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Error Terms of Numerical Integration Formulas

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Abstract. This paper gives a proof that the polynomial function $Q_n(s) = \int_0^s t(t-1)(t-2)\cdots(t-n)dt$ does not change sign on the interval (O,n). Heretofore it was generally believed that $Q_n(s)$ changed sign on the interval (O,n) when n was an odd integer. The technique of proof is to show that when n is odd, $Q_n(s)$ has an upper bound $Q_n(n-1)$ for $\frac{n-1}{2} \leqq s \leqq n$ which is shown to be negative. This result simplifies the treatment of the error terms in certain numerical integration formulas which involve divided differences. The simplified treatment is given here.

A PRELIMINARY LEMMA AND THEOREM

In certain types of numerical integration formulas which are developed later, the error terms are of the form

$$E_n = \int_0^i F[0, 1, \cdots n, t]\, \pi_n(t)dt \tag{1}$$

where $\pi_n(t) = t(t-1)(t-2)\cdots(t-n)$ and $F[0, 1, \cdots n, t]$ is a divided difference of order $n+1$. In order to put this error term in a more tractable form it is necessary to apply the mean value theorem to the integral. This requires that the function

$$Q_n(S) = \int_0^s \pi_n(t)dt \tag{2}$$

not change sign on the interval $(0, n)$. We begin by proving the following lemma, the proof of which, in part, is due to Steffenson.

Lemma: The integral I_v defined by

$$I_v = \int_v^{v+1} \pi_n(t)dt \qquad v = 0, 1, \cdots, n-1 \tag{3}$$

where $\pi_n(t) = t(t-1)(t-2)\cdots(t-n)$ possesses the following properties:

(i) $|I_{v-1}| > |I_v|$ for n even and $v = 1, 2, \cdots \frac{n}{2}-1$

$|I_{v-1}| = |I_v|$ for n even and $v = \frac{n}{2}$

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$|I_{v-1}| < |I_v|$ for n even and $v = \frac{n}{2}+1, \frac{n}{2}+2, \cdots, n-1$

(ii) $|I_{v-1}| > |I_v|$ for n odd and $v = 1, 2, \cdots, \frac{n-1}{2}$

$|I_{v-1}| < |I_v|$ for n odd and $v = \frac{n+1}{2}, \frac{n+3}{2}, \cdots, n-1$

(iii) $I_0, I_2, I_4, \cdots, I_{n-2}$ are positive when n is even
$I_1, I_3, I_5, \cdots, I_{n-1}$ are negative when n is even

(iv) $I_0, I_2, I_4, \cdots, I_{n-1}$ are negative when n is odd
$I_1, I_3, I_5, \cdots, I_{n-2}$ are positive when n is odd

(v) $I_{n-1-v} = (-1)^{n-1} I_v$ for all n and $v = 0, 1, \cdots, n-1$.

Proof: It is helpful to understand the technique of proof to study figures 1, 2, and 3. From the definition of the integral I_v we can write

$$I_{v-1} = \int_{v-1}^{v} \pi_n(t)dt = \int_{v-1}^{v} t(t-1 \cdots (t-n)dt.$$

Let us put $t = s-1$ in the above integral to obtain

$$I_{v-1} = \int_{v}^{v+1} \frac{s-n-1}{s} \pi_n(s)ds.$$

Since $\pi_n(s)$ does not change signs on the interval $(v, v+1)$, we can apply the mean value theorem to this last integral and obtain

$$I_{v-1} = \frac{\alpha-n-1}{\alpha} \int_{v}^{v+1} \pi_n(s)ds = \frac{\alpha-n-1}{\alpha} I_v \text{ for } v < \alpha < v+1.$$

Therefore $|I_{v-1}| = |\frac{\alpha-n-1}{\alpha}| \cdot |I_v|$. For n even and $v = 1, 2, \cdots, \frac{n}{2}-1$ we see that $|\frac{\alpha-n-1}{\alpha}| > 1$ so that $|I_{v-1}| > |I_v|$ for $v = 1, 2, \cdots, \frac{n}{2}-1$. For n even and $v = \frac{n}{2}+1, \frac{n}{2}+2, \cdots, n-1$ we see that $|\frac{\alpha-n-1}{\alpha}| < 1$ so that $|I_{v-1}| < |I_v|$ for $v = \frac{n}{2}+1, \frac{n}{2}+2, \cdots, n-1$. This proves the first and third inequalities of property (i).

Now consider $I_{n-1-v} = \int_{n-1-v}^{n-v} \pi_n(t)dt$. If we make the subsituation $t = n-s$ in this integral, we obtain

$$I_{n-1-v} = \int_{v+1}^{v} \pi_n(n-s) \cdot (-ds) = (-1)^{n+2} \int_{v+1}^{v} \pi_n(s)ds =$$

$$(-1)^{n+1}\int_{v}^{v+1}\pi_n(s)ds.$$

Therefore we have proved property (v) that $I_{n-1-v} = (-1)^{n+1}I_v$. From this result we see that when n is even and $v = \frac{n}{2}$, we get $I_{\frac{n}{2}-1} = -I_{\frac{n}{2}}$ so that $|I_{\frac{n}{2}-1}| = |I_{\frac{n}{2}}|$ which is the second part of property (i).

When n is odd, $\left|\frac{\alpha-n-1}{\alpha}\right| > 1$ for $v = 1, 2, \cdots, \frac{n-1}{2}$ so that $|I_{v-1}| > |I_v|$ for these values of v. Also when n is odd, $\left|\frac{\alpha-n-1}{\alpha}\right| < 1$ for $v = \frac{n+1}{2}, \frac{n+3}{2}, \cdots, n-1$ so that $|I_{v-1}| < |I_v|$ for these values of v. This completes the proof of property (ii).

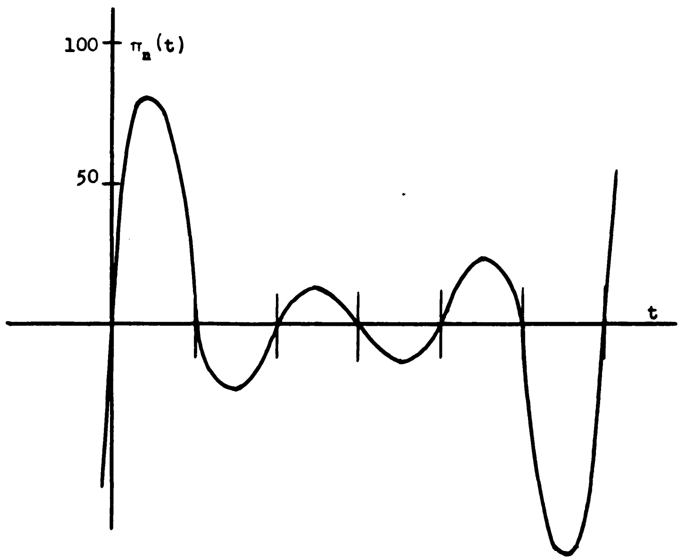

Figure 1. Typical graph of the equation $\pi_n(t) = t(t-1)\cdots(t-n)$ for n even showing symmetry about the point $t = \frac{n}{2}$. The graph shown is for $n = 6$.

Consider the function $\pi_n(t) = t(t-1)(t-2)\cdots(t-n)$ for n even. For t in the interval (0, 1), $\pi_n(t) \geqq 0$ because it is the product of an even number, n, of negative factors. Hence $I_0 = \int_0^1 \pi_n(t)\,dt > 0$ as it is the integral of a function which is zero or positive throughout the interval of integration. If t is in the interval (1, 2) we see that $\pi_n(t)$ is a product of n—1 negative factors so that $\pi_n(t) \leqq 0$. Therefore $I_1 = \int_1^2 \pi_n(t)dt < 0$. By continuing this argument, we can conclude that I_o, I_2 I_4, • • •, I_{n-2} are positive and I_1, I_3, I_5, • • •, I_{n-1} are negative when n is even. This is property (iii). Property (iv) follows by an argument similar to that used to establish property (iii) after making the observation that for n odd, the parity of the products is reversed. This completes the proof of the lemma.

With the aid of the lemma we can prove the following:

Theorem: The function $Q_n(s) = \int_o^s \pi_n(t)dt$

$$= \int_o^s t(t-1)\ (t-2) \cdots (t-n)dt$$

does not change sign for $o \leqq s \leqq n$.

Proof: From the definition of $Q_n(s)$ we see that $Q_n(n-s) = \int_o^{n-s} \pi_n(t)dt$ and if we let $t = n-u$, and $dt = -du$ we obtain

$$Q_n(n-s) = -\int_n^s \pi_n(n-u)du = -(-1)^{n-1}\int_n^s \pi_n(u)du$$

$$= (-1)^n \int_n^s \pi_n(u)du = (-1)^n \left[\int_o^s \pi_n(u)du - \int_o^n \pi_n(u)du\right]$$

so that

$$Q_n(n-s) = (-1)^n[Q_n(s) - Q_n(n)]. \tag{4}$$

If $s = \frac{n}{2}$ in this equation we see that $[1-(-1)^n]Q_n(\frac{n}{2}) = (-1)^{n+1}\ Q_n(n)$ and if n is even this implies $Q_n(n) = 0$ and if n is odd $Q_n(n) = 2Q_n\ (\frac{n}{2})$. Further, we see that

$$Q_n(n-s) = Q_n(s) \text{ when n is even} \tag{5}$$

$$Q_n(n-s) = Q_n(n) - Q_n(s) \text{ when n is odd.}$$

Since $\frac{d}{ds}\ Q_n(s) = \pi_n(s)$ is equal to zero for $s = 0, 1, \cdots, n$ we see that the integral values of s on the interval (0, n) are critical values of the function $Q_n(s)$. By considering the second derivative $\frac{d^2Q_n(s)}{ds^2} = \frac{d}{ds}\ \pi_n(s) = \sum_{i=o}^{n} \frac{\pi_n(s)}{s-i}$ we can easily show that all of these critical values are relative maxima or relative minima.

We now consider two cases.

Case I: n even. In this case $Q_n(s) = Q_n(n-s)$ so that it is sufficient to show that $Q_n(s)$ not change sign for $0 \leqq s \leqq \frac{n}{2}$, for if $s \geqq \frac{n}{2}$, then $n-s \leqq \frac{n}{2}$. To show this we define $v = [s]$, the largest integer contained in s, and observe that $Q_n(v) = I_o + I_1 + \cdots + I_{v-1}$. Therefore if s is such that $v = 0$, then

$$0 \leqq Q_n(s) < I_o,$$

and if s is such that $v = 2, 4, 6, \cdots, n-2$, then

$$I_o + I_1 + \cdots I_{v-1} \leqq Q_n(s) < I_o + I_1 + \cdots + I_v,$$

and when s is such that $v = 1, 3, 5, \cdots, n-1$,

$$I_o + I_1 + \cdots I_{v-1} \geqq Q_n(s) > I_o + I_1 + \cdots + I_v.$$

These inequalities are obvious from property (iii) of the lemma. Further we see that if $s = n$, then $v = n$, and $Q_n(s) = 0$.

From these inequalities, we see that when v is even and $0 \leqq s \leqq \frac{n}{2}$, then

$$Q_n(s) \geqq [I_o+I_1] + [I_2+I_3] + \cdots + [I_{v-2}+I_{v-1}].$$

Each term in the brackets is positive because $I_o, I_2 \cdots I_{v-2}$ are positive and $|I_{j-1}| > |I_j|$ for $j = 1, 2, \cdots \frac{n}{2}-1$ by the lemma.

If v is odd,

$$Q_n(s) > [I_o+I_1] + [I_2+I_3] + \cdots + [I_{v-1}+I_v],$$

and each term in the brackets is positive with the possible exception of the last one which is equal to zero if $v = \frac{n}{2}$ and also odd. Thus, we have shown that when n is even $Q_n(s) \geqq 0$ for $0 \leqq s \leqq n$.

Case II: n odd. In this case, we define v as in case I. If s is such that $v = 0$, then

$$I_o < Q_n(s) \leqq 0,$$

if s is such that $v = 2, 4, 6, \cdots, n-1$ then

$$I_o + I_1 + \cdots + I_v < Q_n(s) \leqq I_o + I_1 + \cdots + I_{v-1},$$

if s is such that $v = 1, 3, 5, \cdots, n-2$ then

$$I_o + I_1 + \cdots + I_v > Q_n(s) \geqq I_o + I_1 + \cdots + I_{v-1},$$

and finally when $s = n$

$$Q_n(n) = I_o + I_1 + \cdots + I_{n-1}.$$

These inequalities follow readily from property (iv) of the lemma. Now suppose $0 \leqq s < \frac{n+1}{2}$. We have, from the inequalities above

$$Q_n(s) \leqq 0 \text{ when } v = 0,$$

$$Q_n(s) \leqq [I_o+I_1] + [I_2+I_3] + \cdots + [I_{v-2}+I_{v-1}]$$

when v is even, and (6)

$$Q_n(s) < [I_o+I_1] + [I_2+I_3] + \cdots + [I_{v-1}+I_v]$$

when v is odd.

Each term in the brackets of these two expressions is negative because I_o, I_2, etc. are negative when n is odd and $|I_{j-1}| > |I_j|$ for $j = 1, 2, \cdots, \frac{n-1}{2}$. Thus $Q_n(s)$ is negative or zero for $0 \leqq s < \frac{n+1}{2}$.

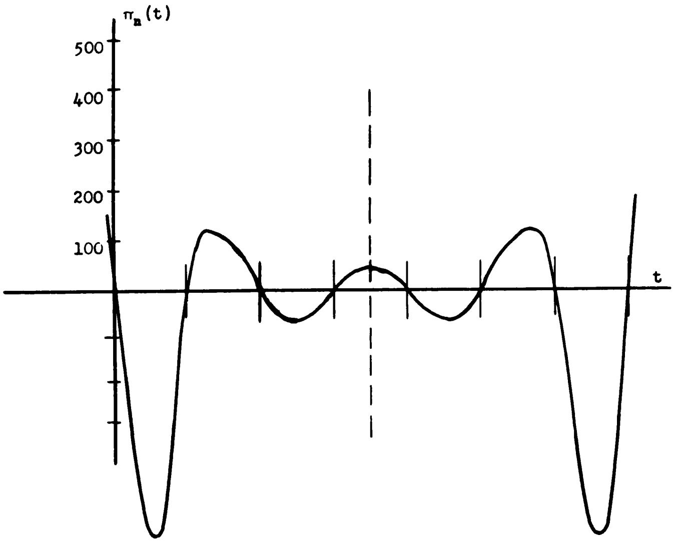

Figure 2. Typical graph of the equation $\pi_n(t) = t(t-1) \cdots (t-n)$ for n odd and $\frac{n-1}{2}$ odd showing symmetry about the vertical line $t = \frac{n}{2}$. The graph shown is for n = 7.

When $\frac{n+1}{2} \leqq s < n$ we write

$$Q_n(s) \leqq I_o + I_1 + \cdots + I_{\frac{n-3}{2}} + I_{\frac{n-1}{2}} + \cdots + I_{v-1} \qquad (7)$$

when $v = 2, 4, 6, \cdots, n-1$ or

$$Q_n(s) < I_o + I_1 + \cdots + I_{\frac{n-3}{2}} + I_{\frac{n-1}{2}} + \cdots + I_{v-1} + I_v$$

when $v = 1, 3, 5, \cdots, n-2$.

If now $\frac{n-1}{2}$ is odd we group the terms in inequalities (7) as follows:

$$Q_n(s) \leqq Q_n(\tfrac{n-1}{2}) + [I_{\frac{n-1}{2}}] + [I_{\frac{n+1}{2}} + I_{\frac{n+3}{2}}] + \cdots + [I_{v-2} + I_{v-1}] \quad (8)$$

when $v = 2, 4, \cdots, n-1$ or

$$Q_n(s) < Q_n(\tfrac{n-1}{2}) + [I_{\frac{n-1}{2}}] + [I_{\frac{n+1}{2}} + I_{\frac{n+3}{2}}] + \cdots + [I_{v-1} + I_v]$$

when $v = 1, 3, \cdots, n-2$.

Each term in the brackets on the right hand side of these inequalities is positive by the lemma. The right hand side of the first of these expressions attains its maximum value for $v = n-1$ and the right hand side of the second of these inequalities attains its maximum value for $v = n-2$ for each has the maximum number of positive terms. In either case, this maximum value is $Q_n(n-1)$.

For $\frac{n-1}{2}$ even we group the terms in inequalities (7) as follows:

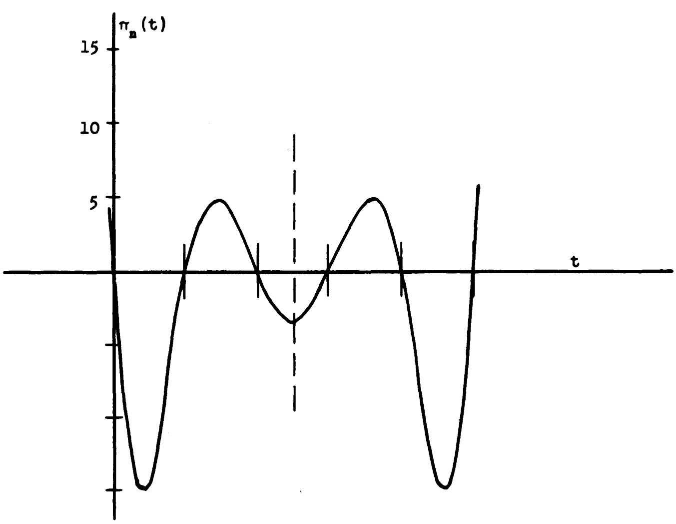

Figure 3. Typical graph of the equation $\pi_n(t) = t(t-1) \cdots (t-n)$ for n odd and $\frac{n-1}{2}$ even. The graph shown is for $n = 5$.

$$Q_n(s) \leqq Q_n(\tfrac{n-1}{2}) + [I_{\frac{n-1}{2}} + I_{\frac{n+1}{2}}] + [I_{\frac{n+3}{2}} + I_{\frac{n+5}{2}}] + \cdots + [I_{v-2} + I_{v-1}]$$

for $v = 2, 4, \cdots, n-1$,

$$Q_n(s) < Q_n(\tfrac{n-1}{2}) + [I_{\frac{n-1}{2}} + I_{\frac{n+1}{2}}] + [I_{\frac{n+3}{2}} + I_{\frac{n+5}{2}}] + \cdots + I[_{v-1} + I_v]$$

for $v = 1, 3, \cdots, n-2$.

Again each term in the brackets is positive by the lemma and again each expression attains the maximum value for the largest permissible value of v. In either case this maximum value is $Q_n(n-1)$.

We have just shown that $Q_n(s) \leqq Q_n(n-1)$ for $\frac{n+1}{2} \leqq s < n$ when n is odd.

$$\text{Now} \quad Q_n(n-1) = \int_0^{n-1} \pi_{n-1}(t) \cdot (t-n)dt$$

$$= \Big[(t-n)Q_{n-1}(t) \Big]_0^{n-1} - \int_0^{n-1} Q_{n-1}(t)dt$$

$$= 0 - \int_0^{n-1} Q_{n-1}(t)dt < 0,$$

since, by case I, $Q_{n-1}(t) \geqq 0$ when $n-1$ is even.

Now since $Q_n(n-1)$ is an upper bound for $Q_n(s)$ when $\frac{n+1}{2} \leqq s < n$, we see that $Q_n(s)$ is negative for $\frac{n+1}{2} \leqq s < n$.

When $s = n$, we write

$$Q_n(n) = Q_n(n-1) + I_{n-1}$$

and we observe that $Q_n(n)$ is negative and less than $Q_n(n-1)$ since $Q_n(n-1)$ was just shown to be negative and I_{n-1} is negative by property (iv) of the lemma.

These results prove that $Q_n(s) \leqq 0$ for $0 \leqq s \leqq n$ when n is odd and that $Q_n(s) \geqq 0$ when n is even which proves the theorem.

A Brief Discussion of the Derivation of Numerical Integration Formulas

This derivation is given in part by Hildebrand in his excellent text on numerical analysis and is given here for completeness.

Suppose $f(x)$ is of class $C^{(n+1)}$ on the closed interval $[a, b]$. Let $x_0, x_1, \cdots, x_n$ be a set of $n+1$ points equally spaced at interval h on $[a, b]$ and such that $x_0=a$ and $x_n=b$. By the change of variable $x=x_0+hs$, the function $f(x)=f(x_0+hs)$ becomes the function $F(s)$ defined for s in the interval $[0, n]$ and the points $x_0, x_1, \cdots, x_n$ become the points $0, 1, \cdots, n$ respectively. We define divided differences of orders $0, 1, 2, \cdots, k$ recursively by the relations

$$
\begin{aligned}
F[0] &= F(0), \\
F[0, 1] &= \frac{F[1] - F[0]}{1 - 0} \\
F[0, 1, 2] &= \frac{F[1, 2] - F[0, 1]}{2 - 0} \qquad (9) \\
&\vdots \\
F[0, 1, \cdots, k] &= \frac{F[1, 2, \cdots, k] - F[0, 1, \cdots, k-1]}{k - 0}
\end{aligned}
$$

It can be shown that the divided difference of any order is a symmetric function of its arguments. Also by using appropriate limiting processes, it can be shown that

$$F[0, 1, \cdots, n, s] = \frac{F[0, 1, \cdots, n-1, s] - F[0, 1, \cdots, n]}{s - n} \qquad (10)$$

for any point s in the interval $[0, n]$. Also it can be shown that

$$\frac{d}{ds} F[0, 1, \cdots, n, s] = F[0, 1, \cdots, n, s, s] \qquad (11)$$

and that

$$\frac{F^{(n+1)}(\alpha)}{(n+1)!} = F[0, 1, \cdots, n, s] \qquad (12)$$

for some α such that $0 \leqq \alpha \leqq n$.

Suppose now that $Y(s)$ is a polynomial of degree n which passes through the points $[0, F(0)], [1, F(1)], [2, F(2)], \cdots, [n, F(n)]$. It is known that the polynomial $Y(s)$ is unique and, in fact, it is given by

$$Y(s) = \sum_{i=o}^{n} \frac{F(i) \prod_{j \neq i} (s-j)}{\prod_{j \neq i} (i-j)}. \qquad (13)$$

Now the function $F(s)$ can be shown to be expressible in terms of the polynomial function $Y(s)$ in the following way:

$$F(s) = Y(s) + E(s) \qquad (14)$$

where

$$E(s) = \frac{F^{(n+1)}(\alpha)\pi_n(s)}{(n+1)!} \tag{15}$$

for some α in $[0, n]$. A set of numerical integration formulas for $\int_0^i F(s)ds$ will result from equation (14) above if both sides are integrated over the interval $[0, i]$ for some integer $i=1, 2, \cdots, n$. These formulas are:

$$\int_0^i F(s)ds = \int_0^i Y(s)ds + \int_0^i \frac{F^{(n+1)}(\alpha)\pi_n(s)ds}{(n+1)!} \tag{16}$$

for $i=1, 2, \cdots, n$.

The numerical value of the integral $\int_0^i Y(s)ds$ can be calculated easily by using equation (13) above and expressing the result as a linear combination of $F(i)$ for $i=0, 1, \cdots, n$. The value of this latter integral can be taken as an approximation of $\int_0^i F(s)ds$ provided that the error term $\int_0^i E(s)ds$ is small.

For more details on the derivation of the numerical integration formulas the reader is referred to the literature cited.

Application of the Theorem to the Simplification of the Error Term

It is the purpose of this paper to apply the theorem which was proved earlier to the simplification of the error term

$$\int_0^i E(s)ds = \int_0^i \frac{F^{(n+1)}(\alpha)\pi_n(s)ds}{(n+1)!} \tag{17}$$

of equation (16).

In view of (10) and (12), this error term may be written

$$\int_0^i F[0, 1, \cdots, n, s]\pi_n(s)ds =$$

$$\int_0^i \frac{\{F[0, 1, \cdots, n-1, s] - F[0, 1, \cdots, n]\}\pi_n(s)ds}{s-n} \tag{18}$$

Now we define $Q_n(s) = \int_0^s \pi_n(s)ds$ and integrate each term of the right hand side of (18) separately to get

$$\int_0^i F[0, 1, \cdots, n, s]\pi_n(s)ds = \int_0^i F[0, 1, \cdots, n-1, s]\pi_{n-1}(s)ds - \int_0^i F[0, 1, \cdots, n]\pi_{n-1}(s)ds \tag{19}$$

$$= \Big\{F[0, 1, \cdots, n-1, s]Q_{n-1}(s)\Big\}_0^i - \int_0^i Q_{n-1}(s)F[0, 1, \cdots, n-1, s, s]ds - \int_0^i F[0, 1, \cdots, n]\pi_{n-1}(s)ds$$

$$= F[0, 1, \cdots, n-1, i]Q_{n-1}(i) - F[0, 1, \cdots, n]Q_{n-1}(i) - \int_0^i F[0, 1, \cdots, n-1, s, s]Q_{n-1}(s)ds,$$

so that finally

$$\int_0^i F[0, \cdots, n, s]\pi_n(s)ds = (i-n)F[0, 1, \cdots, n, i]Q_{n-1}(i) - \int_0^i F[0, 1, \cdots, n-1, s, s]Q_{n-1}(s)ds. \tag{20}$$

Now since it was proven in the theorem that $Q_{n-1}(s)$ does not change signs on (o, i), the mean value theorem may be applied to the last integral in (20) to get

$$\int_0^i F[0, \cdots, n, s]\pi_n(s)ds = (i-n)Q_{n-1}(i)F[0, 1, \cdots, n, i] - F[0, 1, \cdots, n-1, \alpha, \alpha]\int_0^i Q_{n-1}(s)ds$$

$$= (i-n)Q_{n-1}(i)\frac{F^{(n+1)}(\alpha_1)}{(n+1)!} - \frac{F^{(n+1)}(\alpha_2)}{(n+1)!}\int_0^i Q_{n-1}(s)ds, \tag{21}$$

where α_1 and α_2 and α all lie in (o, n).

If n is odd, $Q_{n-1}(s)$ is positive or zero on the interval (o, n). Therefore the coefficient of $F^{(n+1)}(\alpha_1)$ in (21) is negative as well as the coefficient of $F^{(n+1)}(\alpha_2)$ so that there exists a mean value α_3 between α_1 and α_2 such that (21) may be written

$$\int_0^i F[0, \cdots, n, s]\pi_n(s)ds = \frac{F^{(n+1)}(\alpha_3)}{(n+1)!} \cdot \left[(i-n)Q_{n-1}(i) - \int_0^i Q_{n-1}(s)ds\right] \quad (22)$$

$$= \frac{F^{(n+1)}(\alpha_3)}{(n+1)!} \cdot \int_0^i \pi_n(s)ds$$

where α_3 is in [0, n] and $i \leqq n$ and n is odd.

If n is even and $i < n$, the coefficients of $F^{(n+1)}(\alpha_1)$ and $F^{(n+1)}(\alpha_2)$ in equation (21) are both positive since $Q_{n-1}(s)$ is negative or zero on (o, n). Therefore the same mean value argument applies giving the same result as (22) except that $i < n$.

If $i = n$ in equation (22) when n is even the $\int_0^n \pi_n(s)ds$ vanishes so that we must proceed differently. If this is the case we integrate (17) by parts directly to get

$$\int_0^n F[0, 1, \cdots, n, s]\pi_n(s)ds = \Big\{F[0, 1, \cdots, n, s]Q_n(s)\Big\}_0^n - \int_0^n Q_n(s)F[0, 1, \cdots, n, s, s]ds$$

$$= -\int_0^n Q_n(s)F[0, 1, \cdots, n, s, s]ds$$

since $Q_n(n) = Q_n(0) = 0$. Now $Q_n(s)$ is positive or zero on the interval (0, n) from the theorem so that the mean value theorem can be applied directly to get

$$\int_0^n F[0, 1, \cdots, n, s]\pi_n(s)ds = -F[0, 1, \cdots, n, \alpha, \alpha]\int_0^n Q_n(s)ds$$

$$\text{where } 0 \leqq \alpha \leqq n = -\frac{F^{(n+2)}(\alpha_1)}{(n+2)!}\left\{[sQ_n(s)]_0^n - \int_0^n s\pi_n(s)ds\right\}$$

$$= + \frac{F^{(n+2)}(\alpha_1)}{(n+2)!} \int_0^n s\pi_n(s)ds,$$

provided n is even, $i = n$, and both α and α_1 are in $[o, n]$.

Literature Cited

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