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# A Revision of a Completion Method for Inverting Matrices and Its Adaptation to Ill-conditioned Matrices

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*Abstract.* An essentially new method for the inversion of  $n \times n$  matrices, closely related to the method of completion attributed to Sherman, Morrison, and Bartlett, is presented. This revised technique has the added advantages that it can be used for any square matrix, regardless of its conditioning state, and that it can be readily adapted to electronic digital computers.

Among the most efficient methods for obtaining the inverse of an arbitrary, non-singular  $n \times n$  matrix is the completion method formulated by Sherman, Morrison, and Bartlett. Bodewig symbolizes their completion formula in the following way:

$$(A + xe'_k)^{-1} = R - \beta_k Rxe'_k R = R - \beta_k R_x R_k$$

where the inverse of  $A$  is  $A^{-1} = R$ ,  $x$  is a column vector with  $n$  components,  $e'_k$  is the row vector with  $n$  components and all zeros with the exception of a one in the  $k^{\text{th}}$  component,  $R_k$  is the  $k^{\text{th}}$  row

of the matrix  $R$ , and  $\beta_k = \frac{1}{1 + R_k x}$ .

In the practical application of this formula, one chooses a convenient non-singular matrix  $A^{(0)}$  whose inverse  $R^{(0)}$  is known. A logical choice for  $A^{(0)}$  might be the identity or the diagonal matrix whose diagonal elements are those of the matrix  $M$  where  $M^{-1}$  is desired. By making successive column changes in  $A^{(0)}$  and calculating the corresponding changes in  $R^{(0)}$ , one seeks to obtain  $M^{-1}$ .

Let  $A^{(j)}$  denote the  $j^{\text{th}}$  stage in the process where  $R^{(j)} = [A^{(j)}]^{-1}$  is the corresponding inverse at the  $j^{\text{th}}$  stage. It should be pointed out that although  $A^{(0)}$  and  $M$  are both non-singular matrices, it is possible for some  $A^{(j)}$  ( $j = 1, \dots, n$ ) to be a singular matrix. In applying the Sherman, Morrison, and Bartlett formula, one changes a column of  $A^{(0)}$  to agree exactly with the corresponding column of  $M$ . For simplicity, let us suppose that the first column vector of the process,  $x_1$  is so chosen that the  $A^{(1)}$  matrix  $A^{(1)} \equiv M_{.1}$ . Choose the second column vector  $x_2$  such that  $A^{(2)} \equiv M_{.2}$ ; in general, at the  $j^{\text{th}}$  stage, the column vector change

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$x_j$  is such that the  $j^{\text{th}}$  column of  $A^{(j)}$  is equal to the  $j^{\text{th}}$  column of  $M$ , that is,

$$A^{(j)} \equiv M_{.j} \quad (j = 1, \dots, n).$$

In making the changes as suggested, it is possible that some  $A^{(j)}$  will be singular.

It is well known that, if the determinant of a matrix is zero, the matrix is singular. The fact that some  $A^{(j)}$  can be zero will become apparent if one considers the following theorem.

Theorem 1: If  $|A^{(j)}|$  is the determinant of the matrix  $A^{(j)}$ , then

$$|A^{(j+1)}| = (1 + R_k^{(j)}x) |A^{(j)}|.$$

Proof: According to the Sherman, Morrison, and Bartlett formula,

$$[A^{(j+1)}]^{-1} \equiv (A^{(j)} + xe_k')^{-1} = R^{(j)} - \beta_k R^{(j)} x R_k^{(j)} = R^{(j+1)} \text{ where } \beta_k = \frac{1}{1 + R_k^{(j)}x}.$$

It is clear that the change from  $A^{(j)}$  to  $A^{(j+1)}$  will produce a related change in  $R^{(j)}$  as it is transformed to  $R^{(j+1)}$ . Consider the effect on the individual rows of  $R^{(j)}$ . It is apparent that

$$R_i^{(j+1)} = R_i^{(j)} - \beta_k (R_i^{(j)}x) R_k^{(j)}.$$

Thus  $|R^{(j)}|$  will be changed only when a multiple of  $R_k^{(j)}$  is added to  $R_i^{(j)}$  since a multiple of a row may be added to any other row of the determinant without changing the value of the determinant. Hence the value of the determinant will be changed from  $|R^{(j)}|$  to

$$(1 - \beta_k R_k^{(j)}x) |R^{(j)}| = |R^{(j+1)}|.$$

Now  $[A^{(j+1)}] [R^{(j+1)}] = I$ , where  $I$  is the identity matrix, so that  $|A^{(j+1)}| |R^{(j+1)}| = 1$  and

$$\begin{aligned} |A^{(j+1)}| &= \frac{1}{|R^{(j+1)}|} = \frac{1}{(1 - \beta_k R_k^{(j)}x) |R^{(j)}|} = \frac{|A^{(j)}|}{1 - \beta_k R_k^{(j)}x} \\ &= \frac{|A^{(j)}|}{1 - \frac{R_k^{(j)}x}{1 + R_k^{(j)}x}} = (1 + R_k^{(j)}x) |A^{(j)}| = (1 + \alpha) |A^{(j)}| \end{aligned}$$

where  $\alpha = R_k^{(j)}x$ .

If  $R_k^{(j)}x = \alpha = -1$ ,  $|A^{(j+1)}| = 0$   $|A^{(j)}| = 0$  which says that  $A^{(j+1)}$  is singular. As an additional implication, this result shows that when working with a non-singular matrix in the proof of the Sherman, Morrison, and Bartlett formula one can always divide by  $(1 + \alpha)$  since it will never be zero.

Thus it is clear that the Sherman, Morrison, and Bartlett method can be expected to fail when the determinant of  $A^{(j)} = 0$  for some  $j$ .

Ill-conditioning arises when the determinant of a matrix is small relative to the size of the elements of the matrix. By revising this method so that the ill-conditioned stages are avoided, the completion method becomes one of the most practical methods available.

The revised method will control ill-conditioning by controlling the size of the elements of  $M$  and the size of the determinant of  $M$ . The average size of the elements in each row or column of  $M$  can be scaled to approximately one in absolute value by premultiplying or postmultiplying, respectively, by the appropriate diagonal matrix. For convenience, one might choose to make the average size of the elements of  $M$  approximately one in absolute value by using the same scale factor for all rows or columns of the matrix.

Von Neumann and Goldstine have done research on numerical inverting of matrices of high order. Their paper on this research discusses errors and scale factoring. In the following discussion, let us assume that  $M$  has been properly scaled so that the average size of the elements of  $M$  is approximately one in magnitude. In the Sherman, Morrison, and Bartlett completion method, one chooses a convenient non-singular matrix  $A^{(0)}$  whose inverse  $R^{(0)}$  is known. By making successive column changes in  $A^{(0)}$  and calculating the corresponding changes in  $R^{(0)}$  according to their formula, one computes  $M^{-1}$ . The Sherman, Morrison, and Bartlett formula is the following:

$$[A^{(j+1)}]^{-1} = (A^{(j)} + xe'_k)^{-1} = R^{(j)} - \beta_k R^{(j)} x R_k^{(j)} = R^{(j+1)}$$

where  $\beta_k = \frac{1}{1 + R_k^{(j)} x}$  and  $x$  is the desired column vector change

which can be chosen so that  $A_{ij}^{(j)} = M_{ij}$ . If  $R_k^{(j)} x \neq -1$  for some  $j$ , then  $n$  successive applications of this formula will give  $R^{(n)} = M^{-1}$  where  $n$  is the order of  $M$ .

It has been assumed that the matrix  $M$  has been scaled so that the average magnitude of its elements is approximately one. As a precautionary measure to avoid ill-conditioning, one can constrain the determinant at each stage of the process so that it has a value near or equal to  $\pm 1$ . It will be shown that the value of the determinant of  $A^{(j)}$  can be controlled by multiplying the column vector  $x_i$  by a judicious choice of the scalar  $\delta_i$ .

Choose  $A^{(0)}$  so that the determinant of  $A^{(0)}$  is equal to one. Let  $R_i^{(j)} x_i = \alpha_i$ ; then according to the determinant relationship given in Theorem 1,

$$|A^{(0)} + \delta_1 x_1 e'_1| = |A^{(1)}| = (1 + \delta_1 \alpha_1) |A^{(0)}| = 1 + \delta_1 \alpha_1.$$

It will now be required that  $|A^{(1)}| = -1$ . Hence,  $1 + \delta_1 \alpha_1 = -1$  and  $\delta_1 \alpha_1 = -2$ .

Perform a similar operation with  $A^{(2)}$  and  $R^{(2)}$ ; that is, determine  $\vartheta_2$  so that

$$(1 + \vartheta_2 \alpha_2) |A^{(1)}| = |A^{(2)}| = -(1 + \vartheta_2 \alpha_2) = 1$$

which also implies that  $\vartheta_2 \alpha_2 = -2$ . Continue this process requiring that  $(1 + \vartheta_i \alpha_i) = -1$  at each stage. Having placed such a restriction on the vectors  $x_i$ , one will obtain the inverse of a matrix  $A^{(n)}$  rather than the inverse of  $M$ . However, the relationship between  $A^{(n)}$  and  $M$  is a relatively simple one, for  $A^{(n)} = (\vartheta_1 M_{.1}, \vartheta_2 M_{.2}, \dots, \vartheta_n M_{.n})$  where  $\vartheta_i M_{.i}$  is used to denote the  $i^{\text{th}}$  column of  $A^{(n)}$ , that is  $\vartheta_i M_{.i} = A_i^{(n)}$ .

Hence,  $M = A^{(n)}D$  where  $D$  is the diagonal matrix with the reciprocal of the  $\vartheta_i$  on the diagonal. Therefore,

$$D = \begin{pmatrix} \frac{1}{\vartheta_1} & 0 & \dots & 0 \\ 0 & \frac{1}{\vartheta_2} & \dots & 0 \\ \cdot & \cdot & \dots & \cdot \\ \vdots & \cdot & \dots & 0 \\ 0 & \dots & 0 & \frac{1}{\vartheta_n} \end{pmatrix}.$$

The inverse of  $M$  can be computed readily from this equation since  $M^{-1} = D^{-1}R^{(n)}$  where

$$D^{-1} = \begin{pmatrix} \vartheta_1 & 0 & \dots & 0 \\ 0 & \vartheta_2 & \dots & \cdot \\ \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \dots & 0 \\ 0 & \cdot & \dots & \vartheta_n \end{pmatrix}$$

and  $R^{(n)}$  has just been calculated.

If this technique is going to be applicable, one must have a method for determining  $\vartheta_i R_i^{(j)} x_i = \vartheta_i \alpha_i = -2$  from available data. The vector  $\vartheta_i x_i$  was chosen so that  $\vartheta_i x_i = \vartheta_i M_{.i} - A_i^{(j)}$ , and thus

$$R_i^{(j)} \vartheta_i x_i = R_i^{(j)} \vartheta_i M_{.i} - 1 = -2,$$

and  $R_i^{(j)} \vartheta_i M_{.i} = -1$  so that  $\vartheta_i = \frac{-1}{R_i^{(j)} M_{.i}}$ . This gives a useful method for computing  $\vartheta_i$  which can then be used to find  $\vartheta_i x_i$ .

In this revised form of the completion formula, the column change vector  $x$  has been replaced by  $\vartheta_i x_i$  where  $\vartheta_i$  represents a judicious choice of a scalar. Therefore, the revised completion formula is the following:

$$[A^{(j+1)}]^{-1} = (A^{(j)} + \vartheta_i x_i e_i')^{-1} = R^{(j)} - \partial_i R^{(j)} \vartheta_i x_i R_i^{(j)}$$

and thus

$$[A^{(j+1)}]^{-1} = R^{(j)} + R^{(j)} \vartheta_i x_i R_i^{(j)}$$

where  $\partial_i = \frac{1}{1 + \vartheta_i R_i^{(j)} x_i}$ ,  $\vartheta_i = \frac{-1}{R_i^{(j)} M_{.i}}$ , and  $\vartheta_i x_i = \vartheta_i M_{.i} - A_i^{(j)}$ .

It can be shown that  $\partial_1 = -1$  through the use of the above equations; for, as a result of the control of the determinant of  $A^{(j)}$  at every stage,  $\partial_1 = \frac{1}{1 + \delta_i R_i^{(j)} x_i}$  and  $\delta_i R_i^{(j)} x_i = \delta_i \alpha_i = -2$ . Thus  $\partial_1 = \frac{1}{1-2} = -1$ . When applying this formula, one would compute  $\delta_i$ ,  $\delta_i x_i$ , and  $R^{(j)} \delta_i x_i R_i^{(j)}$  respectively; and then, by summing the appropriate matrices, one would obtain  $R^{(j+1)}$ .

This formula will be operative unless  $R_i^{(j)} M_{.i} = 0$  or is small, for this would mean that  $\delta_i$  would be undefined or a large scalar. When programming this technique, one can require the machine to check the size of  $|R_i^{(j)} M_{.i}|$ . If  $|R_i^{(j)} M_{.i}|$  is not within the desired limits, the machine can be commanded to change a different column.

Suppose that one chooses to set up a program which would make successive column changes so that at each stage,  $A_{.i}^{(j)} = A_{.i}^{(n)}$  where  $n$  is the order of  $A$ . It is possible that  $|R_i^{(j)} M_{.i}|$  will not be within the desired limits for all  $i \geq j$ . When such a condition is encountered, one can replace  $M$  by the matrix  $\bar{M} = (M - \epsilon E_{ki})$  where  $E_{ki} = e_k e_i'$  and  $k$  is chosen so that the element  $r_{ik}^{(j)} \neq 0$  where  $r_{ik}^{(j)}$  is the element of  $R^{(j)}$  in the  $i$ th row and  $k$ th column. As a matter of convenience,  $\epsilon$  can be chosen so that

$$R_i^{(j)} \bar{M}_{.i} = R_i^{(j)} M_{.i} - \epsilon r_{ik}^{(j)} = -1,$$

that is

$$\epsilon = \frac{1 + R_i^{(j)} M_{.i}}{r_{ik}^{(j)}}.$$

After making such a change, one will no longer obtain  $A^{(n)}$ , but instead will obtain  $[\bar{A}^{(n)}]^{-1} = \bar{R}^{(n)}$ . In most applications of this revised completion method, it will not be necessary to change more than one element of  $M$ ; however, this process can be repeated.

After making  $n$  column changes in  $A^{(0)}$  and calculating the corresponding changes in  $R^{(0)}$ , one will obtain either  $[A^{(n)}]^{-1} = R^{(n)}$  or  $[\bar{A}^{(n)}]^{-1} = \bar{R}^{(n)}$ . If  $R^{(n)}$  is the result,  $M^{-1}$  can be found by the method, previously described, which used the equation,  $M^{-1} = D^{-1} R^{(n)}$ . On the other hand, if  $\bar{R}^{(n)}$  is the final result,  $\bar{M}^{-1}$  can be obtained from  $\bar{M}^{-1} = D^{-1} \bar{R}^{(n)} = \bar{\bar{R}}$ ; then  $M^{-1}$  can be found from the following relation:

$$M^{-1} = (\bar{M} + \epsilon E_{ki})^{-1} = \bar{\bar{R}} - \frac{\epsilon}{1 + \epsilon \bar{R}_{i.e_k}} \bar{\bar{R}}_k \bar{\bar{R}}_i,$$

and thus

$$M^{-1} = \bar{\bar{R}} - \frac{\epsilon}{1 + \epsilon \bar{r}_{ik}} \bar{\bar{R}}_k \bar{\bar{R}}_i,$$

where  $\overline{\overline{R}} = \overline{\overline{M}}^{-1}$ . This relation is a special case of the Sherman, Morrison, and Bartlett formula.

One should note that by constraining  $|A^{(j)}| = (-1)^j$  through the use of a judicious choice of each  $\varkappa_i$  that the elements in  $A_{i,i}^{(n)} = \varkappa_i M_{i,i}$  have an approximate magnitude of  $\varkappa_i$ , that is  $|\varkappa_i|$ . It is clear that if  $|\varkappa_i|$  becomes too large or too small, the relative size of  $|A^{(j)}| = (-1)^j$  will change correspondingly. The relative size of the elements of a matrix and its determinant is not as essential to the theoretical operation of this technique as it is in the operation of the adjoint-determinant method where one actually divides by the determinant of the matrix. However, by having controlled the size of  $|A^{(j)}|$ , one can place a desired control on the average size of the elements of  $A_{i,i}^{(j)}$  and thereby help control round-off errors.

The details of this revised completion method might seem cumbersome at first observation. It should be noted that in most practical applications one will not find it necessary to change the single element in  $M$ . The method is relatively easy to program and it has the decided advantage that it will produce the inverse for any matrix, even the ill-conditioned ones. The above proposed revision of the completion method has been used by the author to calculate the inverse for the exceedingly ill-conditioned matrix  $M$  which has a determinant value of  $-.000001$  with the following result:

$$M^{-1} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1.01 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1.01 \\ 1 & 1 & .99 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} -100 & 100 & 0 & 0 \\ 101 & -100 & -100 & 100 \\ 100 & 0 & 0 & -100 \\ -100 & 0 & 100 & 0 \end{pmatrix}.$$

When using an electronic digital computer to calculate the inverse of a matrix, one must be concerned with the memory capacity of the machine. It is possible that one might desire the inverse of a matrix  $M$  where the program would require more storage than is available in the memory. In this case, one can resort to the Frobenius-Schur relation discussed by Bodewig. One partitions the matrix  $M$  into four matrices which, in most cases, can be accommodated by the machine. This relation combined with the completion method gives a practical scheme for obtaining the inverse of a matrix of nearly any size.

After studying the capacity of the computer and the accuracy which is desired, a programmer can choose upper and lower bounds for  $|R_i^{(j)} M_{i,i}|$ . Since  $\varkappa_i = \frac{-1}{R_i^{(j)} M_{i,i}}$ , the bounds chosen for  $|R_i^{(j)} M_{i,i}|$  will be approximately the reciprocal of the bounds placed

on the average size of the elements of  $A_i^{(j+1)}$ . Knowledge of a relation such as this is valuable to a programmer.

A possible alternate modification of the completion method of Sherman, Morrison, and Bartlett would be to place limits on  $|R_k^{(j)}x|$  in their formula rather than on  $|\delta_i|$  in the revised formula. The critical point is reached when  $R_k^{(j)}x$  is near  $-1$ , and thus one can choose proper limits for  $|R_k^{(j)}x|$  and use it as the factor which determines when one should change a different column vector or resort to the single element change described in the formula. This modification will usually be more efficient than the suggested revision.

The problem would be to decide on the proper limits for  $|R_k^{(j)}x|$ . In the modified process, the proper choice for the limits of  $|R_k^{(j)}x|$  is not apparent for one does not have a direct relationship between the change in columns of  $A^{(0)}$  and  $|R_k^{(j)}x|$ . It was this very difficulty which led to the derivation of the above revised form for the completion formula of Sherman, Morrison, and Bartlett.

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