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On the Construction of the Measurable Sets

DONOVAN F. SANDERSON

Abstract. Whenever we have a measure function \( \alpha \) defined on some set \( M \) of subsets of a set \( T \), we may determine a binary relation \( \mathcal{Q} \) between the elements of \( M \) by defining, for all members \( A \) and \( B \) of \( M \), \( A \mathcal{Q} B \) if and only if \( \alpha(A) \leq \alpha(B) \). Using such a binary relation, we may derive certain measure theoretic properties independently of the real number system. In particular, if we use what might be termed a process of completion, we may construct, from a system of Borel sets, not only a system of Lebesgue measurable sets, but, in general, a somewhat larger system.

In the following paper, we shall state certain measure theoretic results, without proof, which can be derived using a binary relation rather than a measure function. For typographical reasons, we shall denote the intersection and union of two sets, \( A \) and \( B \), by \( A \cdot B \) and \( A + B \) respectively. If \( A \) is a subset of \( B \), we shall write \( A < B \). The empty set will be denoted by \( \emptyset \).

A measure function \( \alpha \) on a set \( T \) induces a partial order \( \mathcal{Q} \) on the set of measurable subsets \( M \) of \( T \), if we define, for all \( A \) and \( B \in M \), \( A \mathcal{Q} B \) if and only if \( \alpha(A) \leq \alpha(B) \). Such a binary relation can easily be shown to satisfy the following axioms.

Axiom 1. If \( A \) and \( B \in M \), then either \( A \mathcal{Q} B \) or \( B \mathcal{Q} A \) or both.
Axiom 2. If \( A \) and \( B \in M \) and \( A < B \), then \( A \mathcal{Q} B \).
Axiom 3. If \( A, B, \) and \( C \in M \), \( A \mathcal{Q} B \), and \( B \mathcal{Q} C \), then \( A \mathcal{Q} C \).
Axiom 4. If \( A, B, \) and \( \theta \in M \) and \( A \mathcal{Q} \theta \), then \( (A+B) \mathcal{Q} \theta \).
Axiom 5. If \( A_i \) and \( \theta \in M \) and \( A_i \mathcal{Q} \theta \), for all positive integers \( i \), then \( \Sigma A_i \mathcal{Q} M \) and \( (\Sigma A_i) \mathcal{Q} \theta \).

A binary relation which satisfies the above axioms will be termed a measure relation.

In the remainder of the paper, we shall assume that \( M \) is closed under countable unions and set differences, and contains \( \emptyset \). We note this implies that \( M \) is closed under countable intersections. We shall also suppose that a measure relation \( \mathcal{Q} \) is defined on \( M \).

Definition 1. If \( A \) and \( B \in M \), then \( A(\equiv)B \) if and only if \( A \mathcal{Q} B \) and \( B \mathcal{Q} A \).

Definition 2. If \( A \) and \( B < T \), then \( A(\equiv)_1 B \) if and only if there is a \( D \in M \) such that \( D \mathcal{Q} \theta \) and \( (A-B)+B-A) < D \).

A(\equiv)B corresponds to saying that \( A \) and \( B \) have the same measure. A(\equiv)_1 B is analogous to saying that \( A \) and \( B \) are equal

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almost everywhere, that is, except on a set of measure zero.

We may now prove the following theorem.

Theorem 1. (i) \(=\) and \(\sim\) are equivalence relations on their respective domains.

(ii) If \(A\) and \(B\) are members of \(\mathcal{M}\), then \(A \sim B\).

In general, however, we may have \(A \sim B\) without having \(A = B\), even though \(A\) is a member of \(\mathcal{M}\). To eliminate this possibility, we shall now extend the domain of definition of \(\mathcal{Q}\).

Definition 3. (i) \(H(\mathcal{M}) = \{ A \mid \exists B \subseteq \mathcal{M} \text{ such that } A \sim B \}\).

(ii) If \(A \in H(\mathcal{M})\), then \(P(A) = \{ B \subseteq \mathcal{M} \mid B \subseteq A \sim B \}\).

(iii) If \(A \in H(\mathcal{M})\), then \(P(A) = \{ B \subseteq \mathcal{M} \mid B \subseteq A \sim B \}\).

Definition 4. \(A \in \mathcal{L}(\mathcal{M})\) if and only if \(\exists B \subseteq \mathcal{M} \sim B \).

\(\mathcal{L}(\mathcal{M})\) will be called the set of \(\mathcal{Q}\)-measurable sets with respect to \(\mathcal{M}\).

Theorem 2. \(\mathcal{L}(\mathcal{M})\) is closed with respect to countable unions and set differences.

We will now impose a measure relation on \(\mathcal{L}(\mathcal{M})\), with the aid of the following definition.

Definition 5. If \(A\) and \(B\) are members of \(\mathcal{L}(\mathcal{M})\), then \(A \subseteq B\) if and only if for every \(C \subseteq \mathcal{P}(B)\) there is a \(D \subseteq \mathcal{P}(A)\) such that \(\mathcal{Q}C\).

Theorem 3. (i) \(\mathcal{Q}\) is a measure relation on \(\mathcal{L}(\mathcal{M})\).

(ii) \(\mathcal{Q}\) corresponds with \(\mathcal{Q}\) on \(\mathcal{M}\).

The following theorem is now derivable.

Theorem 4. If \(A \sim B\) and \(A \in \mathcal{L}(\mathcal{M})\), then \(B \in \mathcal{L}(\mathcal{M})\) and \(A \subseteq B\). (\(\sim\) is the equivalence relation corresponding to \(\mathcal{Q}\).)

At this point, we shall note a few things. If \(\mathcal{Q}\) had been induced by a Borel measure \(\alpha\) on a set \(\mathcal{M}\) of Borel sets of a topological space \(\mathcal{T}\), then a set \(A \subseteq \mathcal{T}\) would be Lebesgue measurable if and only if there existed a set \(B \subseteq \mathcal{M}\) such that \(A \subseteq B\). Thus, by the previous theorem, \(\mathcal{L}(\mathcal{M})\) would contain the Lebesgue measurable sets. However, in general, \(\mathcal{L}(\mathcal{M})\) will be somewhat larger.

Also, one might ask whether it is possible to extend \(\mathcal{L}(\mathcal{M})\) by using the above procedure. The answer is contained in the following "closure" theorem.

Theorem 5. \(\mathcal{L}(\mathcal{M}) = \mathcal{L}(\mathcal{L}(\mathcal{M}))\).

Needless to say, the previous paragraphs left many questions unanswered. Some of these questions are answered in (1) and, it is hoped, a subsequent series of papers will answer many more.

Literature Cited