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Abstract. The stated reference systems are mathematically defined using three dimensional Cartesian coordinates and sets of functions, and then systematically developed and contrasted. The linearity of the Lorentz transformation equations for special relativity is shown to follow directly from the definition of a Galilean inertial reference system.

While working on a clear and understandable explanation of the special theory of relativity through the usage of undergraduate mathematics one readily finds a major discontinuity. This discontinuity centers around the words reference frame, reference system, Galilean system, Newtonian system, and inertial systems. To bring this problem more clearly to the front a few examples from the many found in a general survey of available publications are presented.

BRIEF SURVEY OF CURRENT STATEMENTS

Tolman (1917) presents the idea that Newtonian mechanics exist in any system which has as axes the three Cartesian coordinates x, y, z. There may be any number of these systems, but the ones in which we normally consider Newtonian mechanics to hold are anchored in space in such a position that the fixed stars are permanently fixed. This is a representative definition of an inertial system—that being any system for which the fixed stars are fixed.

L. Bolton (1920) comments that in studying special relativity we must use only unaccelerated systems, so that F= ma and the other laws of Newton will hold for every system. He also associates the name Galileo with unaccelerated rectangular Cartesian coordinates (Galilean coordinates), with reference to which the laws of mechanics are usually stated.

D'Abro (1927) explains that there are reference systems in which an object at rest will remain at rest and these are called Galilean or inertial frames. Here we have the words Galilean and inertial used interchangeably when describing a reference frame.

[1]This work was done as part of an honors project in mathematics under the direction of Professor William Small, Department of Mathematics, Grinnell College, Grinnell, Iowa.

Peter Bergmann (1949) introduces the concept of an inertial system as a system with respect to which all mass points not subject to forces are unaccelerated.

Synge and Griffith (1959) describe a Galilean frame as a physically rigid body which is isotropic with respect to natural phenomena. Also it is stated that two Galilean frames of reference have a relative uniform velocity less than the speed of light.

In many books dealing with relativity and/or physics in general there is not even an attempt to define the reference systems which are being used, and the understanding of these reference systems is left purely to the intuition of the reader.

As can be seen from the cited references it is not definite whether reference systems are physical, mathematical, or a combination. Therefore, it is felt that a precise statement of reference systems and these other vague terms in a mathematical form will be of value in the understanding of the laws of nature. Throughout the development of these statements the present day major agreements concerning specific terms will be maintained.

Development of a Galilean Inertial Reference System

A frame of reference is an n-dimensional Cartesian coordinate system such that any real ordered n-tuple is a point.

A Newtonian frame of reference is a three-dimensional frame of reference. These dimensions are x, y, and z. Physically, it may be thought of as three rigidly fixed lines, perpendicular to one another, which consists of an infinite number of points on each line such that any point of space may be defined by describing its coordinate on each axis. An absolute Newtonian frame of reference is a unique Newtonian frame of reference. Physically speaking, an absolute Newtonian frame of reference is one which is at rest with respect to absolute space. The most famous absolute Newtonian frame of reference was one considered as anchored in ether by Faraday, Maxwell, and the other men who originally studied the transmission of light.

It should be noted that no mention of natural phenomena as connected with a reference system has been made. These natural phenomena will now be considered. Nature is the set (F) of all possible implicit functions of the four real variables x', y', z', and t'. Any natural phenomenon may be described by these functions.

Newtonian nature is a subset (F_n) of nature consisting of all differentiable implicit functions of the real variables x', y', z' which have continuous derivatives, and which express or are implied by the mathematical formulation of the laws of Newton. These functions may always be written in parametric form with t^* as the parameter. Here t^* is the absolute time for all nature as established by Newton. A common expression of physical nature is seen in the equation $F = ma$, where 'a' is the second derivative with respect to t^* of x', y', or z'. This constitutes the famous second law of Newton.

A Newtonian inertial reference system consists of the open unbounded set of points (N) of the Newtonian frame of reference, and the set of functions (F_n) defined over N. These are defined in such a way that the variables, x', y', z', are now identified with the coordinates x, y, and z. Physically we may think of bringing Newtonian nature into a Newtonian frame of reference and forming a Newtonian inertial reference system. There may be any number of these Newtonian inertial reference systems moving with uniform relative velocities. There are no restrictions placed on the values of these relative velocities.

A Galilean frame of reference is a four-dimensional frame of reference. The four dimensions are x, y, z, and t. There is no absolute Galilean frame of reference.

Galilean nature is a subset (F_g) of nature. It consists of all the implicit, differentiable, homogeneous functions of the real variables x', y', z', t', having continuous derivatives, whose second derivatives with respect to t' are zero, and whose first derivatives' absolute values with respect to t' are less than or equal to a positive constant c, and which express such laws of Newton as may be expressed, or are implied by the mathematical formulation of these laws. Note that the mathematical limitations placed on the functions (F_g) yield a physical situation in which there is no force (thus noting stops or starts) and all motion takes place on a straight line through the origin. Remember also that t' is merely a variable and there is no absolute time. Physically speaking c is the velocity of the propagation of light in a vacuum and no velocity in a vacuum may exceed this limit.

A Galilean inertial reference system consists of the open unbounded set of points (G) of the Galilean frame of reference and the functions (F_g) defined over G. These are defined in a manner such that the variables x', y', z', t' are now identified with the coordinates x, y, z, and t. Physically we may think of placing Galilean nature into a Galilean frame of reference and forming a Galilean inertial reference system. Extending our intuitive feeling for motion in three-dimensional space to motion in four-dimensional space-time, there may be any number of these Galilean inertial reference systems moving with uniform relative velocities. Since the movement of any system may be described in another system, and the above limitation on the first derivatives of the functions holds, it follows that the absolute value of the relative velocity cannot exceed c, the speed of light.

Lorentz Transformation Linearity

One of the clearest statements of the special theory of relativity is given by Albert Einstein and translated by Robert Lawson (1920). He explains that the laws of nature are such that mathematically they maintain the same form regardless of the inertial reference system they are considered in. The equations relating the variables of one system to those of another system are the Lorentz transformation equations.

The derivations of these transformation equations as demonstrated by Einstein, and many others since 1905, all originally require that the transformation equations be linear and proceed from there to derive them. It is this linearity premise which will now be dealt with.

Many authors—Synge and Griffith being a prime example—merely assume the transformation equations to be linear.

Talmey (1941) gives his reason for linearity by saying that x' is conditioned on x and t.

The majority of authors, such as Cunningham (1914), refer the reader to Einstein's original 1905 paper in which he says the equations are clearly linear on account of the homogeneity of space and time. To the common mind this is not as clear as Einstein has assumed it to be.

Along this same line the next comment historically was by A. W. Conway (1915), who explained that the linearity must exist so that the connections between infinitesimal increments of space and time will be independent of the coordinates. Since the origin of a system may be any point in space, he says the homogenity of space is the basis for this increment independency.

As might be expected, the first rigorous demonstration of why the Lorentz transformations must be linear came from Einstein himself in 1922. Here, using the principle of the constancy of the velocity of light and the special principle of relativity, he showed that in order to have the quadratic implicit function, which expresses a small increment in the propagation of light, remain invariant under the transformation between inertial reference systems, the Lorentz transformations must be linear.

Once a Galilean inertial reference system has been specifically defined, as was done earlier in this paper, the subject of linearity may be approachd from a different direction.

Given two Galilean inertial reference systems, S and S′, transformation equations must exist between the two systems such that functional relationships expressing a phenomenon are invariant upon their transformation from one system to the other. The only functions associated with system S are the functions F_g, and for systems S′ and F_g'. By definition these functions are among other things linear. Since the transformation equations exist in Galilean nature and the only functions existing here (F_g and F_g') are linear, the Lorentz transformation equations must be linear.

Literature Cited

D'Abro. 1927. The Evolution of Scientific Thought. Dover Publications, Inc., pp. 106-108.

Bergmann, Peter. 1949. Basic Theories of Physics. Prentice-Hall, Inc., pp. 5-8.

Bolton, L. 1921. An Introduction to the Theory of Relativity. E R. Dutton and Co., p. 40.

Conway, A. W. 1915. Edinburg Mathematical Tracts 3: Relativity.

Cunningham. 1914. The Principle of Relativity. Cambridge University Press, pp. 49-50.

Einstein, Albert. 1905. On the electrodynamics of moving bodies. Dover reprint and translation, 1923, p. 44.

Einstein, Albert. 1920. Relativity. The Special and the General Theory. Henry Holt and Co., translated by Robert Lawson, p. 15.

Einstein, Albert. 1922. The Meaning of Relativity. Princeton University Press, pp. 32-33.

Synge and Griffith. 1959. Principles of Mechanics. McGraw-Hill Company, pp. 509-512.

Talmey, Max. 1941. Relativity Theory Simplified. Darwin Press, p. 41.

Tolman, Richard. 1917. The Theory of the Relativity of Motion. Univ. of California Press, pp. 7-9.