Proceedings of the Iowa Academy of Science

Volume 69 | Annual Issue Article 68

1962

A Class of Vector Functions with Linear Norms

George Seifert
Iowa State University

Thus

$$f(r,u) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} r^{-s}\pi \csc \pi s \, P_{s-1}(\cos u)ds = (1 + 2r \cos u + r^2)^{-1/2};$$
$$u \leqslant V, 0 < r < \infty.$$

ACKNOWLEDGMENT

I would like to thank Dr. Harry J. Weiss for his assistance in this study.

Literature Cited

Erdelyi, A., W. Magnus, F. Oberhettinger and F. Tricomi. Tables of Integral transforms, V. 1. New York, N.Y., McGraw Hill Book Co., Inc. 1954.

A Class of Vector Functions with Linear Norms[1]

GEORGE SEIFERT[2]

Abstract. In some work with systems of ordinary differential equations, a certain compact convex subset of a Banach space of vector-valued functions continuous on the real closed interval [0,1] was introduced [1]. The topology in this Banach space is induced by the supremum norm, while the norm used in the n-dimensional vector space of function values is arbitrary. It is observed in this note that all functions of this class have the same norm, a linear function on the set [0,1]. However, the nature of this subset depends rather markedly on the type of vector norm used.

A theorem due to J. Schauder [3] says, in effect, that a compact convex subset of a Banach space has the fixed point property; i.e., a continuous function mapping the subset into itself necessarily has a fixed point. In applications of this result to certain existence problems in the theory of differential equations, we often deal with a Banach space of continuous vector functions of a real variable, and it then becomes necessary to introduce compact convex subsets of this space. In this note we consider a certain type of subset of such a space which has arisen in some work of the author and D. D. James [1]. It is observed in particular that these convex compact subsets depend rather considerably on the norm used in the n-dimensional vector space

[1] This research was supported in part by the National Science Foundation under grant NSF-G17851.
[2] Iowa State University of Science and Technology, Ames, Iowa.

containing the ranges of the functions making up the Banach space.

Notation:

(i) $\{R^n, |\ |\}$ is a normed linear n-space over the complex field where the norm of $x \varepsilon R^n$ is denoted by $|x|$.

(ii) C[0,1] is the set of continuous functions on the closed real interval [0,1] to R^n.

(iii) If $f \varepsilon C$ [0,1], we define $||f|| = \sup_{[0,1]} |f(u)|$.

(iv) S^α is the subset of C[0,1] consisting of functions f with continuous derivative f′ on [0,1] and such that

$$f(0) = 0,\ f(1) = \alpha.\ |f'(u)| \leqslant b, \text{ where } b = |\alpha|;$$

here if $f = (f_1, \ldots, f_n)$ then $f' = (f'_1, \ldots, f'_n)$.

Remark 1. $\{C[0,1], ||\ ||\}$ is a Banach space over the complex field.

Remark 2. S^α is convex and conditionally compact; i.e., the closure $\bar{S^\alpha}$ of S^α is convex and compact.

Conditional compactness follows directly from Ascoli's lemma (cf. [2], p. 5) since S^α is an equicontinuous set of functions uniformly bounded on [0,1]. Convexity follows easily; we omit the details.

Remark 3. If $f \varepsilon S^\alpha$, then $|f(u)| = bu$ (cf. [1]) and it follows easily that $|f(u)|' = |f'(u)| = b$ on [0,1]. Thus S^α consists of functions with linear norm, and in (iv), the inequality $|f'(u)| \leqslant b$ can in fact be replaced by $|f'(u)| = b$. On the other hand, not all functions f with linear norm on [0,1] and $f(1) = \alpha$ are in S^α, as the following example in R^2 shows. If $x = (x_1,x_2)$, take $|x| = \max(|x_1|, |x_2|)$, $f(u) = (u,u-u^3)$. Then $\alpha = (1,0)$, $b = 1$, $f'(u) = (1, 1-3u^2)$, and since $|f'(1)| = 2$, f is not an element of S^α. However since $u^3 \leqslant u$ for $u \varepsilon [0,1]|$, $f(\ )u| = u$.

Remark 4. S^α clearly contains the linear vector function $u\alpha$, but does not in general consist of only this function, as is always the case if n = 1 and R^1 is real. For consider the case in R^2 with norm as in the preceding example, where now $(f(u) = (u, u-u^2)$. Since $|f'(u)| = \max(1, |1-2u|) = 1$ for $u \varepsilon$ [0,1] it follows that $f \varepsilon S^\alpha$. But f is obviously not linear. Another example in R^2 using the norm $|x| = |x_1| + |x_2|$ is due to D. D. James [1].

1. D. D. James and G. Seifert, Periodic Integral Surfaces for Periodic

Literature Cited

Systems of Differential Equations, (presented to Iowa Section, M.A.A. April 1962)

2. E. A. Coddington and N. Levinson, Theory of Ordinary Differential Equations, McGraw Hill, N. Y. (1955)

3. J. Schauder, Der Fixpunktsatz in Funktionalräumen, Studio Math. 2, 171-179, (1930).