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A Class of Vector Functions with Linear Norms

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Thus

$$f(r,u) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} r^{-s}\pi \csc \pi s \; P_{s-1}(\cos u)ds = (1 + 2r\cos u + r^2)^{-1/2};$$
$$u \leqslant V,\ 0 < r < \infty.$$

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A Class of Vector Functions with Linear Norms[1]

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Abstract. In some work with systems of ordinary differential equations, a certain compact convex subset of a Banach space of vector-valued functions continuous on the real closed interval [0,1] was introduced [1]. The topology in this Banach space is induced by the supremum norm, while the norm used in the n-dimensional vector space of function values is arbitrary. It is observed in this note that all functions of this class have the same norm, a linear function on the set [0,1]. However, the nature of this subset depends rather markedly on the type of vector norm used.

A theorem due to J. Schauder [3] says, in effect, that a compact convex subset of a Banach space has the fixed point property; i.e., a continuous function mapping the subset into itself necessarily has a fixed point. In applications of this result to certain existence problems in the theory of differential equations, we often deal with a Banach space of continuous vector functions of a real variable, and it then becomes necessary to introduce compact convex subsets of this space. In this note we consider a certain type of subset of such a space which has arisen in some work of the author and D. D. James [1]. It is observed in particular that these convex compact subsets depend rather considerably on the norm used in the n-dimensional vector space

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containing the ranges of the functions making up the Banach space.

Notation:

(i) $\{R^n, |\ |\}$ is a normed linear n-space over the complex field where the norm of $x \varepsilon R^n$ is denoted by $|x|$.

(ii) C[0,1] is the set of continuous functions on the closed real interval [0,1] to R^n.

(iii) If $f \varepsilon C\,[0,1]$, we define $||f|| = \sup_{[0,1]} |f(u)|$.

(iv) S^α is the subset of C[0,1] consisting of functions f with continuous derivative f′ on [0,1] and such that

$$f(0) = 0,\ f(1) = \alpha.\ |f'(u)| \leqslant b, \text{ where } b = |\alpha|;$$

here if $f = (f_1, \ldots, f_n)$ then $f' = (f'_1, \ldots, f'_n)$.

Remark 1. $\{C[0,1], ||\ ||\}$ is a Banach space over the complex field.

Remark 2. S^α is convex and conditionally compact; i.e., the closure $\bar{S^\alpha}$ of S^α is convex and compact.

Conditional compactness follows directly from Ascoli's lemma (cf. [2], p. 5) since S^α is an equicontinuous set of functions uniformly bounded on [0,1]. Convexity follows easily; we omit the details.

Remark 3. If $f \varepsilon S^\alpha$, then $|f(u)| = bu$ (cf. [1]) and it follows easily that $|f(u)|' = |f'(u)| = b$ on [0,1]. Thus S^α consists of functions with linear norm, and in (iv), the inequality $|f'(u)| \leqslant b$ can in fact be replaced by $|f'(u)| = b$. On the other hand, not all functions f with linear norm on [0,1] and $f(1) = \alpha$ are in S^α, as the following example in R^2 shows. If $x = (x_1,x_2)$, take $|x| = \max(|x_1|, |x_2|)$, $f(u) = (u, u-u^3)$. Then $\alpha = (1,0)$, $b = 1$, $f'(u) = (1, 1-3u^2)$, and since $|f'(1)| = 2$, f is not an element of S^α. However since $u^3 \leqslant u$ for $u \varepsilon [0,1]|$, $f(\ )u| = u$.

Remark 4. S^α clearly contains the linear vector function $u\alpha$, but does not in general consist of only this function, as is always the case if $n = 1$ and R^1 is real. For consider the case in R^2 with norm as in the preceding example, where now $(f(u) = (u, u-u^2)$. Since $|f'(u)| = \max(1, |1-2u|) = 1$ for $u \varepsilon$ [0,1] it follows that $f \varepsilon S^\alpha$. But f is obviously not linear. Another example in R^2 using the norm $|x| = |x_1| + |x_2|$ is due to D. D. James [1].

1. D. D. James and G. Seifert, Periodic Integral Surfaces for Periodic

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