

Proceedings of the Iowa Academy of Science

Volume 71 | Annual Issue

Article 55

1964

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Recommended Citation

Pilgrim, Donald H. (1964) "Engel Conditions on Groups," *Proceedings of the Iowa Academy of Science*, 71(1), 377-383.

Available at: <https://scholarworks.uni.edu/pias/vol71/iss1/55>

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Engel Conditions on Groups¹

DONALD H. PILGRIM²

Abstract. Let g, c denote positive integers. A group is said to have type $(g \rightarrow c)$ if every subgroup which can be generated by g elements is nilpotent of class at most c . A result of R. H. Bruck shows that groups of type $(4 \rightarrow 5)$ without elements of order 2 are nilpotent of class at most 7. In the present paper the following result is reported: If G is a $(4 \rightarrow 5)$ group on 5 generators without elements of order 2, then G is nilpotent of class at most 6.

Recent work by Kostrikin, [4], on the Burnside problem motivates the following definition: Let g, c denote positive integers. A group has type $(g \rightarrow c)$ if every subgroup which can be generated by g elements is nilpotent of class at most c . R. H. Bruck has studied groups of type $(4 \rightarrow 5)$ in connection with the Burnside problem. A result of Bruck shows that groups of type $(4 \rightarrow 5)$ without elements of order 2 are nilpotent of class at most 7. It has been conjectured that this upper bound on the nilpotency class is not best possible in the case of groups on 5 generators. The main result of this paper is the following: Theorem. If G is a $(4 \rightarrow 5)$ group on 5 generators without elements of order 2, then G is nilpotent of class at most 6.

DEFINITIONS AND NOTATION

Let G be a group and let $(a, b) = a^{-1}b^{-1}ab$ for $a, b \in G$. Let

$$(a, b; 0) = a; (a, b; n) = ((a, b; n-1), b).$$

If H, K are subgroups of G , let (H, K) be the subgroup generated by all commutators (h, k) , where $h \in H$ and $k \in K$. The lower central series of G is a chain $\{G_n\}$ of subgroups defined by

$$G_1 = G; G_{n+1} = (G_n, G).$$

If there exists a non-negative integer n such that $G_{n+1} = 1$, and if c is the least such n , then G is nilpotent of (exact) class c . The center of G is a subgroup Z defined by:

$$Z = \{a \in G: (a, x) = 1 \forall x \in G\}.$$

PRELIMINARY LEMMAS AND THE MAIN THEOREM

Let G be a group generated by u, v, x, y, z . Order the generators by

$$u < v < x < y < z$$

¹ This work was done while the author was a student at the University of Wisconsin under NSF Fellowship No. 62016.

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and form basic commutators as on page 166 of [2], so that the following holds:

$$f \in G_n \implies f \equiv \prod_{c=1}^k c_i^{e_i} \pmod{G_{n+1}}, \text{ for } n \geq 1,$$

where the c_i are basic commutators in G_n and the e_i are integers. Let B denote the set of all basic commutators of length 7. For all integers m, n, p, q, r that pertain, let $T_{m,n,p,q,r}$ denote the subset of B consisting of all basic commutators of length 7 in which one generator occurs m times, another generator occurs n times, another p times, another q times, and the remaining generator occurs r times.

Lemma 1. If G is a $(4 \rightarrow 5)$ group on 5 generators without elements of order 2, then $T_{3,1,1,1,1} = 1$.

Proof. Since G is a $(4 \rightarrow 5)$ group we have in particular that

$$(a, b; 5) = 1 \quad \forall a, b \in G.$$

Then, since G has no elements of order 2, it follows that G/Z has no elements of order 2. Also, since G has type $(4 \rightarrow 5)$, it follows immediately that G/Z has type $(3 \rightarrow 4)$. Now Bruck, [1] p. 5.4 has proved that groups of type $(g \rightarrow 2g-2)$ without elements of order 2 are nilpotent of class at most $3g-3$. With $g = 3$, we conclude that G/Z is nilpotent of class at most 6, whence G is nilpotent of class at most 7. Thus commutators of length 8 reduce to the identity, and also, commutators of length 7 commute.

Next, we note that

$$(1.1) \quad (v, u, u; x, u; z, y) = 1,$$

where $(v, u, u; x, u; z, y)$ is the complex commutator

$$(((v, u, u), (x, u)), (z, y)).$$

For let K be the subgroup generated by the four elements $u, v, x, (z, y)$. Then $(v, u, u) \in K_3$, $(x, u) \in K_2$, and $(z, y) \in K_1$ in the lower central series of K . Therefore we have

$$(v, u, u; x, u; z, y) \in (K_3, K_2, K_1) \subset K_{3+2+1} = K_6 = 1.$$

We observe that (1.1) holds for any permutation of the arguments v, x, z, y

Taking K to be the subgroup generated by the four elements $u, y, z, (v, x)$ we get the identity $(v, x, u; y, u; z, u) = 1$. In terms of basic commutators this is

$$(1.2) \quad (v, u, x; y, u; z, u) (x, u, v; y, u; z, u)^{-1} = 1.$$

On the other hand, $(vx, u, vx; y, u; z, u) = 1$ by $(4 \rightarrow 5)$, whence

$$(1.3) \quad (v, u, x; y, u; z, u) (x, u, v; y, u; z, u) = 1.$$

The factors in (1.2) and (1.3) commute, and hence we conclude that

$$(v, u, x; y, u; z, u)^2 = 1.$$

Since G has no elements of order 2, we conclude that

$$(1.4) \quad (v, u, x; y, u; z, u) = 1.$$

In exactly similar ways, we find that

$$(1.5) \quad (z, y; x, u; v, u, u) = 1,$$

$$(1.6) \quad (z, u; y, u; v, u, x) = 1.$$

Now permuting v, x, y, z , in (1.1), (1.4), (1.5), (1.6) we find that all 3-2-2 complex commutators in $T_{3,1,1,1,1}$ in which u occurs 3 times must be 1. By successively interchanging u with v, x, y, z and changing to basic form, we find that all 3-2-2 commutators in $T_{3,1,1,1,1}$ reduce to the identity.

If we let K be the subgroup generated by the four elements $u, v, x, (z, y)$ and then use (4→5), we find that

$$(1.7) \quad (v, u, u, x; z, y) = 1.$$

By (4→5) we have

$$\begin{aligned} 1 &= (v, u, u, xy, xy; z, u) = (v, u, u, x, y; z, u) (v, u, u, y, x; z, u) \\ &= (v, u, u, x, y; y, u)^2 (v, u, u; y, x; z, u) = (v, u, u, x, y; z, u)^2, \end{aligned}$$

whence

$$(1.8) \quad (v, u, u, x, y; z, u) = 1.$$

Permuting u, v, x, y, z in (1.7) and (1.8) we find that all 5-2 complex commutators in $T_{3,1,1,1,1}$ reduce to 1.

By (4→5) we have

$$(1.9) \quad (v, u, u, u; y, x, z) = 1.$$

Similarly we have $(v, u, ux, y, z, u) = 1$, which in basic form is

$$(1.10) \quad (v, u, u, x; y, u, z) (v, u, u, x; z, u, y)^{-1} = 1.$$

Also, by (4→5) we have $(v, u, u, x; yz, u, yz) = 1$, whence

$$(1.11) \quad (v, u, u, x; y, u, z) (v, u, u, x; z, u, y) = 1.$$

From (1.10), (1.11), and the fact that G has no elements of order 2, we conclude that

$$(2.12) \quad (v, u, u, x; y, u, z) = 1.$$

Again by (4→5) we have

$$\begin{aligned} 1 &= (v, u, xy, xy; z, u, u) = (v, u, x, y; z, u, u) (v, u, y, x; u, u) \\ &= (v, u, x, y; z, u, u)^2 (v, u; y, x; z, u, u) \\ &= (v, u, x, y; z, u, u)^2, \end{aligned}$$

whence

$$(1.13) \quad (v, u, x, y; z, u, u) = 1.$$

Permuting u, v, x, y, z in (1.9), (1.12), (1.13) we see that all 4-3 complex commutators in $T_{3,1,1,1,1}$ reduce to 1.

At this point we have shown that all non-simple complex com-

mutators in $T_{3,1,1,1,1}$ reduce to the identity. The only simple commutators in this set which are trivially 1 by (4→5) are those which are dealt with below.

Now $(z,u,v,y,y,y,x) = 1$ by (4→), which in basic form is

$$(1.14) \quad (z,u,v,x,y,y,y) = 1.$$

We observe that (1.14) holds for any permutation of its arguments. This leaves only (z,u,v,x,y,z,z) left to consider. By (4→5) we have $(z,u,v,x,z,z,y) = 1$, which in basic form yields

$$(z,u,v,x,y,z,z) = 1.$$

This completes the proof of Lemma 1.

Lemma 2. If G is a (4→5) group on 5 generators without elements of order 2, then $T_{2,2,1,1,1} = 1$.

Proof. Taking K to be the subgroup generated by the four elements $u,v,x,(z,y)$ and using (4→5) we find that

$$(2.1) \quad (v,u,u;x,v;z,y) = 1.$$

Similarly, we have

$$(2.2) \quad (v,u,x;v,u;z,y) = 1.$$

$$(2.3) \quad (x,u,v;v,u;z,y) = 1.$$

By (4→5) we have

$$(2.4) \quad (y,x,u;v,u;z,v) = 1,$$

from which we get

$$(2.5) \quad (y,u,x;v,u;z,v)(x,u,y;v,u;z,v)^{-1} = 1.$$

Also by (4→5) we have $(xy,u,xy;v,u;z,v) = 1$, whence

$$(2.6) \quad (y,u,x;v,u;z,v)(x,u,y;v,u;z,v) = 1.$$

From (2.5), (2.6) we conclude that

$$(2.7) \quad (y,u,x;v,u;z,v) = 1.$$

Interchanging u and v in (2.7) we get

$$1 = (x,v,y;u,v;z,u) = (x,v,y;v,u;z,u)^{-1}.$$

whence

$$(2.8) \quad (x,v,y;v,u;z,u) = 1.$$

From (4→5) we have

$$(2.9) \quad (y,x,z;v,u;v,u) = 1.$$

Also from (4→5) we have $(vy,u,x;vy,u;z,vy) = 1$. Since G has no elements of order 2 we may use the linearization process of Heineken, [3], page 699, to obtain

$$(y,u,x;v,u;z,v)(v,u,x;y,u;z,v)(v,u,x;v,u;z,y) = 1.$$

In view of (2.2) and (2.7) we conclude that

$$(2.10) \quad (v,u,x;y,u;z,v) = 1.$$

Again from (4→5)

$$1 = (x,u,vy;vy,u;z,vy)$$

$$= (x,u,y;v,u;z,v)(x,u,v;y,u;z,v)(x,u,v;v,u;z,y).$$

It follows from (2.6), (2.7), (2.3) that

$$(2.11) \quad (x, u, v; y, u; z, v) = 1.$$

From Lemma 1 we have

$$\begin{aligned} 1 &= (x, uv, uv; y, uv; z, v) \\ &= (x, v, u; y, u; z, v)(x, u, v; y, u; z, v)(x, u, u; y, v; z, v) \\ &= (x, u, v; y, u; z, v)^2(v, u, x; y, u; z, v)^{-1}(x, u, u; y, v; z, v). \end{aligned}$$

Then, by (2.10) and (2.11), we conclude that

$$(2.12) \quad (x, u, u; y, v; z, v) = 1.$$

Permuting arguments in (2.1)–(2.12) we find that all 3-2-2 complex commutators in $T_{2^2, 2, 1, 1, 1}$ in which the segment of length 3 comes first reduce to 1. Exactly similar arguments show that all the 3-2-2 complex commutators in which the segment of length 3 occurs last reduce to 1. Thus all the 3-2-2 complex commutators in $T_{2, 2, 2, 1, 1, 1}$ reduce to 1.

From (4→5) we get

$$(2.13) \quad (v, u, u, v, x; z, y) = 1,$$

$$(2.14) \quad (x, u, u, v, v; z, y) = 1.$$

Also from (4→5) we have

$$\begin{aligned} 1 &= (v, u, u, xy, xy; z, v) \\ &= (v, u, u, x, y; z, v)(v, u, u, y, x; z, v) \\ &= (v, u, u, x, y; z, v)^2(v, u, u, y, x; z, v) \\ &= (v, u, u, x, y; z, v)^2, \end{aligned}$$

from which we conclude that

$$(2.15) \quad (v, u, u, x, y; z, v) = 1.$$

From (4→5) we have

$$\begin{aligned} 1 &= (x, u, u, vy, vy; z, vy) \\ &= (x, u, u, y, v; z, v)(x, u, u, v, y; z, v)(x, u, u, v, v; z, y) \\ &= (x, u, u, v, y; z, v)^2(x, u, u, y, v; z, v) \text{ by (2.14)} \\ &= (x, u, u, v, y; z, v)^2 \end{aligned}$$

whence

$$(2.16) \quad (x, u, u, v, y; z, v) = 1.$$

Interchanging u and v in (2.15) we get

$$1 = (u, v, v, x, y; z, u) = (v, u, v, x, y; z, u)^{-1},$$

from which it follows that

$$(2.17) \quad (v, u, v, x, y; z, u) = 1.$$

From Lemma 1 we have

$$\begin{aligned} 1 &= (x, uv, uv, uv, y; z, u) \\ &= (x, u, v, v, y; z, u)(x, v, u, v, y; z, u)(x, v, v, u, y; z, u) \\ &= (x, u, v, v, y; z, u)^2(v, u, x, v, y; z, u)^{-1} \text{ by interchanging} \end{aligned}$$

u and v in (2.16) In view of (2.17) we conclude that

$$(2.18) \quad (x, u, v, v, y; z, u) = 1.$$

From (4→5) we have

$$\begin{aligned} 1 &= (x,u,vy,vy,z;vy,u) \\ &= (x,u,y,v,z;v,u)(x,u,v,y,z;v,u)(x,u,v,v,z;y,u) \\ &= (x,u,v,y,z;v,u)^2 \text{ by interchanging } y \text{ and } z \text{ in} \\ &\quad (2.18). \end{aligned}$$

Therefore we have

$$(2.19) \quad (x,u,v,y,z;v,u) = 1.$$

Also by (4→5) we have

$$\begin{aligned} 1 &= (vx,u,vx,y,z;vx,u) \\ &= (x,u,v,y,z;v,u)(v,u,x,y,z;v,u)(v,u,v,y,z;x,u). \end{aligned}$$

By applying the permutation $\begin{pmatrix} u & v & x & y & z \\ u & v & y & z & x \end{pmatrix}$ to (2.17) and using (2.19) we arrive at

$$(2.20) \quad (v,u,x,y,z;v,u) = 1.$$

Now by permuting the arguments in (2.13)–(2.20) we find that all the 5-2 complex commutators in $T_{2,2,1,1,1}$ reduce to 1.

From (4→5) we get

$$(2.21) \quad (v,u,u,v;y,x,z) = 1.$$

Also from (4→5) we have $(v,u,u,x;y,z,v) = 1$, whence

$$(2.22) \quad (v,u,u,x;y,v,z)(v,u,u,x;z,v,y)^{-1} = 1.$$

On the other hand $(v,u,u,x;yz,v,yz) = 1$, so that

$$(2.23) \quad (v,u,u,x;y,v,z)(v,u,u,x;z,v,y) = 1.$$

From (2.22) and (2.23) and the fact that G has no elements of order 2, we conclude that

$$(2.24) \quad (v,u,u,x;y,v,z) = 1.$$

In a similar way we get

$$(2.25) \quad (x,u,u,v;y,v,z) = 1.$$

Interchanging u and v in (2.24) we get

$$(2.26) \quad \begin{aligned} 1 &= (u,v,v,x;y,u,z) = (v,u,v,x;y,u,z)^{-1}, \text{ whence} \\ &(v,u,v,x;y,u,z) = 1. \end{aligned}$$

Interchanging u and v in (2.25) we get

$$\begin{aligned} 1 &= (x,v,v,u;y,u,z) = (x,v,u,v;y,u,z) \\ &= (x,u,v,v;y,u,z)(v,u,x,v;y,u,z)^{-1} \\ &= (x,u,v,v;y,u,z)(v,u,v,x;y,u,z)^{-1}, \end{aligned}$$

which in view of (2.26) means that

$$(2.27) \quad (x,u,v,v;y,u,z) = 1.$$

By (4→5) we have

$$\begin{aligned} 1 &= (x,u,u,vy;z,vy,vy) \\ &= (x,u,u,y;z,v,v)(x,u,u,v;z,y,v)(x,u,u,v;z,v,y) \\ &= (x,u,u,y;z,v,v)(x,u,u,v;z,y,v) \text{ by interchanging } y \end{aligned}$$

and z in (2.25). We conclude that

$$(2.28) \quad (x,u,u,y;z,v,v) = 1.$$

Permuting the arguments in (2.21) and (2.24)–(2.28) we find that all the 4-3 complex commutators in $T_{2,2,1,1,1}$ reduce to 1.

Finally, since all the non-simple complex commutators in $T_{2,2,1,1,1}$ reduce to the identity, it follows that all the simple commutators therein also reduce to the identity. This completes the proof of Lemma 2.

Now by Lemmas 1 and 2 the fact that G has type $(4 \rightarrow 5)$ we see that all the basic commutators of length 7 in G reduce to 1. It follows that all commutators of length 7 in G reduce to 1, and thus we have the main theorem of this paper.

Theorem. If G is a $(4 \rightarrow 5)$ group on 5 generators without elements of order 2, then G is nilpotent of class at most 6.

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