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Abstract. Consider an n-cell containing the origin and suppose its boundary S^{n-1} has the following property: every point on S^{n-1} contains a neighborhood in S^{n-1} such that any ray from the origin intersecting this neighborhood intersects it exactly once. Then it is shown by using the concept of a covering space that any ray from the origin intersects S^{n-1} exactly once.

In some recent work of the author in differential equations, the following geometric problem arose. Suppose the surface S^{n-1} of an n-cell E^n which contains the origin in such that for each point $x \varepsilon S^{n-1}$ there is a surface neighborhood N_x containing x such that if $y \varepsilon N_x$, the ray from the origin through y intersects N_x exactly once. Then it is true that every ray from the origin intersects S^{n-1} exactly once? The answer is in the affirmative, and it can be proved directly. It is the purpose of this note, however, to formulate the problem as an application of covering spaces. The following two definitions are as given in Chevalley's "Theory of Lie Groups I," p. 40. We assume the topological spaces involved are locally connected; i.e., each neighborhood of a point contains a connected neighborhood of that point.

Definition 1. Let f be a continuous function on a space $\tilde{V}$ to a space V. The set $E \subset V$ is said to be evenly covered by $\tilde{V}$ with respect to f if $f^{-1}(E)$ is not empty, and every component of $f^{-1}(E)$ is mapped homeomorphically onto E by f.

Definition 2. A covering space $(\tilde{V}, f)$ of a topological space V is a pair such that $\tilde{V}$ is a connected topological space, and f is a continuous function of $\tilde{V}$ onto V which has the property that each point of V has a neighborhood which is evenly covered by $\tilde{V}$ with respect to f.

We state some further definitions.

Definition 3. Let S be a subset of R^n, the n-space over the reals. A neighborhood in S of the point $x \varepsilon S$ is any set of the form $D_x \cap S^{n-1}$, where D_x is an open set in R^n containing x.

Definition 4. A set $S \subset R^n$ is r-convex if for each $x \varepsilon S$ such that $x \neq 0$, the set of points $\{\lambda x\}$, $\lambda \geqq 0$, satisfies $\{\lambda x\} \cap S = x$.

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Definition 5. A set $S \subset R^n$ is locally r-convex if for each $x \varepsilon S$, there exists a neighborhood N_x in S containing x such that N_x is r-convex.

We assume the topology in R^n to be defined by a norm, and denote by $|x|$ the norm of $x \varepsilon R^n$; since all norms for R^n are equivalent, it is no loss of generality to assume this norm is Euclidean norm.

It is known that the boundary of any n-cell in R^n is locally connected.

Let E^n be an n-cell containing the origin, and let S^{n-1} be its boundary. Let K^{n-1} be the unit sphere in R^n; i.e., the set of x such that $|x| = 1$. Define the function of f on S^{n-1} to K^{n-1} by $f(x) = x/|x|$. Clearly f takes S^{n-1} onto K^{n-1} and is continuous.

We now show that if S^{n-1} is locally r-convex (S^{n-1}, f) is a covering space of K^{n-1}. Fix $y \varepsilon K^{n-1}$; since the set $\{\lambda y\} \cap S^{n-1}$, $\lambda \geqq 0$, is compact in R^n, $f^{-1}(y)$ is compact since, in fact, $f^{-1}(y) = \{\lambda y\} \cap S^{n-1}$, $\lambda \geqq 0$. Since S^{n-1} is locally r-convex, there exists for each point $x \varepsilon f^{-1}(y)$ a neighborhood D_x in R^{n-1} such that if $N_x = D_x \cap S^{n-1}$, then f is one-to-one on N_x. By the Brouwer "Theorem on Invariance of Domain" (cf. Hurewicz and Wallman, "Dimension Theory," p. 95) $f(N_x)$ is a neighborhood in K^{n-1} of y. By the Heine-Borel theorem, there exists a finite set $\{x_k\}$, $k = 1,2$ . . . , m, of points, $x_k \varepsilon f^{-1}(y)$ such that $\underset{k=1}{\overset{m}{}} D_{xk} \supset f^{-1}(y)$. Using the fact that fis one-to-one on N_{xk}, $k = 1,2, \ldots$, it follows that $f^{-1}(y)$ consists of the finite set $\{x_k\}$, $k = 1, \ldots, m$. We now define $N_y = \bigcap_{k=1}^{m'} f(N_{xk})$; N_y is clearly a neighborhood in K^{n-1} of y, and each component of $f^{-1}(N_y)$ is mapped homeomorphically onto N_y by f; this also follows readily from the Brouwer theorem mentioned above. Hence (S^{n-1}, f) is a covering of K^{n-1}. We will now apply the following result (Lemma 1, p. 45, Chevalley, ibid.):

If $(\tilde{V}, f)$ is a covering space of V, and A is an open subset of $\tilde{V}$ which is mapped one-to-one onto V by f, then f is a homeomorphism of $\tilde{V}$ with V, an din fact, $A = \tilde{V}$.

We define our set A as follows: let $y \varepsilon K^{n-1}$, and denote by $g(y)$ the point of $f^{-1}(y)$ nearest the origin. This point $g(y) \varepsilon S^{a-1}$ clearly exists, since $f^{-1}(y)$ is finite. We denote by A the set $\{g(y)\}$, $y \varepsilon K^{n-1}$. Suppose A is not open in S^{n-1}; then there exists an $x \varepsilon A$, $x_k \varepsilon A$, such that $x_k \rightarrow x$ as $k \rightarrow \infty$. Hence, by the definition of A, there exists for each k, a point $y_k \varepsilon A$ such that $|y_k| < |x_k|$, and $f(x_k) = f(y_k)$. Since S^{n-1} is compact, it is no loss of generality to suppose that $y_k \to y \varepsilon S^{n-1}$ as $k \to \infty$. If $y \neq x$, then clearly $|y| < |x|$, a contradiction, since $x \varepsilon A$. On the other hand, if $y = x$, there exist in each neighborhood in S^{n-1} of x distinct points x_k, y_k for k sufficiently large; this contradicts the hypothesis that S^{n-1} is locally r-convex. Thus A must be open, and since clearly $f(A), = K^{n-1}$, the above lemma applies. Since $A = S^{n-1}$, it follows that S^{n-1} is r-convex.

A number of short remarks are in order. It is not true that a compact locally r-convex set in R^n is necessarily r-convex; for example, consider a spiral in the R^2 plane about the origin.

On the other hand, a compact locally r-convex set in R^n is a locally connected space; this can be proved by arguments along the lines of the central one in the proof that (S^{n-1}, f) is a covering space of K^{n-1}. This suggests that S^{n-1} could be more general than the boundary of an n-cell; in fact; if S^{n-1} is a compact (n-1)-surface in R^n which is the boundary of a domain in R^n containing the origin, then its local r-convexity implies its r-convexity.

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