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Application of the Mellin Transform to Boundary Value Problems

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Abstract. The Mellin transform is investigated with special emphasis on its applications to the solution of boundary value problems. A technique is given for solution of Laplace's equation in plane polar and spherical polar coordinates.

One of the methods of solving boundary value problems is by use of an integral transform and its corresponding inversion formula. The integral transform T $\{f(x)\}$ of a function f(x) with respect to a kernel K(x,s) is defined by

$$T\{f(x)\}=\bar{f}(s) = \int_a^b f(x)\ K(x,s)dx$$

where a and b may be real or complex, finite, or infinite, and s is a parameter, real or complex. The inversion formula gives f(x) in terms of f(s) as

$$f(x) = \int_c^d \bar{f}(s)\ H(x,s)ds$$

where c and d may also be real or complex, finite or infinite.

The integral transform is useful in solving boundary value problems involving partial differential equations, as its use generally reduces the complexity of the differential equation by removing one independent variable. The solution of the transformed equation is found by any convenient method and the desired function is obtained from this solution by means of the inversion formula.

Integral transforms are particularly useful in solving boundary value problems with discontinuous or mixed boundary conditions. An example will follow that illustrates this use with discontinuous boundary conditions.

The specific transform considered here is the Mellin transform which has the kernrel

$$K(x,s) = x^{s-1}$$

and integration is from zero to infinity along the real x-axis.

THE MELLIN TRANSFORM

The Mellin inversion formula is usually obtained by a change of variables in the exponential form of the Fourier integral

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theorem. This method yields the following:

Theorem 1

Let f(x) satisfy the Dirichlet conditions in $(0,\infty)$, $\int_0^\infty |f(x)|x^{k-1}dx$ exist for some $k > 0$, and let

$$\bar{f}(s) = \int_0^\infty f(x)x^{s-1}dx.$$

Then

$$f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \bar{f}(s)x^{-s}ds$$

for $c > k$.

Although this transform bears Mellins name, the original inversion theorem was given by Poisson in a memoir read before the Academy of Sciences, Paris, in 1815. Mellin gave a rigorous discussion of it in 1896.

A useful property of the Mellin transform

$$T\{xf'(x)\} = -s\bar{f}(s)$$

follows from consideration of

$$\int_0^\infty xf'(x)x^{s-1}dx = f(x)x^s\Big]_0^\infty - s\int_0^\infty f(x)x^{s-1}dx$$

and the assumption that the evaluated term vanishes. Similarly the formula

$$T\left\{x^N f^{(N)}(x)\right\} = (-1)^N \frac{\Gamma(s+N)}{\Gamma(s)}\bar{f}(s)$$

follows by repeated use of integration by parts and the assumption that all evaluated terms vanish.

The Mellin transform can be used to solve the ordinary differential equation.

$$\sum_{k=1}^{n} A_k x^{k+p} f^{(k)}(x) = q(x)$$

if $\int_0^\infty q(x)x^{s-p-1}dx$ converges, as use of the previous formula will reduce it to an algebraic equation in $\overline{f}(s)$.

$$T\left\{x^N \frac{\partial^N f(x,y,z,\ldots t)}{\partial x^N}\right\} =$$

$$\frac{(-1)^N\,\Gamma(s+N)}{\Gamma(s)}\,\overline{f}(s,y,z,\ldots t)$$

is useful in removing terms of the form

$$x^N \frac{\partial f^N(x,y,z,\ldots t)}{\partial x^N}$$

from a partial differential equation. Thus equations like Poisson's, Laplace's or the biharmonic equation in polar coordinates may be attacked by this method. The procedure is to multiply the differential equation by x^q where q is determined such that all partial derivatives with respect to x have the form

$$x^{m+s-1}\frac{\partial^{m+p}f}{\partial x^m \partial y^r \partial z^s \ldots \partial t^u}.$$

Then integration with respect to x from zero to infinity eliminates x and leaves a differential equation, called the auxiliary equation, in the variables y,z, . . ., t with parameter s. If the differential equation is a part of a boundary value problem, the conditions on the unknown function must be transformed if they are to be applied to the solution of the auxiliary equation. The solution to the original equation is obtained from the solution to the auxiliary equation by use of the inversion theorem.

Assumptions concerning the nature of the unknown function must occasionally be made in reducing the original differential equation and in the use of the inversion theorem. Sometimes the physical nature of the problem makes the needed assumptions reasonable, but to be completely rigorous, either such assumptions must be checked for validity after the solution is obtained, or the solution must be checked to see if it satisfies the differential equation and the given conditions. In the latter case, the details in obtaining the solution are regarded as formal manipulative techniques in obtaining a formal solution, which is then checked. In any particular case, one can decide which method is easier in checking the validity of the solutions.

Solution of Laplaces Equation

Plane Polar of Coordinates

The problem considered here is that of determining a function f which satisfies Laplace's equation in the interior of an infinite

2-dimensional wedge of angle 2V, $V \leqslant \pi/2$, with conditions given on the boundary.

Consider the wedge referred to plane polar coordinates (r,u) with the polar axis bisecting the wedge angle and the pole at the apex of the wedge. Assume the boundary conditions to be

$$f(r, \pm V) = \begin{cases} 1 & 0 < r < a \\ 0 & r > a \end{cases}$$

$f(r.u), \frac{\partial f}{\partial r}(r.u) \to 0$ as $r \to \infty$ for $|u| \leqslant V$. In plane polar coordinates, Laplace's equation is

$$\nabla^2 f = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial u^2} = 0$$

Multiply the equation by r^{s+1} and integrate with respect to r from zero to infinity. Integration by parts and the assumptions that the bracketed terms vanish and that the order of differentiation and integration may be interchanged reduces the equation to

$$\frac{d^2\bar{f}}{du^2} + s^2\bar{f} = 0$$

with solution

$$\bar{f}(s,u) = A(s)\cos su + B(s)\sin su$$

Because of the symmetric geometry and boundary conditions it is reasonable to assume that f is a symmetric function of u, and thus so is $\bar{f}$. Thus, $B(s) = 0$ and

$$\bar{f}(s,u) = A(s)\cos su$$

The transformed boundary conditions are

$$\bar{f}(s, \pm V) = \int_0^a r^{s-1}\,dr = \frac{a^s}{s}$$

Thus

$$A(s) = \frac{a^s}{s\cos sV}$$

and

$$\bar{f}(s,u) = \frac{a^s\cos su}{s\cos sV}.$$

This gives

$$f(r,u) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{1}{s}\,\frac{\cos su}{\cos sV}\left(\frac{a}{r}\right)^s ds$$

where, because of the nature of the integrand it follows that the line integral must be taken along a line $0 < \text{Re}(s) = c < \pi/2V$.

The integrand has poles at $s = 0, \pm\frac{(2k-1)\pi}{2V}$, $k = 1,2,\ldots$ To evaluate the contour integral consider a sequence of contour integrals, where a generic contour c_N, $N = 1,2,\ldots$, consists of the boundary of a rectangle with vertices at $(\frac{\pi}{4V}, N)$, $(\frac{N\pi}{V}, N)$, $(\frac{N\pi}{V}, -N)$, and $(\frac{\pi}{4V}, -N)$. On any one contour, the integrand is analytic; and within the contour has only a finite number of simple positive poles at $s = \frac{(2k-1)\pi}{2V}$, $k=1,2,\ldots,N$. It can be shown that

$$f(r,u) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{1}{s}\,\frac{\cos su}{\cos sV}\left(\frac{a}{r}\right)^{s} ds,$$

$$= \lim_{n\to\infty} \frac{1}{2\pi i}\oint_{c_n} \frac{1}{s}\,\frac{\cos su}{\cos sV}\left(\frac{a}{r}\right)^{s} ds.$$

And using Cauchy's integral formula this is

$$f(r,u) = \frac{2}{\pi}\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2k-1}\left(\frac{a}{r}\right)^{(2k-1)\pi/2V} \cos (2k-1)\frac{\pi u}{2V}, \quad \text{for } r > a.$$

Similarly, by taking a sequence of rectangles with vertices at $(-\frac{N\pi}{V}, N)$, $(\frac{\pi}{4V}, N)$, $(\frac{\pi}{4V}, -N)$, and $(-\frac{N\pi}{V}, -N)$ it can be shown that

$$f(r,u) = 1 - \frac{2}{\pi}\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2k-1}\left(\frac{a}{r}\right)^{-(2k-1)\pi/2V} \cos (2k-1)\frac{\pi u}{2V},$$

for $r < a$.

It can be shown that the two series and their derivatives needed in Laplace's equation are uniformly convergent and that the formal solution is the actual solution. Note that the solution obviously satisfies the boundary conditions.

Spherical Polar Coordinates

The problem here is to determine a function f which satisfies Laplace's equation in the interior of an infinite right circular cone of vertex angle 2V, $V \leqslant \pi/2$, and given conditions on the boundary.

Consider the cone referred to spherical coordinates (r,θ,u) with the vertex of the cone at the origin. Assume the solution is independent of the meridian angle and $f(r,u) = h(r)$ for $u = V$. Thus the boundary value problem is

$$\frac{1}{r}\frac{\partial^2(rf)}{\partial r^2} + \frac{1}{r^2 \sin u}\frac{\partial}{\partial u}\left(\sin u \frac{\partial f}{\partial u}\right) = 0,$$

$$o < r < \infty, 0 \leqslant u < V,$$

with boundary condition

$$f(r,V) = h(r).$$

Application of the Mellin transform reduces the differential equation to

$$\frac{1}{\sin u}\frac{d}{du}\left(\sin u \frac{d\bar{f}}{du}\right) + s(s-1)\bar{f} = 0$$

which has as solution

$$\bar{f}(s,u) = A(s)P_{s-1}(\cos u) + B(s)Q_{s-1}(\cos u)\,.$$

In order that the solution be bounded for $u = 0$, let $B = 0$. Applying the boundary condition gives

$$A(s) = \frac{\bar{h}(s)}{P_{s-1}(\cos V)}$$

and

$$f(r,u) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} r^{-s}\frac{\bar{h}(s)P_{s-1}(\cos u)}{P_{s-1}(\cos V)}\,ds.$$

$P_{s-1}(\cos V)$ has an infinite number of real simple zeros for each choice of V, whose numerical values are known only approximately. Except for very special cases the integral can only be evaluated numerically and approximately.

As one of these special cases consider the boundary condition

$$h(r) = (1 + 2r\cos V + r^2)^{-1/2}$$

Then from a table of integral transforms [1]

$$\bar{h}(s) = \pi \csc \pi s\, P_{s-1}(\cos V)$$

$$0 < \mathrm{Re}(s) < 1.$$

Thus

$$f(r,u) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} r^{-s}\pi \csc \pi s \, P_{s-1} (\cos u)ds = (1 + 2r \cos u + r^2)^{-1/2};$$
$$u \leqslant V, 0 < r < \infty.$$

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A Class of Vector Functions with Linear Norms[1]

George Seifert[2]

Abstract. In some work with systems of ordinary differential equations, a certain compact convex subset of a Banach space of vector-valued functions continuous on the real closed interval [0,1] was introduced [1]. The topology in this Banach space is induced by the supremum norm, while the norm used in the n-dimensional vector space of function values is arbitrary. It is observed in this note that all functions of this class have the same norm, a linear function on the set [0,1]. However, the nature of this subset depends rather markedly on the type of vector norm used.

A theorem due to J. Schauder [3] says, in effect, that a compact convex subset of a Banach space has the fixed point property; i.e., a continuous function mapping the subset into itself necessarily has a fixed point. In applications of this result to certain existence problems in the theory of differential equations, we often deal with a Banach space of continuous vector functions of a real variable, and it then becomes necessary to introduce compact convex subsets of this space. In this note we consider a certain type of subset of such a space which has arisen in some work of the author and D. D. James [1]. It is observed in particular that these convex compact subsets depend rather considerably on the norm used in the n-dimensional vector space

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