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# Engel Conditions on Groups


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*Abstract.* Let g,c denote positive integers. A group is said to have type $(g \to c)$ if every subgroup which can be generated by g elements is nilpotent of class at most c. A result of R. H. Bruck shows that groups of type $(4 \to 5)$ without elements of order 2 are nilpotent of class at most 7. In the present paper the following result is reported: If G is a $(4 \to 5)$ group on 5 generators without elements of order 2, then G is nilpotent of class at most 6.

Recent work by Kostrikin, [4], on the Burnside problem motivates the following definition: Let g,c denote positive integers. A group has type $(g \to c)$ if every subgroup which can be generated by g elements is nilpotent of class at most c. R. H. Bruck has studied groups of type $(4 \to 5)$ in connection with the Burnside problem. A result of Bruck shows that groups of type $(4 \to 5)$ without elements of order 2 are nilpotent of class at most 7. It has been conjectured that this upper bound on the nilpotency class is not best possible in the case of groups on 5 generators. The main result of this paper is the following: Theorem. If G is a $(4 \to 5)$ group on 5 generators without elements of order 2, then G is nilpotent of class at most 6.

## DEFINITIONS AND NOTATION

Let G be a group and let $(a,b) = a^{-1}b^{-1}ab$ for $a,b \in G$. Let

$$(a,b;0) = a; \ (a,b;n) = ((a,b;n-1),b).$$

If H, K are subgroups of G, let (H,K) be the subgroup generated by all commutators (h,k), where $h \in H$ and $k \in K$. The lower central series of G is a chain $\{G_n\}$ of subgroups defined by

$$G_1 = G; \quad G_{n+1} = (G_n,G).$$

If there exists a non-negative integer n such that $G_{n+1} = 1$, and if c is the least such n, then G is nilpotent of (exact) class c. The center of G is a subgroup Z defined by:

$$Z = \{a \in G: (a,x) = 1 \ \forall x \in G\}.$$

## PRELIMINARY LEMMAS AND THE MAIN THEOREM

Let G be a group generated by u,v,x,y,z. Order the generators by

$$u < v < x < y < z$$

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and form basic commutators as on page 166 of [2], so that the following holds:

$$f \varepsilon G_n \Longrightarrow f \equiv \prod_{c=1}^{k} c_i^{e_i} \bmod G_{n+1}, \text{ for } n \geq 1,$$

where the $c_i$ are basic commutators in $G_n$ and the $e_i$ are integers. Let B denote the set of all basic commutators of length 7. For all integers m,n,p,q,r that pertain, let Tm,n,p,q,r denote the subset of B consisting of all basic commutators of length 7 in which one generator occurs m times, another generator occurs n times, another p times, another q times, and the remaining generator occurs r times.

Lemma 1. If G is a (4→5) group on 5 generators without elements of order 2, then $T_{3,1,1,1,1} = 1$.

Proof. Since G is a (4→5) group we have in particular that

$$(a,b;5) = 1 \ \forall \ a,b \varepsilon G.$$

Then, since G has no elements of order 2, it follows that G/Z has no elements of order 2. Also, since G has type (4→5), it follows immediately that G/Z has type (3→4). Now Bruck, [1] p. 5.4 has proved that groups of type (g→2g–2) without elements of order 2 are nilpotent of class at most 3g–3. With g = 3, we conclude that G/Z is nilpotent of class at most 6, whence G is nilpotent of class at most 7. Thus commutators of length 8 reduce to the identity, and also, commutators of length 7 commute.

Next, we note that

(1.1) $$(v,u,u;x,u;z,y) = 1,$$

where (v,u,u;x,u;z,y) is the complex commutator

$$(((v,u,u),(x,u)),(z,y)) .$$

For let K be the subgroup generated by the four elements u,v,x, (z,y). Then (v,u,u) $\varepsilon K_3$, (x,u) $\varepsilon K_2$, and (z,y) $\varepsilon K_1$ in the lower central series of K. Therefore we have

$$(v,u,u;x,u;z,y) \ \varepsilon \ (K_3,K_2,K_1) \subset K_{3+2+1} = K_6 = 1.$$

We observe that (1.1) holds for any permutation of the arguments v,x,z,y

Taking K to be the subgroup generated by the four elements u,y,z,(v,x) we get the identity (v,x,u;y,u;z,u) = 1. In terms of basic commutators this is

(1.2) $$(v,u,x;y,u;z,u)\ (x,u,v;y,u;z,u)^{-1} = 1.$$

On the other hand, (vx,u,vx;y,u;z,u) = 1 by (4→5), whence

(1.3) $$(v,u,x;y,u;z,u)\ (x,u,v;y,u;z,u) = 1.$$

The factors in (1.2) and (1.3) commute, and hence we conclude that

$$(v,u,x;y,u;z,u)^2 = 1.$$

Since G has no elements of order 2, we conclude that

(1.4) $$(v,u,x;y,u;z,u) = 1.$$

In exactly similar ways, we find that

(1.5) $$(z,y;x,u;v,u,u) = 1,$$

(1.6) $$(z,u;y,u;v,u,x) = 1.$$

Now permuting v,x,y,z, in (1.1), (1.4), (1.5), (1.6) we find that all 3-2-2 complex commutators in $T_{3,1,1,1,1}$ in which u occurs 3 times must be 1. By successively interchanging u with v,x,y,z and changing to basic form, we find that all 3-2-2 commutators in $T_{3,1,1,1,1}$ reduce to the identity.

If we let K be the subgroup generated by the four elements u,v,x,(z,y) and then use (4→5), we find that

(1.7) $$(v,u,u,x;z,y) = 1.$$

By (4→5) we have

$$\begin{aligned} 1 &= (v,u,u,xy,xy;z,u) = (v,u,u,x,y;z,u)\ (v,u,u,y,x;z,u) \\ &= (v,u,u,x,y;y,u)^2(v,u,u;y,x;z,u) = (v,u,u,x,y;z,u)^2, \end{aligned}$$

whence

(1.8) $$(v,u,u,x,y;z,u) = 1.$$

Permuting u,v,x,y,z in (1.7) and (1.8) we find that all 5-2 complex commutators in $T_{3,1,1,1,1}$ reduce to 1.

By (4→5) we have

(1.9) $$(v,u,u,u;y,x,z) = 1.$$

Similarly we have $(v,u,ux;y,z,u) = 1$, which in basic form is

(1.10) $$(v,u,u,x;y,u,z)(v,u,u,x;z,u,y)^{-1} = 1.$$

Also, by (4→5) we have $(v,u,u,x;yz,u,yz) = 1$, whence

(1.11) $$(v,u,u,x;y,u,z)(v,u,u,x;z,u,y) = 1.$$

From (1.10), (1.11), and the fact that G has no elements of order 2, we conclude that

(2.12) $$(v,u,u,x;y,u,z) = 1.$$

Again by (4→5) we have

$$\begin{aligned} 1 &= (v,u,xy,xy;z,u,u) = (v,u,x,y;z,u,u)(v,u,y,x;u,u) \\ &= (v,u,x,y;z,u,u)^2(v,u;y,x;z,u,u) \\ &= (v,u,x,y;z,u,u)^2, \end{aligned}$$

whence

(1.13) $$(v,u,x,y;z,u,u) = 1.$$

Permuting u,v,x,y,z in (1.9),(1.12),(1.13) we see that all 4-3 complex commutators in $T_{3,1,1,1'1}$ reduce to 1.

At this point we have shown that all non-simple complex commutators in $T_{3,1,1,1,1}$ reduce to the identity. The only simple commutators in this set which are trivially 1 by $(4 \rightarrow 5)$ are those which are dealt with below.

Now $(z,u,v,y,y,y,x) = 1$ by $(4 \rightarrow)$, which in basic form is

(1.14) $\quad (z,u,v,x,y,y,y) = 1.$

We observe that (1.14) holds for any permutation of its arguments. This leaves only $(z,u,v,x,y,z,z)$ left to consider. By $(4 \rightarrow 5)$ we have $(z,u,v,x,z,z,y) = 1$, which in basic form yields

$$(z,u,v,x,y,z,z) = 1.$$

This completes the proof of Lemma 1.

Lemma 2. If G is a $(4 \rightarrow 5)$ group on 5 generators without elements of order 2, then $T_{2,2,1,1,1} = 1$.

Proof. Taking K to be the subgroup generated by the four elements $u,v,x,(z,y)$ and using $(4 \rightarrow 5)$ we find that

(2.1) $\quad (v,u,u;x,v;z,y) = 1.$

Similarly, we have

(2.2) $\quad (v,u,x;v,u;z,y) = 1$

(2.3) $\quad (x,u,v;v,u;z,y) = 1.$

By $(4 \rightarrow 5)$ we have

(2.4) $\quad (y,x,u;v,u;z,v) = 1,$

from which we get

(2.5) $\quad (y,u,x;v,u;z,v)(x,u,y;v,u;z,v)^{-1} = 1.$

Also by $(4 \rightarrow 5)$ we have $(xy,u,xy;v,u;z,v) = 1$, whence

(2.6) $\quad (y,u,x;v,u;z,v)(x,u,y;v,u;z,v) = 1.$

From (2.5), (2.6) we conclude that

(2.7) $\quad (y,u,x;v,u;z,v) = 1.$

Interchanging u and v in (2.7) we get

$$1 = (x,v,y;u,v;z,u) = (x,v,y;v,u;z,u)^{-1}.$$

whence

(2.8) $\quad (x,v,y;v,u;z,u) = 1.$

From $(4 \rightarrow 5)$ we have

(2.9) $\quad (y,x,z;v,u;v,u) = 1.$

Also from $(4 \rightarrow 5)$ we have $(vy,u,x;vy,u;z,vy) = 1$. Since G has no elements of order 2 we may use the linearization process of Heineken, [3], page 699, to obtain

$$(y,u,x;v,u;z,v)(v,u,x;y,u;z,v)(v,u,x;v,u;z,y) = 1.$$

In view of (2.2) and (2.7) we conclude that

(2.10) $\quad (v,u,x;y,u;z,v) = 1.$

Again from $(4 \rightarrow 5)$

$$\begin{aligned} 1 &= (x,u,vy;vy,u;z,vy) \\ &= (x,u,y;v,u;z,v)(x,u,v;y,u;z,v)(x,u,v;v,u;z,y). \end{aligned}$$

It follows from (2.6), (2.7), (2.3) that

$$(2.11) \qquad (x,u,v;y,u;z,v) = 1.$$

From Lemma 1 we have

$$\begin{aligned} 1 &= (x,uv,uv;y,uv;z,v) \\ &= (x,v,u;y,u;z,v)(x,u,v;y,u;z,v)(x,u,u;y,v;z,v) \\ &= (x,u,v;y,u;z,v)^2(v,u,x;y,u;z,v)^{-1}(x,u,u;y,v;z,v). \end{aligned}$$

Then, by (2.10) and (2.11), we conclude that

$$(2.12) \qquad (x,u,u;y,v;z,v) = 1.$$

Permuting arguments in (2.1)–(2.12) we find that all 3-2-2 complex commutators in $T_{2'2,1,1,1}$ in which the segment of length 3 comes first reduce to 1. Exactly similar arguments show that all the 3-2-2 complex commutators in which the segment of length 3 occurs last reduce to 1. Thus all the 3-2-2 complex commutators in $T_{2,2,1,1,1}$ reduce to 1.

From $(4 \rightarrow 5)$ we get

$$(2.13) \qquad (v,u,u,v,x;z,y) = 1,$$

$$(2.14) \qquad (x,u,u,v,v;z,y) = 1.$$

Also from $(4 \rightarrow 5)$ we have

$$\begin{aligned} 1 &= (v,u,u,xy,xy;z,v) \\ &= (v,u,u,x,y;z,v)(v,u,u,y,x;z,v) \\ &= (v,u,u,x,y;z,v)^2(v,u,u;y,x;z,v) \\ &= (v,u,u,x,y;z,v)^2, \end{aligned}$$

from which we conclude that

$$(2.15) \qquad (v,u,u,x,y;z,v) = 1.$$

From $(4 \rightarrow 5)$ we have

$$\begin{aligned} 1 &= (x,u,u,vy,vy;z,vy) \\ &= (x,u,u,y,v;z,v)(x,u,u,v,y;z,v)(x,u,u,v,v;z,y) \\ &= (x,u,u,v,y;z,v)^2(x,u,u;y,v;z,v) \text{ by } (2.14) \\ &= (x,u,u,v,y;z,v)^2 \end{aligned}$$

whence

$$(2.16) \qquad (x,u,u,v,y;z,v) = 1.$$

Interchanging u and v in (2.15) we get

$$1 = (u,v,v,x,y;z,u) = (v,u,v,x,y;z,u)^{-1},$$

from which it follows that

$$(2.17) \qquad (v,u,v,x,y;z,u) = 1.$$

From Lemma 1 we have

$$\begin{aligned} 1 &= (x,uv,uv,uv,y;z,u) \\ &= (x,u,v,v,y;z,u)(x,v,u,v,y;z,u)(x,v,v,u,y;z,u) \\ &= (x,u,v,v,y;z,u)^2(v,u,x,v,y;z,u)^{-1} \end{aligned}$$

by interchanging u and v in (2.16) In view of (2.17) we conclude that

$$(2.18) \qquad (x,u,v,v,y;z,u) = 1.$$

From $(4 \rightarrow 5)$ we have

$$\begin{aligned} 1 &= (x,u,vy,vy,z;vy,u) \\ &= (x,u,y,v,z;v,u)(x,u,v,y,z;v,u)(x,u,v,v,z;y,u) \\ &= (x,u,v,y,z;v,u)^2 \text{ by interchanging and } y \text{ and } z \text{ in } (2.18). \end{aligned}$$

Therefore we have

$$(x,u,v,y,z;v,u) = 1. \tag{2.19}$$

Also by $(4 \rightarrow 5)$ we have

$$\begin{aligned} 1 &= (vx,u,vx,y,z;vx,u) \\ &= (x,u,v,y,z;v,u)(v,u,x,y,z;v,u)(v,u,v,y,z;x,u). \end{aligned}$$

By applying the permutation $\begin{pmatrix} u & v & x & y & z \\ u & v & y & z & x \end{pmatrix}$ to (2.17) and using (2.19) we arrive at

$$(v,u,x,y,z;v,u) = 1. \tag{2.20}$$

Now by permuting the arguments in (2.13)–(2.20) we find that all the 5-2 complex commutators in $T_{2,2,1,1,1}$ reduce to 1.

From $(4 \rightarrow 5)$ we get

$$(v,u,u,v;y,x,z) = 1. \tag{2.21}$$

Also from $(4 \rightarrow 5)$ we have $(v,u,u,x;y,z,v) = 1$, whence

$$(v,u,u,x;y,v,z)(v,u,u,x;z,v,y)^{-1} = 1. \tag{2.22}$$

On the other hand $(v,u,u,x;yz,v,yz) = 1$, so that

$$(v,u,u,x;y,v,z)(v,u,u,x;z,v,y) = 1. \tag{2.23}$$

From (2.22) and (2.23) and the fact that G has no elements of order 2, we conclude that

$$(v,u,u,x;y,v,z) = 1. \tag{2.24}$$

In a similar way we get

$$(x,u,u,v;y,v,z) = 1. \tag{2.25}$$

Interchanging u and v in (2.24) we get

$$1 = (u,v,v,x;y,u,z) = (v,u,v,x;y,u,z)^{-1}, \text{ whence}$$

$$(v,u,v,x;y,u,z) = 1. \tag{2.26}$$

Interchanging u and v in (2.25) we get

$$\begin{aligned} 1 &= (x,v,v,u;y,u,z) = (x,v,u,v;y,u,z) \\ &= (x,u,v,v;y,u,z)(v,u,x,v;y,u,z)^{-1} \\ &= (x,u,v,v;y,u,z)(v,u,v,x;y,u,z)^{-1}, \end{aligned}$$

which in view of (2.26) means that

$$(x,u,v,v;y,u,z) = 1. \tag{2.27}$$

By $(4 \rightarrow 5)$ we have

$$\begin{aligned} 1 &= (x,u,u,vy;z,vy,vy) \\ &= (x,u,u,y;z,v,v)(x,u,u,v;z,y,v)(x,u,u,v;z,v,y) \\ &= (x,u,u,y;z,v,v)(x,u,u,v;z,y,v) \quad \text{by interchanging } y \end{aligned}$$

and z in (2.25). We conclude that

$$(x,u,u,y;z,v,v) = 1. \tag{2.28}$$

Permuting the arguments in (2.21) and (2.24)–(2.28) we find that all the 4-3 complex commutators in $T_{2,2,1,1,1}$ reduce to 1.

Finally, since all the non-simple complex commutators in $T_{2,2,1,1,1}$ reduce to the identity, it follows that all the simple commutators therein also reduce to the identity. This completes the proof of Lemma 2.

Now by Lemmas 1 and 2 the fact that G has type $(4 \rightarrow 5)$ we see that all the basic commutators of length 7 in G reduce to 1. It follows that all commutators of length 7 in G reduce to 1, and thus we have the main theorem of this paper.

Theorem. If G is a $(4 \rightarrow 5)$ group on 5 generators without elements of order 2, then G is nilpotent of class at most 6.

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