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A Note on Dialectics in Mathematics

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Abstract. A complete dialectical process is defined; it is shown that such a process is a function, and that every real function is a complete dialectical process. Some general implications of this result are discussed.

We consider the science of modern dialectics to be the form of reasoning which analyzes phenomena of any kind by means of a variable triadic form, which form we shall call a dialectical triadic form, or briefly a triad. Such a dialectical triad consists of a thesis, a unique corresponding antithesis, and a resulting synthesis. The elements of a given triadic form are undefined concepts, but intuitively they may be considered as exhibiting the following relationship: the thesis and the antithesis oppose each other, and this opposition is resolved in the synthesis. Any three elements related in this way form a triadic form, or a triad; and conversely, in any triad, the component elements are thus related.

A simple example of informal modern dialectics is found in an ordinary conversation between two persons; here, any statement by one person may be considered to be a thesis, and the succeeding statement by the other person to be the corresponding antithesis. The variability of this form results from the fact that it is a function of time. In such an example, the opposition of the two statements is implied by the distinct, empirically-opposed identities of the participants. The synthesis consists of the reception in some way of the two statements, by either participant. It should be noted that each participant may receive the two statements in a different way, thereby showing that the synthesis achieved may not be unique, but instead is subjective, and a function of the receiver. The validity of this example as an example of modern dialectics is shown in the section below.

We may call such a verbal interchange as discussed above "verbal dialectics"; we do not demand that it be formalized beyond the level of capability of the participants in the conversation. In the example given, and in fact in general, it is a matter of indifference as to the content or meaning of the conversation.

The science of modern dialectics has developed in many areas and many ways since some of its principles were first propounded by the philosopher Hegel (translation 1874) in the early part of the nineteenth century. If we attempt to state in detail what place was

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assigned to mathematics by Hegel in his general dialectical theory of logic, this would take us too far afield. However, we may say concisely that Hegel considered that mathematics deals in general with quantity; and by means of geometry, mathematics deals with abstract representations of physical space. We also note that Hegel considered that some concepts, such as freedom, cannot be expressed by a mathematical formula. This point of view resulted in his characterization of mathematics as being logically equivalent to a materialism.

The concept of *Relation* is basic to the Hegelian theory of dialectic. We shall use the mathematical meaning of relation in its particular meaning of *function* to investigate briefly the meaning of modern dialectics in mathematics.

Dialectical Processes and Functions

The fundamental relationship between modern dialectics and mathematics was noted by Frederick Engels (translation, 1954 edition) in the latter part of the nineteenth century. Engels considered Rene Descartes' idea of a *variable* as the means of expression of *motion* in mathematics. Since Engels also considered that motion (in its broadest sense) and dialectics are logically equivalent, this implied that the introduction of the concept of a variable brought dialectical reasoning into mathematics. However, Engels restricted the meaning of a variable in mathematics to a quantitative variable, and therefore did not employ the mathematical view in its most general sense. On the other hand, we note that wherever Engels observed motion, he observed variability, so that his observations on motion may be put into terms involving variables, by substituting "variability" for "motion" throughout his writings in dialectics.

By employing some concepts from elementary set theory, we may generalize this idea of Engels, and at the same time make it more precise.

Definition I: A *complete dialectical process* is a non-empty set of dialectical triads; that is, it is a set of theses, a set of corresponding antitheses, and the set of resulting syntheses.
We note that such processes provide the formal basis of modern dialectics.

Theorem I: A complete dialectical process is a function.

Proof: Let X be a non-empty set of elements x, each of which denotes a given thesis. x is called the *thetic variable*. Then for each thesis x there is a corresponding unique antithesis which we may denote by y, and set $y = f(x)$, by which we mean that y is the antithesis that corresponds to the thesis x; and we may let Y denote the set of those corresponding antitheses. y is called the *antithetic*

variable. Then each dialectical triad of the process may be denoted by the symbol (x,y), x ϵ X, y ϵ Y, y = f(x), which symbol (x,y) is called the *synthetic variable.* The set of triads (x,y) is a set of ordered pairs of elements, no two distinct pairs of which have the same first element, and is therefore a function. This proves Theorem I.

Note that each synthetic element (x,y) is of a more complex structure than either x or y, and that it exhibits a union of x and y. This represents abstractly the resolution of the opposition of these two elements, on a more complex level than the original elements.

This shows that a complete dialectical process is essentially a mathematical concept, that of a function. This function need not be numerical; but when it is discovered in a phenomenon, if it can be expressed numerically as in many cases it has been in mechanics, it then becomes amenable to analysis.

We now demonstrate the converse of Theorem I, with regard to real functions.

Theorem II: Every real function is a complete dialectical process.

Proof: Given the real function y = (f(x). Let X be the domain, with independent variable x; and let Y be the range with dependent variable y. Then the set of ordered pairs (x,y), x ϵ X, y ϵ Y, y = f(x), is the function. In each ordered pair of this function, the characteristic thesis represented by x is its *independent* nature, or position as the first (thetic) element in the pair; and the y- characteristic which is the corresponding antithesis to a given x is its *dependent* nature, or the position of y as the second (antithetic) element in the pair. The unity of the pair in the function (synthetic) element (x,y) is the synthesis of the two opposing elements. Thus the set of ordered pairs (x,y) is a non-empty set of dialectical triads and is therefore a complete dialectical process. This proves Theorem II.

From the proof of Theorem II we may infer that the nature of the opposition between a thesis and an antithesis need not appear in their intrinsic meaning, such as in their numerical values; but it may and does often appear in their order of occurrence or of appearance or of position. This illustrates a statement of Engels that the opposition between the variables appears only in their relation and not in each element separately. This also shows the validity of our example in verbal dialectics.

If the given function of Theorem II is a one-to-one correspondence, then it is clear that either of the two variables x and y may be designated as the thetic variable, and then the remaining variable is

the antithetic variable. In either case, the same synthetic pair is obtained as far as the opposing nature of the positions of x and y and their unity in the synthesis (x,y) are concerned. The example given by Engels in which he used the elements of heredity and adaptation as defined in the theory of evolution of organic life is a special case of this idea.

On the other hand the fact that there exists such a real function (complete dialectical process) as $y = x^2$, for any real number, shows that not every dialectical process is one-to-one. For given the two theses $x = 1$, and $x = -1$, the same antithesis $y = 1$ corresponds to both.

If the component variables x,y of a given real function $y = f(x)$ are functions of a real parameter t, we may consider t as time, and infer that the function itself (x,y) is a function of the time. From this it follows that any such function is a set of ordered pairs of elements $(t, (x(t), y(t)))$ and is a complete dialectical process for which, generally speaking, the thesis is time, and the corresponding antithesis is the synthesis of the original function pair (x,y). Then the resulting synthesis $(t, (x(t), y(t)))$ represents the resolution of the dialectical conflict between time and any time-dependent process.

CONCLUSION

From the foregoing discussion is follows that whatever phenomenon can be analyzed by modern dialectics (i.e., stated as a dialectical process) can also be analyzed mathematically by expressing the process in function form. Conversely, whatever can be analyzed mathematically by using the function concept can also be analyzed by modern dialectics by expressing the function as a dialectical process. Because of this formal equivalence, we see that the modern dialectical approach to any problem is limited in the same way that the function approach to the same problem is limited, and conversely.

Other applications of our theory provide further examples. Thus we may say that the Hegelian search in phenomena for Being, Essence, and the Notion, appears to be the search for the functions and related variables which characterize the stated phenomena. In this application, the Hegelian Notion is seen to be the function itself.

As another example, we may conclude that any society which is built upon the modern dialectical philosophy is therefore built implicitly upon a mathematical and thus scientifically sound philosophy; and that such a philosophical basis ensures the development of a scientifically sound educational system.

Since the modern dialectical philosophy is strictly objective and impersonal, as is a mathematical philosophy, it may not be a satisfactory philosophy upon which to build a society. But such an inference, as to whether or not this philosophy is satisfactory as a basis for a society, cannot be made within the system of modern dialectics itself, unless it be based on a statistical, and therefore dialectical, analysis of the results of the society. This lack within the system may indicate a place for the metaphysical approach; namely, to criticize and evaluate the dialectical philosophy.

If, with Hegel, we identify the *mathematical approach* with a *materialism,* then the phrase *dialectical materialism* is redundant in the sense that modern dialectics and materialism (the mathematical approach) are logically equivalent.

Literature Cited

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