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Solutions for Diffusion Equations With Integral Type Boundary Conditions[1]

K. L. Deckert and C. G. Maple

This work was suggested by a paper by V. W. Bolie[2] in which he develops a non-linear impedance theory for bioelectrodes. In his work, he arrives at a pair of diffusion equations with the interesting feature that one of the boundary conditions is expressed as a definite integral. Since we are unaware of any work in which a parabolic equation is subjected to this type of boundary condition, we naturally asked the question as to whether such a problem has a solution; and if it does, is the solution unique? In this paper we answer these questions for the linear diffusion problem.

The explicit problem we consider here is the following:

Problem D: Find a function u(x,t), continuous on the closure $\bar{R}$ of the domain $R(0 < x < 1, t > 0)$, which satisfies the conditions

$$\begin{aligned} u_{xx} - u_t &= F(x,t) && \text{on } R, \\ u(x,0) &= g(x) && 0 \leqslant x \leqslant 1, \\ u(0,t) &= f(t) && t \geqslant 0, \\ \int_0^1 u(x,t)\,dx &= h(t) && t \geqslant 0, \end{aligned}$$

where F,g,f,h are continuous functions, g is representable by its Fourier cosine series,

$$f(0) = g(0) \text{ and } h(0) = \int_0^1 g(x)\,dx.$$

In order to discuss this problem, we make use of its linearity to reduce it to the consideration of several simpler problems. These problems are designated as Problems H_i $(i = 0, 1, 2, 3, 4)$ and are given in the following tabular array.

Problem:	H_0	H_1	H_2	H_3	H_4	
$u_{xx} - u_t =$	0	0	0	0	$F(x,t)$	on R
$u(x,0) =$	0	0	0	$g(x)$	0	$0 \leqslant x \leqslant 1$
$u(0,t) =$	0	0	$f(t)$	$g(0)$	0	$t \geqslant 0$
$\int_0^1 u(x,t)\,dx =$	0	$h(t)$	0	$\int_0^1 g(x)\,dx$	0	$t \geqslant 0$
		$h(0) = 0$	$f(0) = 0$			

[1] Iowa State University.

[2] Bolie, Victor W. On Impedance Theory for Bioelectrodes (unpublished).

We first show that if there is a solution of Problem D, then it is unique. For if $u_1(x,t)$ and $u_2(x,t)$ are two solutions, then the difference function $u(x,t) = u_1(x,t) - u_2(x,t)$ must be a solution of Problem H_o. We show that Problem H_o possesses only the trivial solution.

The proof of the uniqueness depends on a theorem due to E. C. Titchmarsh[3] which we quote without proof.

Theorem (Titchmarsh).

If i) $f(t)$ and $g(t)$ are Lebesgue integrable functions,

ii) $\int_0^t f(s)g(t-s)ds = 0$ almost everywhere for $0 < t < k$,

then $f(t) = 0$ a.e. for $0 < t < a$ and $g(t) = 0$ a.e. for $0 < t < b$ where $a + b \geqslant k$.

The Lebesgue integral is used in the above theorem, however this theorem is applicable to Riemann integrals when the functions involved are such that their absolute values are Riemann integrable.

We will also use the solutions of the following two classical problems.

Problem C_o		*Problem C_1*	
$u_{xx} - u_t = 0$	on R,	$u_{xx} - u_t = 0$	on R,
$u(x,0) = 0$	$0 \leqslant x < t$,	$u(x,0) = 0$	$0 \leqslant x < 1$
$u(0,t) = f_o(t)$	$t > 0$,	$u(0,t) = 0$	$t \geqslant 0$
$u(1,t) = 0$	$t \geqslant 0$.	$u(1,t) = f_1(t)$	$t > 0$.

In general the solutions of these problems are continuous on $\overline{R}$ only if $f_o(0) = 0$ for Problem C_o and $f_1(0) = 0$ for Problem C_1.

Theorem 1.

If i) $f_1(t)$ is continuous for $t \geqslant 0$,

ii) $Q(x,t) = \sum_{n=1}^{\infty} 2n\pi(-1)^{n+1}\exp(-n^2\pi^2 t)$ sin, $n\pi x$,

iii) $u(x,t) = \int_0^t f_1(s)Q(x,t-s)ds$,

then u is the solution of Problem C_1 where it is understood that $u(x,0) = \lim_{t \to 0^+} u(x,t)$ and $u(1,t) = \lim_{x \to 1^-} u(x,t)$. If $f_1(0) = 0$, then these limits also hold at $(1,0)$.

The proof of this theorem is well known: the details are given by Epstein[4].

The function Q used above is related to the well known Jacobi theta functions and in fact $Q(x,t) = -\frac{1}{2}\frac{\partial \theta_3}{\partial x}\left(\frac{1-x}{2},t\right)$ where θ_3

[3] Titchmarsh, E. C. The zeros of certain integral functions. Proc. London Math. Soc. 25: 283-302 (1926).

[4] Epstein, Bernard. Partial differential equations, New York, N. Y., McGraw-Hill Book Co. (1962) see pp. 227-231.

is the theta function of type 3. Because of the highly singular nature of Q at $t = 0$, care must be taken in differentiating or integrating the function $u(x,t)$ of Theorem 1. The following Lemma shows that it is permissible to integrate this integral representation of u and then interchange the order of integration.

Lemma 1.

If i) $f(t)$ is continuous for $t \geqslant 0$ and $f(0) = 0$,

If i) $f(t)$ is continuous for $t \geqslant 0$ and $f(0) = 0$,

$$\text{ii) } u(x,t) = f * Q = \int_0^t f(s)Q(x,t-s)ds,$$

$$\text{then } \int_0^1 u(x,t)dx = \int_0^t f(s)\left(\int_0^1 Q(x,t-s)dx\right)ds \text{ for } t \geqslant 0.$$

Proof. We first show that this lemma is valid for $f(t) = t^n$ for any positive integer n and then extend this result to any continuous function $f(t)$ for which $f(0) = 0$ by use of the Weierstrass approximation theorem.

If we let

$$(1) \qquad v_n(x,t) = t^n * Q(x,t) = \int_0^t s^n Q(x,t-s)ds,$$

then v_n is the solution of Problem C_1 with $f_1(t) = t^n$. By the *maximum* principle (see Epstein, p. 226), we have $|v_n(x,t)| \leqslant t^n$ for $0 \leqslant x \leqslant 1$ and therefore $v_n(x,t)$ has a Laplace transform which is uniformly convergent in x for $0 \leqslant x \leqslant 1$. If we define

$$(2) \qquad V_n(x,s) = L\{v_n(x,t)\} = \int_0^\infty e^{-st} v_n(x,t)dt,$$

then an application of the convolution theorem gives

$$(3) \qquad V_n(x,s) = L\{t^n * Q(x,t)\} = \frac{n!}{s^{n+1}} \; \frac{\sinh x\sqrt{s}}{\sinh\sqrt{s}}$$

Since $V_n(x,s)$ is uniformly convergent in x, we have

$$(4) \qquad \int_0^1 V_n(x,s)dx = \int_0^\infty e^{-st}\left(\int_0^1 v_n(x,t)dx\right)dt$$

$$= \frac{n!}{s^{n+1}} \; \frac{\cosh\sqrt{s} - 1}{\sqrt{s}\,\sinh\sqrt{s}}.$$

If we make use of some of the results of Doetsch[5] to observe that

$$(5) \qquad L^{-1}\left\{\frac{\cosh\sqrt{s} - 1}{\sqrt{s}\,\sinh\sqrt{s}}\right\} = \theta_3(0,t) - \theta_3(\tfrac{1}{2},t) = R(t),$$

[5] Doetsch, Gustav. Theorie and Anwendung der Laplace-Transformation. Berlin, Julius Springer (1937).

we may take the inverse transform of equations (4) to get

$$(6) \qquad \int_0^1 v_n(x,t)dx = \int_0^t s^n R(t-s)ds.$$

Then since $Q(x,t) = -\frac{1}{2}\frac{\partial\theta_3(\frac{1-x}{2},t)}{\partial x}$, we have

$$(7) \int_0^1 Q(x,t)dx = -\frac{1}{2}\int_0^1 \frac{\partial\theta_3(\frac{1-x}{2},t)}{\partial x}dx = R(t).$$

Substitution of this result into (6) establishes the lemma for $f(t) = t^n$, n a positive integer, and in fact for any polynomial in t which vanishes at $t = 0$.

To extend this result to any continuous function $f(t)$ such that $f(0) = 0$, we use the Weierstrass theorem to uniformly approximate $f(t)$ by a sequence of polynomials in the interval $0 \leqslant t \leqslant c$ for any $c > 0$. Hence, given any $\varepsilon>0$ there is a $\rho\varepsilon(t)$ such that $|f(t) - \rho\varepsilon(t)| < \varepsilon$ for $\leqslant t \leqslant c$. In order to show that

$$(8) \qquad \int_0^1\int_0^t f(s)Q(x,t-s)dsdx = \int_0^t f(s)R(t-s)ds$$

holds for $0 \leqslant t \leqslant c$, we consider

$$(9) \qquad \left|\int_0^1\int_0^t f(s)Q(x,t-s)dsdx - \int_0^t f(s)R(t-s)ds\right|.$$

Since the interchange of orders of integration is permissible for f replaced by the polynomial $\rho\varepsilon$, we may write (9) in the form

$$(10) \quad \left|\int_0^1\int_0^t (f(s)-\rho\varepsilon(s))Q(x,t-s)dsdx + \int_0^t (\rho\varepsilon(s)-f(s))R(t-s)ds\right|.$$

The integral $\int_0^t (f(s) - \rho\varepsilon(s))Q(x,t-s)ds$ is the solution of Problem C_1 with f_1 replaced by $f - \rho\varepsilon$ and consequently its absolute value is less then or equal to $|f(t) - \rho\varepsilon(t)$ by the maximum principle.

We next show that $\int_0^t |R(t-s)|ds \leqslant \frac{2}{3}$. From the definition of

$\theta_3 = 1 + 2\sum_{n=1}^{\infty} \exp(-n^2\pi^2 t)\cos 2n\pi x$ we have that

$$(11)\ R(t) = \theta_3(0,t) - \theta_3(\tfrac{1}{2},t) = 4\sum_{k=1}^{\infty} \exp\left[-(2k-1)^2\pi^2 t\right].$$

Now

$$(12)\qquad \int_0^t R(s)ds = \lim_{a\to 0}\int_a^t R(s)ds$$

$$= \lim_{a\to 0} 4\sum_{k=1}^{\infty} \frac{\exp\left[-(2k-1)^2\pi^2 a\right] - \exp\left[-(2k-1)^2\pi^2 t\right]}{(2k-1)^2\pi^2}$$

$$= 4\sum_{k=1}^{\infty} \frac{1-\exp\left[-(2k-1)^2\pi^2 t\right]}{(2k-1)^2\pi^2}$$

$$\leq \frac{4}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2} \leq \frac{2}{3}.$$

If we use these results in conjunction with (10), we can get a bound on (9).

$$(13)\qquad \left|\int_0^1\int_0^t f(s)Q(x,t-s)dsdx - \int_0^t f(s)R(t-s)ds\right|$$

$$\leqslant \int_0^1 |f(t)-\rho\varepsilon(t)|\,dx + \int_0^t |f(t)-\rho\varepsilon(t)|\,|R(t-ds)|$$

$$\leqslant \varepsilon + \frac{2}{3}\varepsilon = \frac{5}{3}\varepsilon, \qquad ((0\leqslant t\leqslant c).$$

Since ε is arbitrarily small, this establishes equation (8).

We are now in position to prove uniqueness.

Theorem 2. The only solution of Problem H_o is the trivial solution $u(x,t) = 0$ on $\bar{R}$.

Proof. Suppose there is a non-trivial solution $u(x,t)$. Then by the maximum principle, the function $u(l,t)$ is not identically zero. Now u is the unique solution of Problem C_1 with $f_1(t) = u(l,t)$ and so $u(x,t) = u(l,t) * Q(x,t)$. Then by Lemma 1, we have $\int_0^1 u(x,t)dx = u(1,t) * R(t)$. But by hypothesis $\int_0^1 u(x,t)dx = 0$ for $t \geqslant 0$. Now $R(t)$ is a positive function for $t > 0$ such that $\int_0^t R(s)ds$ exists for all t as shown in the proof Lemma 1 and $u(l,t)$ is a continuous function. Hence by Titchmarsh's theorem, $u(l,t) = 0$ for $t \geqq 0$. This contradicts the assumption that $u(x,t)$ is not identically zero and completes the proof of the theorem.

We now consider the question of existence of solutions for Problem D, and as suggested earlier we shall use the superposition principle to build a solution from the solutions of problems H_1, H_2, H_3, H_4. The most novel feature of these problems is the integral condition and we consider it in some detail.

It would seem reasonable that the continuity of $h(t)$ should be sufficient for the existence of a solution for Problem H_1. However this is not the case.

Theorem 3. There is a continuous function u which satisfies Problem H_1 if and only if there exists a continuous function $H(t)$ such that $h = H * R$. (* is convolution)

Proof. Suppose there exists a continuous solution u of Problem H_1. Then $u = u(l,t) * Q(x,t)$, and Lemma 1 implies that $\int_0^1 u(x,t)dx = u(l,t) * R(t)$. Hence we may take $H(t) = u(l,t)$.

Conversely, suppose H is continuous and such that $h = H*R$. Then $u(x,t) = H(t)*Q(x,t)$ is a solution of Problem H_1 since Theorem 1 assures us that u satisfies the homogeneous diffusion equation and the condition $u(x,0) = 0$, $u(0,t) = 0$. Then by Lemma 1, we have $\int_0^1 u(x,t)dx = H(t) * R(t) = h(t)$.

At this stage we might ask the question as to what class of functions is representable by $H * R$ for continuous H. The following theorem gives a partial answer to this question.

Theorem 4. If $h(t)$ is such that $h(0) = 0$ and its first derivative is bounded and integrable on every finite interval, then there is a continuous function H such that $h = H * R$.

Proof. We show that

$$H(t) = \int_0^t h'(s)[\theta_3(0,t-s) + \theta_3(\tfrac{1}{2},t-s)]ds$$

is such that $h = H * R$. For convenience we set $\theta_3(0,t) = a(t)$ and $\theta_3(\tfrac{1}{2},t) = b(t)$, so that $R = a - b$ and $H = h' * (a + b)$. Then from the properties of the convolution integral, we have

$$H * R = h' * (a + b) * (a - b) = h' * (a * a - b * b).$$

Again, by the convolution theorem we have that $L(a *) = L(a) \cdot L(a)$. (L is the Laplace transform). So making use of the results from Doetsch[6],

[6] op. cit., pp. 141-144.

so that

$$L(a) = \frac{\cosh \sqrt{s}}{\sqrt{s}\ \sin h \sqrt{s}}, \qquad L(b) = \frac{1}{\sqrt{s}\ \sin h \sqrt{s}},$$

we have

$$L(a * a) = \frac{\cosh^2 \sqrt{s}}{s \sin h^2 \sqrt{s}} \quad \frac{1}{s} \quad \frac{1}{s \sin h^2 \sqrt{s}}$$

$$= L(1) + L(b * b)$$

Hence $a * a = 1 + b * b$, and it follows that

$$H * R = h' * 1 = \int_0^t h(s)ds = h(t) - h(0) = h(t).$$

From the previous two theorems, we see that if h has a first derivative which is bounded and integrable on every finite subinterval and if $h(0) = 0$, then the solution in Problem H_1 is given by

$$u(x,t) = h * [a + b] * Q.$$

Problem H_2 is solved by observing that if u is the solution to Problem C_o with $f_o(t) = f(t)$, then $v(x,t) = u(1\text{–}x,t) - u(x,t)$ satisfies the differential equation, initial and boundary condition of Problem H_2 and

$$\int_0^1 v(x,t)dx = \int_0^1 [u(1-x,t) - u(x,t)]dx = 0.$$

To solve Problem H_3, we assume that in addition to being continuous, g is representable in a Fourier cosine series

$$g(x) = \sum_{k=0}^{\infty} a_k \cos k\pi x, \qquad (0 \leq x \leq 1).$$

Then the function

$$w(x,t) = \sum_{k=0}^{\infty} a_k \exp(-k^2\pi^2 t)\cos k\pi x$$

satisfies the diffusion equation and the initial condition $w(x,o) = g(x)$. Further, since this series may be integrated termwise, we have

$$\int_0^1 w(x,t)dx = \sum_{k=0}^{\infty} a_k \exp(-k^2\pi^2 t)\int_0^1 \cos k\pi x \ dx = a_o.$$

Since $a_o = \int_0^1 g(x)dx$, the function $u(x,t) = w(x,t) - v(x,t)$ is the solution of Problem H_3, where $v(x,t)$ is the solution of Problem H_2 with $f(t) = -g(0) + \sum_{k=0}^{\infty} a_k \exp(-k^2\pi^2 t)$.

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In order to show that there exists a unique solution of Problem H_4 we need the following two lemmas which we state without proof.

Lemma 2. If $f(y)$ is a continuous even periodic function of period 2 for all y, then $h(y) = \int_0^1 f(x-y)dx - \int_0^1 f(x)dx$ is an odd periodic function of period 2.

Lemma 3. If i) $f(x,t)$ is continuous for all x and $t \geqslant 0$,

ii) $f(x,t)$ is an even periodic function of period 2 in x for each $t \geqslant 0$,

$$\text{iii)} \quad u(x,t) = -\int_0^t \int_{-\infty}^{\infty} f(y,s)K(x - y, t - s)dyds,$$

$$\text{where} \quad k(x,y) = \frac{\exp(-x^2/4t)}{\sqrt{4\pi t}}$$

then $\int_0^1 u(x,t)dx = -\int_0^t \int_0^1 f,(x,s)dxds.$

Theorem 5. If $F(x,t)$ is continuous on $\overline{R}$, then there exists a unique solution of Problem H_4.

Proof. The solution is unique by Theorem 2. If we let $F^*(x,t)$ be the even extension on x of $F(x,t)$ for each t to the interval $-1 \leqslant x \leqslant 1$, and $f(x,t)$ the periodic extension of $F^*(x,t)$ for all x, then $f(x,t)$ satisfies the conditions of Lemma 2. Moreover. if we let $h(t) = \int_0^t \int_0^1 f(x,s)dxds$, then h satisfies the conditions of Theorem 4 and thus there is a solution $w(x,t)$ of Problem H_1 such that $\int_0^1 w(x,t)dx = h(t)$. Thereforeo the function u given by

$$u(x,t) = -\int_0^t \int_{-\infty}^{\infty} f(y,s)K(x-y, t-s)dyds + w(x,t)$$

is the solution of Problem H_4

The solution of Problem D can now be constructed by superposition of solutions of problems similar to H_i ($i = 1, 2, 3, 4$).