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Abstract. The generalized mean $M(x,y)$ is defined to be $\Psi^{-1}[p\Psi(x) + \alpha\Psi(y)]$ where $p, \alpha > 0$, $p + \alpha = 1$ and $\Psi(x)$ is monotone and continuous.
This mean is applied to the second difference.
$\triangle^2(f\colon x, h) = f(x+h) + (f-h) - 2f(x)$
to form a generalized second difference
$\triangle^2\Psi(f\colon x, h) = M\Psi[f(x+h), f(x-h)] - f(x)$..
A study is made of functions whose generalized second differences satisfy certain conditions. Maxima of classes of generalized quasi-smooth functions are examined.

It is the purpose of this note to apply the generalized mean to the study of second differences. A generalized second difference will be defined and certain properties of the second difference will be examined under this generalization.

MEANS DEFINED

A generalized mean is defined to be a single valued function $M(x,y)$ of two variables x and y ($\alpha \leqq x,y \leqq \beta$) if $M(x,y)$ satisfied the postulates:

(i) Strictly monotonic: This means that if $x < x'$, then $M(x,y) < M(x',y)$ and likewise for y.
(ii) Continuous:
(iii) Bisymmetric: This means that $M[M(x_1,x_2), M(y_1,y_2)] = M[M(x_1,y_1), M(x_2,y_2)]$
(iv) Reflexive: $M(x,x) = x$
(v) Symmetric: $M(x,y) = M(y,x)$.

It follows immediately from postulates (i) and (iv) that any $M(x,y)$ will have the property that if $x < y$, then $x < M(x,y) < y$.

Aczel [1] has proved that postulates (i) through (v) are necessary and sufficient conditions for the existence of a strictly monotone, continuous function $\Psi(x)$ ($\alpha \leqq x \leqq \beta$) by which $M(x,y)$ has the form

$$M(x,y) = \Psi^{-1}\left[\frac{\Psi(x) + \Psi(y)}{2}\right] \tag{1}$$

Further, a necessary and sufficient condition for the function $M(x,y)$ to satisfy postulates (i) through (iv) is that there exists a strictly monotone, continuous function $\Psi(x)$ ($\alpha \leqq x \leqq \beta$) and a pair of positive numbers p, q such that $p + q = 1$ and by which $M(x,y)$ has the form

$$M(x,y = \Psi^{-1}[p\Psi(x) + q\Psi(y)]. \tag{2}$$

The function $\Psi(x)$ is said to generate the mean M_Ψ **(x, y).** It is easy to see that $\Psi(x)$ is not unique. That is, if the mean M_Ψ is generated by the function $\Psi(x)$, it is also generated by the function $\Phi(x) = A\Psi(x) + B$ (A,B constants).

The well known arithmetic, geometric and harmonic means are means satisfying (i) through (v) and can be generated by the functions x, log x, 1/x respectively. Non-symmetric means are referred to as weighted means. The weighted arithmetic mean is written $M_x(x,y) = px + qy$, the weighted geometric mean is written $M_{\log x}(x,y) = x^p y^q$ and the weighted harmonic mean is written $M_{1/x}(x,y) = \frac{xy}{py + qx}$, where $p + q = 1$, and $p,q > 0$ in all these cases.

Means Applied to the Second Difference

Consider the second difference of a function f

$$(3) \qquad \triangle^2(f;x,h) = f(x+h) + f(x-h) - 2f(x).$$

Then

$$(4) \qquad \frac{\Delta^2(f;x,h)}{2} = \frac{f(x+h) - f(x-h)}{2} - f(x).$$

This equation is an expression of the difference between f(x) and the arithmetic mean of f(x + h) and f(x — h). This might suggest a generalized form of the second difference by use of other means. Therefore the generalized second difference is defined by the expression

$$(5) \qquad \underset{\Psi}{\triangle^2}(f;x,h) = M_\Psi\,[f(x+h), f(x-h] - f(x)$$

where the domain of f is an interval $[\alpha,\beta]$ and the range of f is a subset of the domain of Ψ.

Let $\wedge\ (\alpha,\beta)$, M be the class of all continuous functions f(x) $(\alpha \leqq x \leqq \beta)$ which satisfy the condition

$$|\, f(x+h) + f(x-h) - 2f(x) \,| \leqq 2Mh,\ M,\ h > 0.$$

Such functions are called quasi-smooth. Timan [2] has shown that

$$(6) \qquad \omega^*(h) = \sup_{f\epsilon\wedge(\alpha,\beta)M} \{\omega(f,h)\} = \sup_{f\epsilon\wedge(\alpha,\beta)M} \left\{ \sup_{|x_1-x_2|\leqq h} |f(x_1)-f(x_2)| \right\}$$

possesses the property

$$(7) \qquad \omega^*(h) = \frac{M}{\ln 2}\left(h \ln \frac{1}{h}\right) + 0(h).$$

Since $g(x) = f(x) + Ax + B$ is also quasi-smooth if f(x) is, and since $f(x)\epsilon \wedge (\alpha,\beta)M$ implies that $\frac{1}{M} f(x) \epsilon \wedge (\alpha,\beta)\ 1$, a normalization leads one to consider the class $\wedge^*(-1,1)$ where $\wedge^*(-1,1)$ is the class of functions $\wedge(-1,1)$ which takes on the value zero at $+1$ and -1.

In the course of the proof of the above relation, Timan shows that if

$$K = \sup_{f \epsilon \wedge^*(-1,1)} \left\{ \max_{-1 \leqq x \leqq 1} | f(x) | \right\},$$

then

$$K \leqq 4/3.$$

This result can be generalized by considering the class $\wedge^*_p(-1,1)$ of all continuous $f(x)$ $(-1 \leqq x \leqq 1)$ such that if $(1) = f(-1) = 0$ and

(8) $\quad | pf(x+h) + qf(x-h) - f(x) | \leqq h, \qquad h > 0.$

Theorem: Let $K = \sup_{f \epsilon \wedge^*_p(-1,1)} \left\{ \max_{-1 \leqq x \leqq 1} | f(x) | \right\}$

Then $K \leqq \frac{4}{3} \left\{ \max[p,q] + 1/2 \right\} = \frac{4}{3} \left[1 + |p - \frac{1}{2}| \right]$

Proof: Take $f(x) \epsilon \wedge^*_p (-1,1)$ and let $\max_{-1 \leqq x \leqq 1} f(x) = f(x_0) = K - \epsilon$, $\epsilon > 0$. Let $x_1 \leqq x_2 \leqq \ldots \leqq x_n$ be the points in $[-1,1]$ at which $f(x) = L$, $0 < L < K - \epsilon$. Then there exits two points x_i x_{i+1} such that $x_i \leqq x_0 \leqq x_{i+1}$. Now consider the function

$\Psi(x) = \frac{2}{x_{i+1} - x_i} \left\{ f \left[\frac{x_{i+1} - x_i}{2} x + \frac{x_{i+1} + x_i}{2} \right] - L \right\}$. Then it is easy to show that $\Psi(1) = \Psi(-1) = 0$ and that for $h > 0$, $| p\Psi(x+h) + q\Psi(x-h) - \Psi(x) | \leqq h$ $(-1 \leqq x \leqq 1)$. Therefore $\Psi(x) \epsilon \wedge^*_p (-1,1)$. Hence $\max_{-1 \leqq x \leqq 1} \Psi(x) = \frac{2}{x_{i+1} - x_i}$

$$\left\{ \max_{-1 \leqq x \leqq} f(x) - L \right\} = \frac{2}{x_{i+1} - x_i} \left\{ K - \epsilon - L \right\} \leqq K.$$

from which it follows that

$$K \leqq \frac{2(L + \epsilon)}{2 - (x_{i+1} - x_i)} \tag{9}$$

Now let $x = 0$, $h = 1$, to obtain

$$| pf(1) + qf(-1) - f(0) | \leqq 1, \tag{10}$$

set $x = \frac{1}{2}$, $h =$ to obtain

$$| pf(1) + qf(0) - f(\tfrac{1}{2}) | \leqq \tfrac{1}{2}, \tag{11}$$

and set $X = -\frac{1}{2}$, $h = \frac{1}{2}$ to obtain

$$| pf(0) + qf(-1) - f(-\tfrac{1}{2}) | \leqq 1/2 \tag{12}$$

Then from (10), (11) and (12) it follows that

$$| f(0) | \leqq 1, | f(\tfrac{1}{2}) | \leqq q + \tfrac{1}{2}, | f(-\tfrac{1}{2}) | \leqq p + \tfrac{1}{2}. \tag{13}$$

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Hence if in (15), $L = \max[p,q] + 1/2$ then $x_{i+1} - x_i \leqq 1/2$.

Therefore $K \leqq \dfrac{2\{\max[p,q] + \frac{1}{2} + \epsilon\}}{2 - 1/2} = \dfrac{4}{3}\left\{\max[p,q] + \dfrac{1}{2} + \epsilon\right\}$

and it follows that $K \leqq \dfrac{4}{3}\left\{\max[p,q] + \dfrac{1}{2}\right\}$.

Convex Functions

A function f(x) is said to be convex if

$$(14) \qquad \frac{f(x_1) + f(x_2)}{2} \geqq \frac{f(x_1 + x_2)}{2}$$

for every x_1, x_2 in the domain of f. This leads one to consider "generalized convex" functions. A generalized convex function will be defined as one which satisfies

$$(15) \qquad M_{\Psi}[f(x_1), f(x_2)] \geqq f[M_{\Phi}(x_1, x_2)],$$

for $x_1 > x_2$, $x_1, x_2 \epsilon [\alpha,\beta]$.

In particular, if M_{Ψ} is the weighted arithmetic mean and if M_{Φ} is the arithmetic mean, (15) becomes

$$(16) \quad pf(x_1) + qf(x_2) \geqq f(\frac{x_1+x_2}{2}), p, q > 0, p + q = 1, x_1 > x_2.$$

The case where $p = q = \dfrac{1}{2}$ reduces to ordinary convex functions.

It is easy to see that equality holds in (14) if and only if $f(x) = Ax + B$, and equality holds in (16) if $f(x) = C$. This leads to the question of whether there exists non-constant solutions to the functional equation

$$(17) \; pf(x) + qf(y) = f\left(\frac{x+y}{2}\right) = 0, p + q = 1, p,q > 0, x > y.$$

Let f(x) be a non-constant solution of (17) for $a \leqq x \leqq b$, and let $x_{0} \epsilon (a,b)$. Then for a positive h sufficiently small, $x_0 - 2h$ and $X_0 + 2h$ are contained in the interval (a,b). Also, $g(x) = f(x) - f(x_0)$ is also a solution of (17). Hence

$$(18) \quad \begin{array}{lllll} pg(x_0+2h) & + qg(x_0-2h) & & & = 0 \\ pg(x_0+2h) & & - g(x_0+h) & & = 0 \\ & qg(x_0-2h) & & - g(x_0-h) & = 0 \\ & & pg(x_0+h) & + qg(x_0-h) & = 0 \end{array}$$

Since there exists a non-trivial solution of this system of equations, the determinant of the coefficients $\triangle = pq(q - p) = 0$. Therefore if (17) has a non-constant solution then $p = q = 1/2$. If $p \neq q$ then (17) has only a constant solution.

Now since there are no non-constant solutions of (17) in the case $p \neq q$, what types of functions are such that the inequality holds? This question is answered by the following theorem.

Theorem: If $f(x)$ is continuous for $\alpha \leqq x \leqq \beta$ and if 16 holds then $f(x)$ is monotone.

Proof: Let $a,b \in (\alpha,\beta)$, $a < b$. Then subdivide the interval (a,b) by the points $a + h, a + 2h, \ldots, a + nh = b$ and let h be such that $a - h, b + h \in [\alpha,\beta]$. Then

$$
(19) \quad \begin{cases}
pf(a + h) + qf(a - h) \geqq f(a) \\
pf(a + 2h) + qf(a) \geqq f(a + h) \\
pf(a + 3h) + qf(a + h) \geqq f(a + 2h) \\
\cdot \\
\cdot \\
\cdot \\
pf(b + qf(b - 2h) \geqq f(b - h) \\
pf(b + h) + qf(b - h) \geqq f(b).
\end{cases}
$$

addition of these inequalities yields

$$qf(a-h) + qf(a) + pf(b) + pf(b+h) \geqq f(a) + f(b)$$

This inequality is equivalent to

$$q[f(a-h)-f(a)]+p[f(b+h)-f(b)] \geqq (p-q)\ [f(a)-f(b)].$$

By continuity of f, for any $\epsilon > 0$, h can be made sufficiently small so that $\epsilon \geqq (p-q)\ [f(a)-f(b)]$. Therefore $(p-q)[f(a)-f(b)] \leqq 0$ and $f(x)$ is monotone.

Corollary: If M_Ψ is any weighted mean generated by Ψ, $p \neq q$, and if f is such that

$$M_\Psi [f(x + h), f(x - h)] \geqq f(x)$$

then if $\Psi f(x)$ is continuous it will be a monotone function.

Proof: Since $\Psi^{-1}[P\Psi f(x + h) + q\Psi f(x - h)] \leqq f(x)$ and since $\Psi(t)$ is monotone, we have $p\Psi f(x + h) + q\Psi f(x - h) \geqq \Psi f(x)$ Application of the previous theorem yields the desired result.

Literature Cited

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