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Abstract. It is shown that the usual method of proof of the Remainder Theorem is open to serious objection. A rigorous proof using the limit concept is suggested.

The Remainder Theorem may be stated as follows:

(i) $f(x) = \sum_{i=0}^{n} A_i x^i$

(ii) $f(x) = (x-r)\,Q(x) + R,\ x \neq r$

(iii) R independent of x

$\Longrightarrow R = f(r).$

Almost without exception college algebra texts submit the following argument as proof:

"By definition of division,

(1) $f(x) = (x-r)\ Q\ (x) + R.$

Since this is an identity, we may set $x = r$ to obtain:

(2) $f(r) = (r-r)\ Q(r) + R = R.$"

In the opinion of this author, the use of the phrase "since this is an identity" is open to serious criticism on the ground that it involves circular logic, insofar as the freshman algebra student is concerned. The definition of division allows us to write:

(3) $f(x) = (x-r)\ Q\ (x) + R,\ x \neq r.$

Although it is true that (3) is an identity, the freshman student usually has no background on which to base such a statement. The usual sequence of events is to make use of the Remainder Theorem to prove that if two polynomials of degree no greater than N are equal for more than N values of the unknown, then the two polynomials are identical. This last theorem could be used by the student to show that (3) is an identity, but this constitutes circular reasoning.

Morgan (1943) has avoided this difficulty in an elegant and completely algebraic manner. However, it would seem that this is a good opportunity to let the students see the power of the methods of analysis. Since at this point in the course the student might well have been introduced to the concept of the limit, th author would like to suggest the following approach:

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We shall first need the Lemma;

If f(x) is a polynomial, then $\lim_{x \to r} f(x) = f(r)$.

This may conveniently be shown by writing:

$$(4) \quad \lim_{x \to r} f(x) = \lim_{\delta \to 0} f\,(r+\delta),$$

and expanding by means of the Binomial Theorem.

Proof of the Remainder Theorem

By the definition of division:

$$(5) \quad f(x) = (x-r)\,Q\,(x) = R,\; x \neq r.$$

Then:

$$(6) \quad f(r) = \lim_{x \to r} f(x) = \lim_{x \to r} [\,(x-r)\,Q\,(x) + R\,] = R.$$

Q.E.D.

The author has been using this method in his classes and finds that it is well-accepted by the students and assists materially in their understanding of this Theorem.

Reference

Morgan, Frank M. 1943. *College Algebra.* American Book Company, New York. P. 278.