-1) = 0 \) and that for \( h > 0, \)

\[ |p\Psi(x + h) + q\Psi(x - h) - \Psi(x)| \leq h (-1 \leq x \leq 1). \]

Therefore

\[ \Psi(x) \in \mathcal{F} \] and

\[ \left\{ \max_{-1 \leq x \leq 1} f(x) - L \right\} = \frac{2}{x_{i+1} - x_i} \left\{ K - \epsilon - L \right\} \leq K. \]

from which it follows that

\[ K \leq \frac{2(L + \epsilon)}{2 - (x_{i+1} - x_i)} \]

Now let \( x = 0, h = 1, \) to obtain

\[ |pf(1) + qf(-1) - f(0)| \leq 0, \]

set \( x = \frac{1}{2}, h = 1, \) to obtain

\[ |pf(1) + qf(-1) - f(0)| \leq \frac{1}{2}, \]

and set \( X = \frac{1}{2}, h = \frac{1}{2} \) to obtain

\[ |pf(0) + qf(-1) - f(0)| \leq 1/2 \]

Then from (10), (11) and (12) it follows that

\[ |f(0)| \leq 1, |f(\frac{1}{2})| \leq q + \frac{1}{2}, \]

\[ |f(-\frac{1}{2})| \leq p + \frac{1}{2}. \]
Hence if in (15), \( L = \max [p,q] + 1/2 \) then \( x_{i+1} - x_i \leq 1/2 \).

Therefore \( K \leq \frac{2\{\max[p,q] + \frac{1}{2} + \epsilon\}}{2 - 1/2} = \frac{4}{3}\{\max[p,q] + \frac{1}{2} + \epsilon\} \)

and it follows that \( K \leq \frac{4}{3}\{\max[p,q] + \frac{1}{2}\} \).

**Convex Functions**

A function \( f(x) \) is said to be convex if

\[
\frac{f(x_1) + f(x_2)}{2} \geq f\left(\frac{x_1 + x_2}{2}\right)
\]

for every \( x_1, x_2 \) in the domain of \( f \). This leads one to consider "generalized convex" functions. A generalized convex function will be defined as one which satisfies

\[
M_{\Psi}[f(x_1), f(x_2)] \geq f\left[M_{\Phi}(x_1, x_2)\right],
\]

for \( x_1 > x_2, x_1, x_2 \in [\alpha, \beta] \).

In particular, if \( M_{\Psi} \) is the weighted arithmetic mean and if \( M_{\Phi} \) is the arithmetic mean, (15) becomes

\[
pf(x_1) + qf(x_2) \geq f\left(\frac{x_1 + x_2}{2}\right), p, q > 0, p + q = 1, x_1 > x_2.
\]

The case where \( p = q = \frac{1}{2} \) reduces to ordinary convex functions.

It is easy to see that equality holds in (14) if and only if \( f(x) = Ax + B \), and equality holds in (16) if \( f(x) = C \). This leads to the question of whether there exists non-constant solutions to the functional equation

\[
pf(x) + qf(y) = f\left(\frac{x + y}{2}\right), 0, p + q = 1, p, q > 0, x > y.
\]

Let \( f(x) \) be a non-constant solution of (17) for \( a \leq x \leq b \), and let \( x_0 \in (a,b) \). Then for a positive \( h \) sufficiently small, \( x_0 - 2h \) and \( X_0 + 2h \) are contained in the interval \( (a,b) \). Also, \( g(x) = f(x) - f(x_0) \) is also a solution of (17). Hence

\[
\begin{align*}
pg(x_0 + 2h) + qg(x_0 - 2h) &= 0 \\
pg(x_0 + 2h) - g(x_0 + h) &= 0 \\
qg(x_0 - 2h) - g(x_0 - h) &= 0 \\
pqg(x_0 + h) + qg(x_0 - h) &= 0
\end{align*}
\]

Since there exists a non-trivial solution of this system of equations, the determinant of the coefficients \( \Delta = pq(q - p) = 0 \). Therefore if (17) has a non-constant solution then \( p = q = 1/2 \). If \( p \neq q \) then (17) has only a constant solution.
Now since there are no non-constant solutions of (17) in the case

\( p \neq q \), what types of functions are such that the inequality holds?

This question is answered by the following theorem.

**Theorem:** If \( f(x) \) is continuous for \( \alpha \leq x \leq \beta \) and if (16) holds

then \( f(x) \) is monotone.

**Proof:** Let \( a, b, \alpha, \beta \in \epsilon \), \( a < b \). Then subdivide the interval \((a, b)\) by

the points \( a + nh \), \( a + nh \), \ldots, \( a + nh = b \) and let \( h \) be such that

\( a - h, b + h \in \epsilon \{\alpha, \beta\}\). Then

\[
\begin{align*}
pf(a + h) + qf(a - h) & \geq f(a) \\
pf(a + 2h) + qf(a) & \geq f(a + h) \\
pf(a + 3h) + qf(a + h) & \geq f(a + 2h) \\
& \cdots \\
pf(b + qf(b - 2h) & \geq f(b - h) \\
pf(b + h) + qf(b - h) & \geq f(b).
\end{align*}
\]

(19)

Addition of these inequalities yields

\[ qf(a-h) + qf(a) + pf(b) + pf(b+h) \geq f(a) + f(b) \]

This inequality is equivalent to

\[ q[f(a-h)-f(a)] + p[f(b+h)-f(b)] \geq (p-q) [f(a)-f(b)]. \]

By continuity of \( f \), for any \( \epsilon > 0 \), \( h \) can be made sufficiently small

so that \( \epsilon \geq (p-q) [f(a)-f(b)] \). Therefore

\( (p-q) [f(a)-f(b)] \leq 0 \) and \( f(x) \) is monotone.

**Corollary:** If \( M_\Psi \) is any weighted mean generated by \( \Psi, p \neq q, \)
and if \( f \) is such that

\[ M_\Psi [f(x + h), f(x - h)] \geq f(x) \]

then if \( \Psi(x) \) is continuous it will be a monotone function.

**Proof:** Since \( \Psi^{-1} [p\Psi f(x + h) + q\Psi f(x - h)] \leq f(x) \) and since

\( \Psi(t) \) is monotone, we have \( p\Psi f(x + h) + q\Psi f(x - h) \geq \Psi f(x) \)

Application of the previous theorem yields the desired result.

**Literature Cited**


**Department of Mathematics**

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