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# Some Short-Cut Adaptations of Complex Numbers in Intermediate Physics

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*Abstract.* A few expedient, useful and well known, but seldom fully utilized adaptations of complex numbers are developed. These should interest teachers and students of intermediate physics courses.

The following items are fundamental:

(i) Any "complex number," c, can be written in the form

$$c = \mathrm{Re}(c) + j\,\mathrm{Im}(c) \tag{1}$$

where $j = (-1)^{1/2}$ and Re(c) and Im(c) are real numbers.

(ii) The number Re(c) is called the "real part of c" and the number Im(c) is called the "imaginary part of c."

(iii) The number $c^*$ is called the "complex conjugate of c" and can be written

$$c^* = \mathrm{Re}(c) - j\,\mathrm{Im}(c). \tag{2}$$

(iv) The quantity $|c|$ is used to denote the "amplitude of c" and can be characterized by

$$|c| = \left\{[\mathrm{Re}(c)]^2 + [\mathrm{Im}(c)]^2\right\}^{1/2} \tag{3}$$

or more conveniently

$$|c| = (cc^*)^{1/2}.$$

(v) The quantity $\Phi_c$ is used to denote the "phase of c" and is characterized by

$$\Phi_c = \tan^{-1}\left(\frac{\mathrm{Im}(c)}{\mathrm{Re}(c)}\right) \quad (-\pi < \Phi_c \leq \pi). \tag{4}$$

## THE EXPONENTIAL FORM

We wish to show that any complex number can be written in an exponential form

$$c = |c|\, e^{j\Phi_c}. \tag{5}$$

In order to show (5), the real and imaginary parts of c are chosen as

$$\mathrm{Re}(c) = |c| \cos\Phi_c \text{ and } \mathrm{Im}(c) = |c| \sin\Phi_c. \tag{6}$$

Then,

$$c = |c| \cos\Phi_c + j\,|c| \sin\Phi_c, \tag{7}$$

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which is sometimes referred to as the polar form of a complex number. Now from (7),

$$\frac{d(c/|c|)}{d\,\Phi_c} = -\sin\Phi_c + j\cos\Phi_c = j\,(c/|c|)\ . \tag{8}$$

Integrating, we have the desired result,

$$c = |c|\, e^{j\Phi_c},$$

where the integration constant has been determined to satisfy the condition from (7) that $c = |c|$ when $\Phi_c = 0$. When (5) is written in terms of its real and imaginary parts we have

$$Re(c) + j\,Im(c) = \left\{[Re(c)]^2 + [Im(c)]^2\right\}^{1/2} e^{j\tan^{-1}\left[\frac{Im(c)}{Re(c)}\right]} \tag{5-a}$$

Note, neither an Argand diagram nor a series expansion of sine and cosine was necessary to obtain (5). Similarly, it can be shown that

$$c^* = |c|\, e^{-j\Phi_c}. \tag{9}$$

Results such as

$$j = e^{j\pi/2} \tag{10}$$

and

$$\cos\theta + j\sin\theta = e^{j\theta} \tag{11}$$

follow immediately from (5). In (10) we see that multiplication of any complex number by j simply produces a positive phase shift of $\frac{\pi}{2}$ in that number. In (11) we have an expression convenient for proving numerous trigonometric relationships.

## A Particular Solution to the Second Order Linear Differential Equation With Constant Coefficients

The exponential form of a complex number lends itself readily to short cuts in physical applications involving sinusoids. For instance, a particular solution to the differential equation

$$\frac{d^2x}{dt^2} + 2a\frac{dx}{dt} + \omega_o^2 x = A\cos(\omega t + \phi) \tag{12}$$

is readily found. First consider the differential equation

$$\frac{d^2x_c}{dt^2} + 2a\frac{dx_c}{dt} + \omega_o^2 x_c = Ae^{j(\omega t + \phi)} \tag{13}$$

where $x_c$ is obviously complex. By equating the real parts on both sides in (13) we have[3],

$$\frac{d^2[\mathrm{Re}(x_c)]}{dt^2} + 2a\frac{d[\mathrm{Re}(x_c)]}{dt} + \omega_o^2[\mathrm{Re}(x_c)] = A\cos(\omega t + \phi). \qquad (14)$$

Then comparing (14) to (12),

$$x = \mathrm{Re}(x_c). \qquad (15)$$

Now, if a particular solution

$$x_c = x_c(t) \qquad (15)$$

can be found to (13) a solution to (12) is readily found from (15), that is,

$$x(t) = \mathrm{Re}[x_c(t)]. \qquad (15\text{-a})$$

From the exponential nature of the non-homogeneous term in (13), a solution of the form

$$x_c = X \quad e^{j\omega t} \qquad (16)$$

might be expected. Substitution of this proposed solution intc (13) yields,

$$-\omega^2X + j2\alpha\omega X + \omega^2{}_0X = Ae^{j\Phi}.$$

Therefore (16) is a particular solution provided

$$X = \frac{Ae^{j\Phi}}{(\omega^2{}_0 - \omega^2) + j2\alpha\omega\,.} \qquad (17)$$

Employing equation (5-a) with respect to the denominator of (17) and combining exponentials

$$X = \frac{A}{\sqrt{(\omega_o^2 - \omega^2)^2 + 4a^2\omega^2}}\, e^{j[\phi - \tan^{-1}(\frac{2a\omega}{\omega_o^2 - \omega^2})]} = |X|e^{j\phi_X}. \qquad (17\text{-a})$$

Returning to the original problem of finding a particular solution to (12),

$$x = \mathrm{Re}(x_c) = \mathrm{Re}\,[Xe^{j\omega t}].$$

Employing (17-a),

$$x = \mathrm{Re}\left\{\frac{A}{\sqrt{(\omega_o^2 - \omega^2)^2 + 4a^2\omega^2}}\, e^{j[\omega t + \phi - \tan^{-1}(\frac{2\alpha\omega}{\omega_o^2 - \omega^2})]}\right\}.$$

[3] Note that since

$$\frac{dx_c}{dt} = \frac{d[\mathrm{Re}(x_c) + j\,\mathrm{Im}(x_c)]}{dt} = \frac{d[\mathrm{Re}(x_c)]}{dt} + j\frac{d[\mathrm{Im}(x_c)]}{dt},$$

then

$$\mathrm{Re}\left(\frac{dx_c}{dt}\right) = \frac{d[\mathrm{Re}(x_c)]}{dt}.$$

Finally,

$$x = \frac{A}{\sqrt{(\omega_o^2 - \omega^2)^2 + 4a^2\omega^2}} \cos\left[\omega t + \Phi - \tan^{-1}\left(\frac{2a\omega}{\omega_o^2 - \omega^2}\right)\right]. \quad (18)$$

In terms of X we can write (18) conveniently as

$$x = |X| \cos (\omega t + \Phi_X) .$$

It is important to note that only a knowledge of X is necessary to determine the solution, since X determines both the amplitude and the phase constant. Note that this particular solution contains no arbitrary constants. Consequently, being independent of any initial conditions of the motion, the solution (18) cannot represent a complete or general solution. Since this part of the complete solution neither dies out with time nor depends upon the initial conditions, (18) is called the "steady state solution" of (12). The solution (18) fails when $\alpha = 0$ and $\omega = \omega_0$. In this case, however, a solution of the form

$$x_c = X \; t \; e^{j\omega t}$$

is substituted for (16) and will yield a particular solution. The technique is essentially the same as that following (16).

## Application to AC Circuits

Consider now the steady state solution for a.c. circuits having sinusoidal signals. For a loop containing a resistance R, capacitance C, inductance L and emf $\varepsilon \cos (\omega t + \phi)$, Kirchoff's law (conservation of energy) leads to the differential equation

$$\frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{1}{LC} q = \frac{\epsilon}{L} \cos (\omega t + \phi) . \quad (20)$$

This equation is of the same form as (12). By analogy to (18),

$$q = \frac{\epsilon / L}{\sqrt{(\frac{1}{LC} - \omega^2)^2 + (\frac{R}{L}\omega)^2}} \cos\left[\omega t + \phi - \tan^{-1}\left(\frac{\frac{R}{L}\omega}{\frac{1}{LC} - \omega^2}\right)\right] \quad (21)$$

and

$$i = \frac{dq}{dt}. \quad (22)$$

We wish to proceed, however, in another manner for reasons which will be obvious later. By analogy to (17),

$$Q = \frac{\frac{\epsilon}{L} e^{j\phi}}{\left(\frac{1}{LC} - \omega^2\right) + j\frac{R}{L}\omega} \quad . \tag{23}$$

Since[4]

$$i_c = \frac{dq_c}{dt} = j\omega Q e^{j\omega t} = I e^{j\omega t} \ ,$$

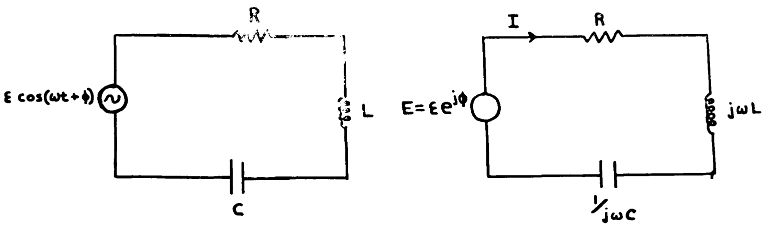
then

$$I = j\omega Q = \frac{j\left(\frac{\omega}{L}\right)\epsilon e^{j\phi}}{j\left(\frac{\omega}{L}\right)R - \left(\omega L - \frac{1}{\omega C}\right)\frac{\omega}{L}} \tag{24}$$

Multiplying numerator and denominator of (24) by $-\frac{jL}{\omega}$ yields

$$I = \frac{E}{R + j\,(\omega L - 1/\omega C)} \tag{25}$$

where $E = \varepsilon\, e^{j\phi}$. Note I is independent of time.[5] Compare the original circuit with the "complex circuit" in Fig. 1 below.




Original circuit Complex xcircuit

Figure 1.

If we apply Kirchhoff's law to the complex circuit,

$$E = I\,(R + j\omega L + 1/j\omega C)$$

or

$$I = \frac{E}{R + j\,(\omega L - 1/\omega C)}$$

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Hence, we may write $i_c = \frac{dq_c}{dt}$ since we are ultimately interested only in the real parts of the complex quantities.

$$\mathrm{Re}\left(\frac{dq_c}{dt}\right) = \frac{d[\mathrm{Re}(q_c)]}{dt} = \frac{dq}{dt} = i = \mathrm{Re}(i_c)$$

[5] Although the reader should appreciate at this point that $i = |I| \cos(\omega t + \Phi_I)$ will yield (22), this is not the motivation for arriving at (25).

which is equation (25). This suggests that the problem of solving for the steady state in a.c. networks with sinusoidal signals might be reduced to a problem of only slightly greater mathematical difficulty than that of solving d.c. networks. This was our sole motivation for obtaining (25). Multiloop networks could be handled similarly by using circulating currents $I_1$, $I_2$, . . . , $I_N$ for each of the N "complex loops." Nonsinusoidal signals could be treated by Fourier analysis. Remember that this technique yields only the steady state solutions. A summary of the technique appears as follows.

1) Write the original circuit in terms of:
   a) the resistance R
   b) the complex inductive reactances, $j\omega L$
   c) the complex capacitive reactances, $1/j\omega L$
   d) the complex currents, I
   e) the amplitudes and phase constants of the sinusoidal sources of emf, $E = \varepsilon e^{j\phi}$
2) Write Kichhoff's loop equations for the "complex network" and solve for the unknown complex quantities.
3) Make use of the exponential form,
   $$c = |c| e^{j\Phi_c}$$
   to obtain the amplitude $|c|$ and phase $\Phi_c$ of the desired unknown complex quantities. Since no information remains to be gained about the sinusoids in the steady state, the problem is solved.

### Addition of N Sinusoids of the Same Frequency

Consider now the addition of N sinusoidal functions of various amplitudes and phase constants but all of the same frequency. The reader can readily show that the sum of N sinusoids of a certain frequency will be sinusoid of that same frequency[6]. The problem is then to determine A and $\alpha$ in the equation

$$A_1\cos(\omega t + \alpha_1) + A_2\cos(\omega t + \alpha_2) + \cdots + A_N\cos(\omega t + \alpha_N)$$
$$= A\cos(\omega t + \alpha) \quad . \tag{26}$$

Consider instead,

[6] This conclusion is justified by using the following result repeatedly.

$$A_1\cos(\omega t + \alpha_1) + A_2\cos(\omega t + \alpha_2) \ .$$

$$= (A_1\cos\alpha_1 + A_2\cos\alpha_2)\cos\omega t - (A_1\sin\alpha_1 + A_2\sin\alpha_2)\sin\omega t$$

$$= \sqrt{A_1^2 + A_2^2 + 2\cos(\alpha_2 - \alpha_1)}\cos\left[\omega t + \tan^{-1}\left(\frac{A_1\sin\alpha_1 + A_2\sin\alpha_2}{A_1\cos\alpha_1 + A_2\cos\alpha_2}\right)\right]$$

$$A_1e^{j(\omega t+\alpha_1)} + A_2e^{j(\omega t+\alpha_2)} + \cdots + A_Ne^{j(\omega t+\alpha_N)}$$
$$= Ae^{j(\omega t+\alpha)}.$$

Of course the quantities in the latter expression bear the same relationship to each other as those in (26) as can readily be seen from the real part of both sides.

Dividing by $e^{j\omega t}$

$$A_1e^{j\alpha_1} + A_2e^{j\alpha_2} + \cdots + A_Ne^{j\alpha_N} = Ae^{j\alpha} \tag{27}$$

Multiply both sides of (27) by its complex conjugate.[7]

$$\left(\sum_{m=1}^{N} A_m e^{j\alpha_m}\right)\left(\sum_{n=1}^{N} A_n e^{-j\alpha_n}\right) = Ae^{j\alpha}Ae^{-j\alpha}$$

or

$$A^2 = \sum_{m=1}^{N}\sum_{n=1}^{N} A_mA_ne^{j(\alpha_m-\alpha_n)} \tag{28}$$

Taking the real part of both sides of (28) yields

$$A^2 = \sum_{m=1}^{N}\sum_{n=1}^{N} A_mA_n\cos(\alpha_m-\alpha_n). \tag{29}$$

Now in $Ae^{j\alpha}$

$$\alpha = \tan^{-1}\left(\frac{\operatorname{Im}(Ae^{j\alpha})}{\operatorname{Re}(Ae^{j\alpha})}\right)$$

by (4). But from (27)

$$\alpha = \tan^{-1}\left[\frac{\operatorname{Im}\sum_{m=1}^{N} A_m e^{j\alpha_m}}{\operatorname{Re}\sum_{m=1}^{N} A_m e^{j\alpha_m}}\right]$$

or finally,

$$\alpha = \tan^{-1}\left[\frac{\sum_{m=1}^{N} A_m\sin\alpha_m}{\sum_{m=1}^{N} A_m\cos\alpha_m}\right]$$

[7] The index n on the conjugate summation is used in order to provide appropriately for cross terms.

Note that in (29) there are N terms, for m=n, or of the form $A^2{}_m$. For m $\neq$ n there are $\frac{N(N-1)}{2}$ terms of the form $2A_mA_n$ cos $(\alpha_m - \alpha_n)$.

The authors are aware that the above results and even the general technique of obtaining these results can be found in a number of textbooks. We further feel, however, that the manner of obtaining these results (and others) are not always clearly and succinctly developed at the intermediate level if indeed developed by a complex number technique at all. Since in more advanced studies complex numbers provide for a conciseness and clarity that can scarcely be sacrificed, it is of great advantage to the student to become familiar with some of the fundamental techniques at the intermediate level of his studies in physics.