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# A Theoretical Study of a Randomly Excited Mechanical Oscillator

VICTOR W. BOLIE and FRANCIS M. LONG<sup>1</sup>

*Abstract.* A mathematical investigation is made of the feasibility of extracting usable mechano-electric power from a critically damped and randomly excited mechanical oscillator. Two different analytic functions are used to approximate practical squared-velocity spectra for the motion of the vehicle supporting the oscillator, and the corresponding expressions for extractable power are developed. Numerically computed results are presented graphically to illustrate the effects of varying the oscillator parameters.

Recent developments in miniaturized radio equipment have made possible the telemetering of physiological data from active subjects (Beenken and Dunn, 1958; Mackay, 1959). However, the present requirements of battery replacement pose a serious problem in chronic surgery if the transmitter is located between the body wall and the digestive tract. A potential solution to this problem is the extraction of energy from random motions of the subject. The purpose of this discussion is to consider the response of a viscously-damped mechanical oscillator to the random motion of its carrier, and to develop a mathematical relationship between the power delivered to the damper and the statistical parameters of the carrier motion. Two restrictions to be used are that the mechanical oscillator is critically damped and the carrier motion is one-dimensional.

## OSCILLATOR DYNAMICS

A sketch of a mechanical oscillator is shown in Figure 1. The frame  $F$  is restricted to one-dimensional motion with  $x$  representing

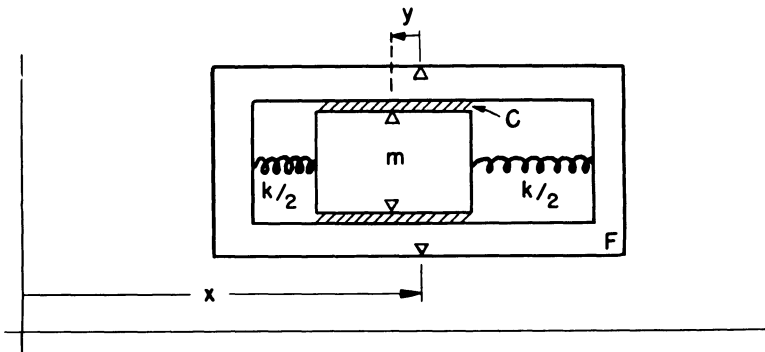


Figure 1. Mechanical oscillator.

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the distance between its center and a stationary reference. The standard symbols  $m$ ,  $c$ , and  $k$  are used to denote the coefficients of mass, viscous damping, and spring restoring force, respectively. The position of the center of the mass  $m$  with respect to the center of the frame  $F$  is indicated by the distance  $y$ , which is considered positive if in the direction opposite that of  $x$ .

The displacement  $y$  of the mass  $m$  from the center of the frame  $F$  is described by the differential equation

$$m \frac{d^2y}{dt^2} + c \frac{dy}{dt} + ky = m \frac{d^2x}{dt^2} .$$

For the condition of critical damping ( $c^2 = 4 km$ ), the transfer function relating the response  $y$  to the excitation  $x$  (also the velocity  $v_y$  to the velocity  $v_x$ ) using operational notation is seen to be  $T(s) =$

$$\frac{1}{\left[ s + \sqrt{\frac{k}{m}} \right]^2} .$$

#### CARRIER MOTION

For purposes of this discussion the random movements of the carrier are most conveniently described by the square of the velocity spectrum  $V^2(\omega)$ , sometimes referred to as the power spectrum of the velocity  $G_v(\omega)$ . According to the Wiener-Khinchine Theorem, the velocity autocorrelation function  $\Phi(\tau)$  is the inverse Fourier transform of the squared velocity spectrum  $V^2(\omega)$ , i.e.,

$$\Phi(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} V^2(\omega) \cos \omega\tau \, d\omega .$$

Two velocity autocorrelation functions often used to describe random motions are given below with their respective squared velocity spectra found by taking the Fourier transform of  $\Phi(\tau)$ :

$$\Phi_1(\tau) = \frac{\sigma^2}{1 + (\omega_1 \tau)^2} \qquad V_1^2(\omega) = \frac{\pi \sigma^2}{\omega_1} \left| \frac{\omega}{\omega_1} \right|$$

$$\Phi_2(\tau) = \sigma^2 e^{-\omega_1 |\tau|} \qquad V_2^2(\omega) = \frac{2 \sigma^2 \omega_1}{\omega_1^2 + \omega^2} ,$$

where  $\sigma$  and  $\omega$  are the rms velocity and particular angular frequency respectively while  $\left( \frac{1}{\omega_1} \right)$  is the correlation time constant and  $\tau$  is the

time delay. These characteristics are shown in Figure 2 and Figure 3.

EXTRACTABLE POWER

When the displacement of the oscillation is random, with  $V^2(\omega)$  defining the squared velocity spectrum and  $|T(j\omega)|$  the absolute value of  $T(s)$  for  $s = j\omega$ , the average power  $P$  dissipated in the damper may be found through application of the formula (James, *et al.*, 1947)

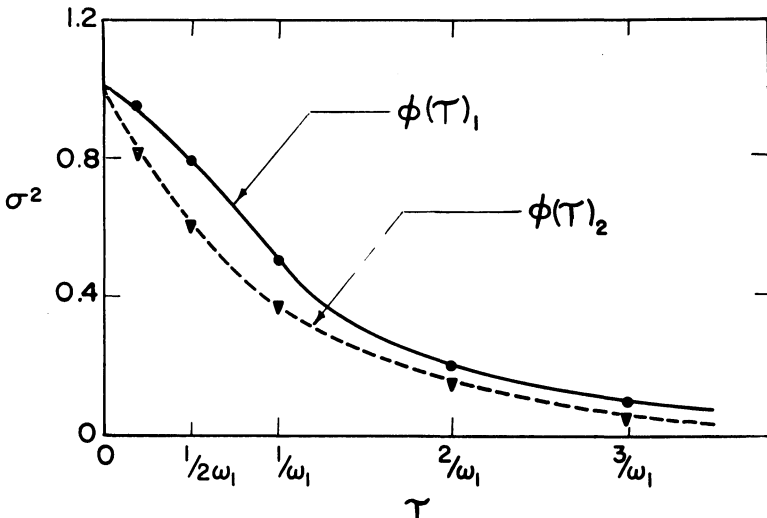


Figure 2. Autocorrelation function.

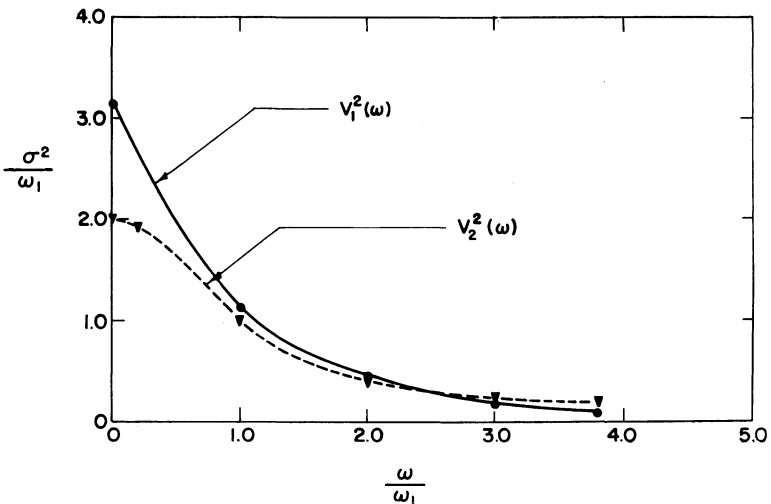


Figure 3. Squared velocity spectrum.

$$P = \frac{c}{2\pi} \int_{-\infty}^{\infty} |T(j\omega)|^2 V^2(\omega) d\omega.$$

Substituting the previously defined functions gives, for  $c^2 = 4 \text{ km}$ ,

$$P_1 = \frac{\sqrt{\text{km}}}{\pi} \int_{-\infty}^{\infty} \frac{\pi \sigma^2}{\omega_1} \left[ \frac{\omega^2}{\left(\frac{k}{m}\right) + \omega^2} \right]^2 e^{-\left(\frac{\omega}{\omega_1}\right)} d\omega$$

and

$$P_2 = \frac{\sqrt{\text{km}}}{\pi} \int_{-\infty}^{\infty} \frac{\omega^4 d\omega}{2 \sigma^2 \omega_1 \left[ \left(\frac{k}{m}\right) + \omega^2 \right]^2 \left[ \omega_1^2 + \omega^2 \right]}.$$

The variation of  $|T(j\omega)|^2$  with  $\omega$  is shown in Figure 4 and a plot of the above two integrands is given in Figure 5 for  $\left(\frac{k}{m}\right) = \omega_0^2$ .

In order to arrive at numerical quantities for comparison purposes, let the following parameters be fixed at the given values:

$$\sigma = 0.305 \text{ meter/sec.}$$

$$\omega_1 = 2\pi \text{ rad/sec.}$$

$$m = 0.1 \text{ kilogram}$$

$$c = 2\sqrt{\text{km}} \text{ critical damping.}$$

Then  $\omega_0$  becomes the parameter available for maximizing the power.

The first integral becomes

$$P_1 = 2 \sigma^2 m \omega_1 \left(\frac{\omega_0}{\omega_1}\right) \int_{-\infty}^{\infty} \left[ \frac{\left(\frac{\omega}{\omega_1}\right)^2}{\left(\frac{\omega_0}{\omega_1}\right)^2 + \left(\frac{\omega}{\omega_1}\right)^2} \right]^2 e^{-\left(\frac{\omega}{\omega_1}\right)} d\left(\frac{\omega}{\omega_1}\right).$$

Evaluation of this expression by graphical means for values of  $\left(\frac{\omega_0}{\omega_1}\right)$  from 0.7 to 1.0 shows a broad maximum in the neighborhood of  $\left(\frac{\omega_0}{\omega_1}\right) = 0.900$  (see Figure 6). The second integral may be evaluated directly by the method of residues and the the result is

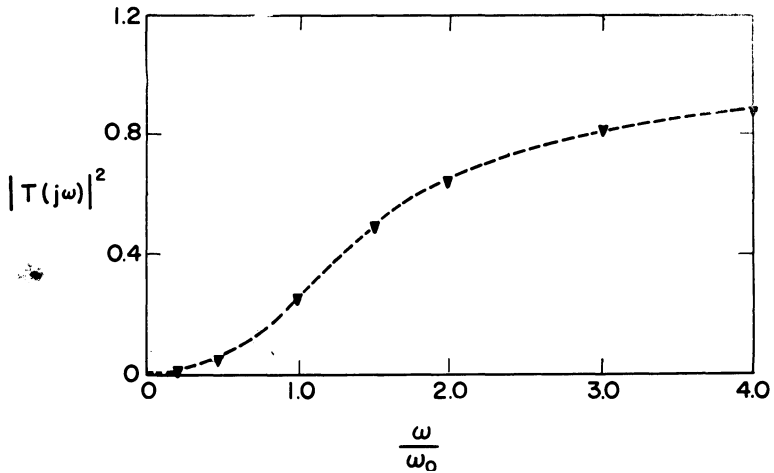


Figure 4. Transfer function.

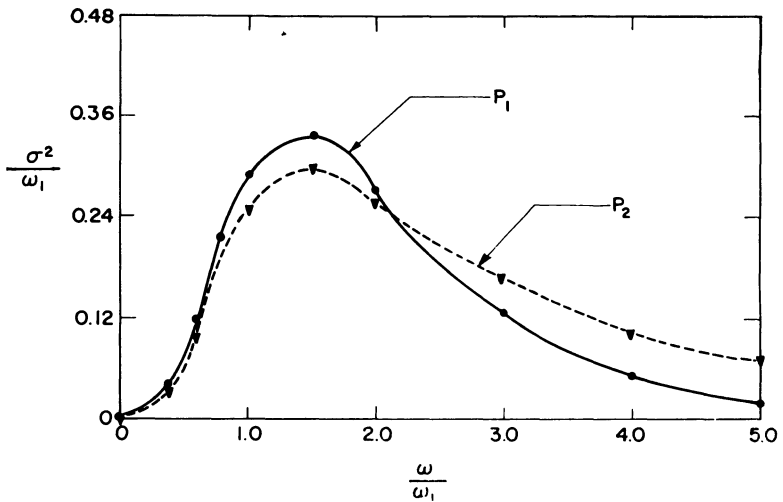


Figure 5. Integrand of integral for power.

$$P_2 = \sigma^2 m \omega_1 \frac{2 \left( \frac{\omega_0}{\omega_1} \right) + \left( \frac{\omega_0}{\omega_1} \right)^2}{1 + 2 \left( \frac{\omega_0}{\omega_1} \right) + \left( \frac{\omega_0}{\omega_1} \right)^2}.$$

This result increases monotonically as  $\left( \frac{\omega_0}{\omega_1} \right)$  increases and ap-

proaches the limiting value of  $\sigma^2 m \omega_1$  for large values of  $\left(\frac{\omega_0}{\omega_1}\right)$ .

Thus, making  $\omega_0$  larger leads to diminishing returns for  $\left(\frac{\omega_0}{\omega_1}\right)$

larger than about 1.4. Since the damping constant  $c$  is directly proportional to  $\omega_0$ , this indicates that there is some latitude in the choice of the related constants of the physical system ( $k, m$ ) for which a nearly optimum power output can be realized. This information is shown in Figure 7. In particular, for  $\omega_0 = \omega_1$ ,  $P_1 = 27.9 \times 10^{-3}$  watts and  $P_2 = 43.8 \times 10^{-3}$  watts.

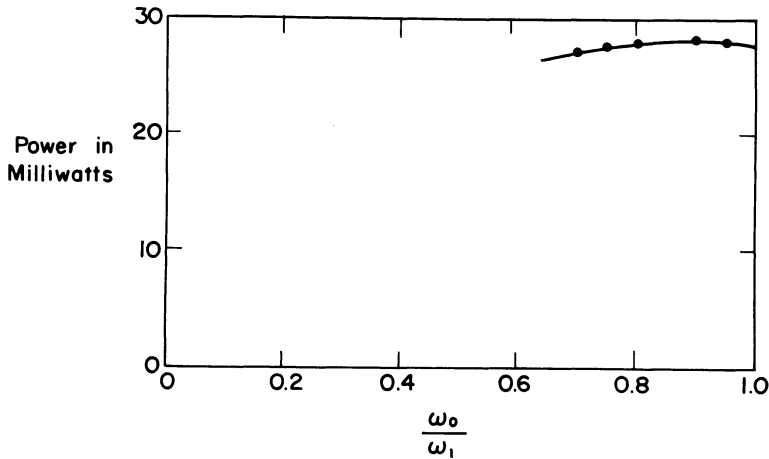


Figure 6. Extractable power.

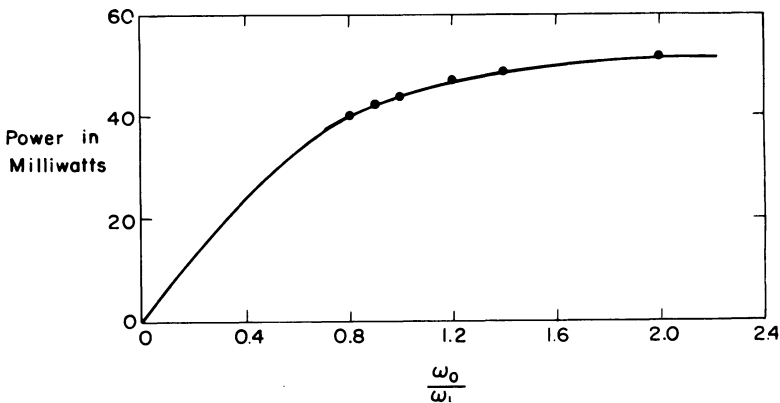


Figure 7. Extractable power.

## SUMMARY

A simple mechanical oscillator making use of the forces of acceleration of its carrier to convert the energy of its random motion to a more readily usable form has been proposed as a potential solution of the battery replacement problem. A study of this system for two different squared velocity spectra functions indicates that for critical damping, the average power output from this system for a random input will be nearly maximum for  $k/m$  approximately equal to  $\omega_1$ , and that this power is sufficient to operate a miniaturized transistor oscillator.

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