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An exploration in Ramsey theory

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An Exploration in Ramsey Theory

An Abstract of a Thesis

Submitted

in Partial Fulfillment

of the Requirement for the Degree

Master of Arts

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ABSTRACT

We present several introductory results in the realm of Ramsey Theory, a subfield of Combinatorics and Graph Theory. The proofs in this thesis revolve around identifying substructure amidst chaos. After showing the existence of Ramsey numbers of two types, we exhibit how these two numbers are related. Shifting our focus to one of the Ramsey number types, we provide an argument that establishes the exact Ramsey number for $h(k, 3)$ for $k \geq 3$; this result is the highlight of this thesis. We conclude with facts that begin to establish lower bounds on these types of Ramsey numbers for graphs requiring more substructure.

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Degree of Master of Arts.

Date	Dr. Adrienne Stanley, Chair, Thesis Committee
Date	Dr. Douglas Shaw, Thesis Committee Member
Date	Dr. Douglas Mupasiri, Thesis Committee Member
Date	Dr. Jennifer Waldron, Dean, Graduate College

I would like to dedicate this thesis to Dr. Adrienne Stanley. Dr. Stanley has been my instructor, research advisor, thesis committee chair, and greatest academic supporter over the last five years of my career. She has encouraged me in my pursuit of a greater understanding of the world of mathematics, inspiring me to find my own niche within the field. She exposed me to different perspectives on teaching and how to learn best myself, helping to deepen my understanding of content. Much of what I have learned from her has gotten me to where I am today. Thank you for your unwavering support throughout it all. Woot!

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I would like to thank the members of my thesis committee: Dr. Adrienne Stanley, Dr. Douglas Shaw, and Dr. Douglas Mupasiri. They have given their time and expertise to ensure a worthwhile product is constructed. Thank you for your contribution to my success at the University of Northern Iowa.

I must acknowledge my family: Patrick, Julie, Maggie, and Cole. These individuals have always supported my dreams and aspirations, lending me their ear whenever I needed a partner to bounce ideas and thoughts off of.

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CHAPTER 0

Introduction

Often as individuals, we come across situations where we wonder what it would take to guarantee a certain outcome. Just as often, mathematicians ask the same type of question. By asking such a question, we are asserting that there is some sort of underlying substructure that can explain when an outcome occurs.

Mathematicians love to research and uncover substructure. The problems addressed in this thesis are no different in that we look for substructure within a graph.

Graphs are extremely useful tools. A graph is a collection of vertices and edges. These vertices sometimes can be thought of as objects with the edges between them indicating a relationship between the two. This characterization of a graph might sound quite familiar. In fact, informally throughout our lives, we have been familiarizing ourselves with graphs. For instance, a map is made up of cities (vertices) that have roads between them (edges describing the ability to travel from a given city to another). When in a maze, one stands at a vertex and chooses a direction to walk, selecting which edge should be taken. Or even cooking a meal can be seen as a graph event; each vertex could represent a state of the dinner (what is ready to eat and what is not), and each edge could represent an action of preparing something for the meal; there are often many orders in which things can be completed in order to finish the dinner.

Problems involving graphs characterized in this way can be traced back to 1735; the branch of mathematics called Graph Theory originated with the Königsburg Bridge problem. Königsburg was a thriving city located along the Pregel River in Prussia, and the city was partitioned into four main land masses by the river. The four regions of the city were connected by a series of seven bridges, and it was common for a citizen of Königsburg to ask if they could go for a walk and cross each bridge exactly once. Eventually, Leonard Euler, a famous and prolific mathematician, had the problem proposed to him.

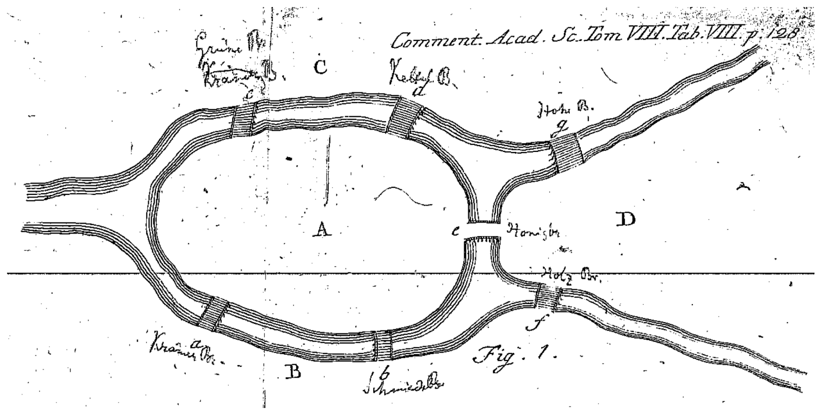


Figure 1

Euler, after some thought, concluded that there was no way to cross each bridge exactly once. He solved the problem by determining how many times (8) the regions of Königsburg would be visited if only crossing each bridge once, and he determined how many times each individual region must be visited based off of the number of bridges that lead to the given region. Each viewpoint led to a different number of regions visited, and so it was not possible to cross each bridge exactly once. The regions and bridges of Königsburg are shown in fig. 1 [8] .

After another 190 years or so, giving graph theory some time to develop and stand on its own, a question that kicked off my thesis inquiry was posed. Frank Ramsey, a young British mathematician, proved that graphs with large enough vertex sets guarantee an induced subgraph that is either complete or independent.

One can phrase a special case of his conclusion as such. If nine people are gathered into a room, Ramsey guaranteed that some four of the nine will all be familiar with each other or some three of the nine will all be unfamiliar with each other. In our math phrasing, each person translates to a vertex, and each type of relationship (familiar or unfamiliar) corresponds to an edge, or lack there of.

In the thesis to follow, we consider these questions of Ramsey and focus on a related question; instead of a complete or independent induced subgraph being guaranteed, can one guarantee a cycle subgraph or an independent induced subgraph? A special case of a conclusion of this form can be phrased like so. If Pat, Julie, Maggie, Jake, Cole, Lou, and Mary Ann are in a room together, we can guarantee that some four of the seven, say Pat; Julie; Maggie; and Cole, will be familiar in the following way: Pat knows Julie, and Julie knows Maggie, and Maggie knows Cole, who knows Pat. It is similar to being four degrees of separation away from someone else who knows you. Otherwise, if this chain of familiarity is not present, some three of the seven will all be unfamiliar with each other.

With this cute party anecdote in the back of our minds, let us dive into the representations and language that describe these underlying graphs, subgraphs, and relationships.

CHAPTER 1

Language and Conventions

Let us begin by laying a foundation for effective and precise communication. The language established in this chapter will be valuable in understanding the expression of thought found within. With no intent of reinventing the wheel, we use definitions and notation consistent with Brualdi [3].

We start the definitions section by first defining what a graph is, its components (vertices and edges), and how these components may relate to each other.

Definition 1.1. *A **graph** G is composed of two types of objects. It has a finite set $V = \{x_0, x_1, x_2, \ldots, x_{k-1}\}$ of elements called **vertices** and a set E of pairs of distinct vertices called **edges**. We denote the graph whose vertex set is V and whose edge set is E by $G = (V, E)$. When more than one graph is present, we denote the the vertex set of G as $V(G)$ and the edge set of G as $E(G)$.*

Definition 1.2. *The number n of vertices in the set V is called the **order** of the graph G.*

Definition 1.3. *If $\alpha = \{x, y\} = \{y, x\}$ is an edge of graph E, then we say that α **joins** x and y and that x and y are **adjacent**. We also say that x and α are **incident**, and y and α are **incident**. Lastly, we refer to x and y as the vertices of the edge α.*

Definition 1.4. *The **degree** of vertex x in a graph G is the number $\deg(x)$ of edges that are incident with x.*

Definition 1.5. *The **total degree** of a graph G is the sum of the degrees of all vertices of G.*

In our main results, we often consider a graph and look for some type of substructure. If some substructure is present, then it is observable in a subset of vertices and edges of the given graph. Next we define subgraph and induced subgraph; each gives us a different lens through which to view a given graph. An induced subgraph preserves the most information from a given graph where a subgraph need not.

Definition 1.6. *Let $G = (V, E)$ be a graph. Let U be a subset of V and F a subset of E such that the vertices of each edge in F belong to U. We call the graph $S = (U, F)$ a* ***subgraph*** *of G. Notice, some edges of G might be omitted in the subgraph S.*

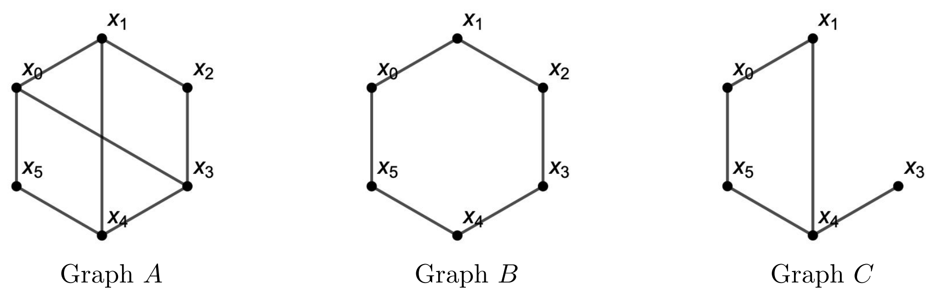


Figure 1.1

In fig. 1.1, B and C are subgraphs of A. Notice, B omits edges $\{x_0, x_3\}$ and $\{x_1, x_4\}$. Similarly, C omits edges $\{x_0, x_3\}$, $\{x_1, x_2\}$, and $\{x_2, x_3\}$.

Definition 1.7. *Let $G = (V, E)$ be a graph. Let $S = (U, F)$ be a subgraph of G such that F consists of all edges of G that join vertices in U. We call S an* ***induced subgraph*** *of G. Here no edges of G are omitted in the subgraph S.*

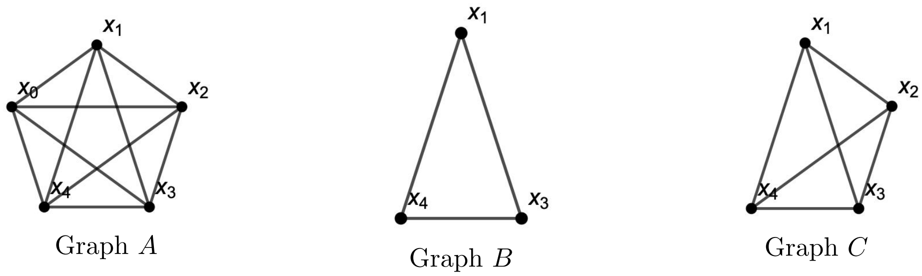


Figure 1.2

In fig. 1.2, B and C are induced subgraphs of A. Notice, B includes all edges connecting vertices x_1, x_3, and x_4. Similarly, C includes all edges connecting vertices x_1, x_2, x_3, and x_4. Additionally, B and C are subgraphs of A as all induced subgraphs are subgraphs.

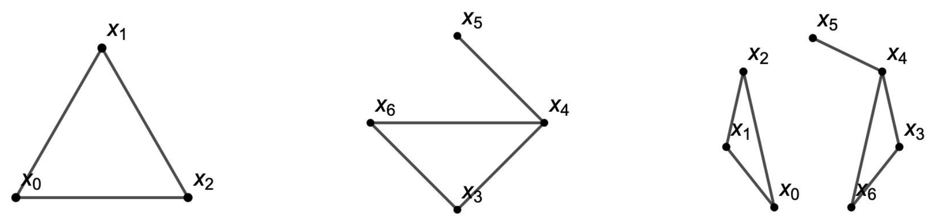
To reiterate, subgraphs are important because they help us understand how the larger graph is built so to speak. We often consider subgraphs with particular structures. Before diving into specific definitions, let us informally describe some of these useful and interesting graphs/structures.

We are interested in disjoint graphs. Vertices of these graphs can be separated into groups such that there are no edges connecting vertices from different groups. We also talk about independent graphs; vertices of these graphs have no edges between them.

Not only do we focus on graphs that have some sense of separateness, but we look at graphs that have some notion of togetherness. In a walk, we describe a sequence of edges that connect a beginning vertex to an ending vertex, and we further develop this idea of a walk by defining a cycle. A cycle is exactly what you would expect; it is a sequence of edges that connects a sequence of vertices, starting and ending at the same vertex. Lastly, complete graphs are those in which all vertices are pairwise adjacent.

In this thesis, we analyze the underlying structure of graphs with respect to the following types of subgraphs: disjoint, independent, cycle, and complete.

Definition 1.8. *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be graphs. We say G_1 and G_2 are* ***disjoint*** *if $V_1 \cap V_2 = \emptyset$ and for any vertex $x_1 \in V_1$ and any vertex $x_2 \in V_2$, no edge joins x_1 and x_2.*



Graph G_1 Graph G_2 Graph G

Figure 1.3

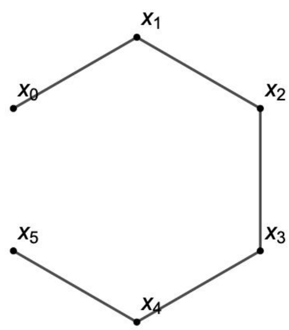
From fig. 1.3, we say G_1 and G_2 are disjoint. Often, we consider a pair of disjoint graphs that are subgraphs of a larger graph. To illustrate this point, notice G has both G_1 and G_2 for subgraphs.

Definition 1.9. *Let $G = (V, E)$ be a graph. A sequence of k edges of the form*

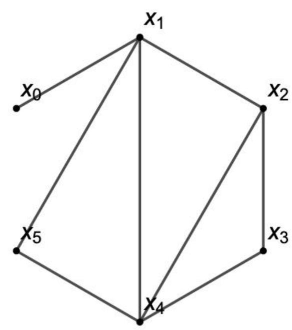
$$\{x_0, x_1\}, \{x_1, x_2\}, \ldots, \{x_{k-1}, x_k\}$$

is called a ***walk*** *of length k, and this walk joins the vertices x_0 and x_k. We also denote the walk by*

$$x_0 - x_1 - x_2 - \cdots - x_k.$$



Graph A



Graph B

Figure 1.4

In fig. 1.4, $x_0 - x_1 - x_2 - x_3 - x_4 - x_5$ is an example of a walk of length 5 found in A. As well, $x_0 - x_1 - x_2 - x_3 - x_4 - x_1 - x_5 - x_4 - x_2$ is an example of a walk of length 8 found in B.

Definition 1.10. *A* ***cycle*** *of order k is a graph whose edges form a walk with the following properties.*

1. *The length of this walk is k.*
2. *If $x_0, x_1, x_2, \ldots, x_{k-1}$ are the vertices of this walk with x_0 and x_{k-1} being the respective beginning and ending vertices, then $x_0 = x_{k-1}$.*
3. *Other than the beginning and ending vertices of this walk, every other vertex is distinct.*

Thus a cycle can be denoted by

$$x_0 - x_1 - x_2 - \cdots - x_{k-1} - x_0.$$

We denote a cycle with k vertices by C_k.

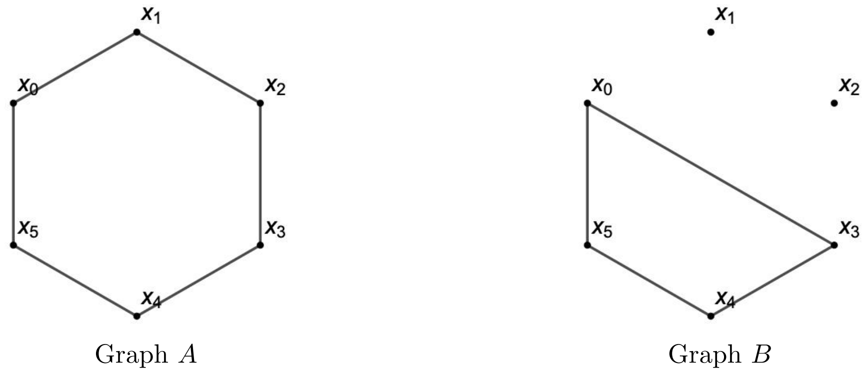


Figure 1.5

In fig. 1.5, $x_0 - x_1 - x_2 - x_3 - x_4 - x_5 - x_0$ is an example of a cycle of length 6 found in A. In addition, $x_0 - x_3 - x_4 - x_5 - x_0$ is an example a cycle of length 4 found in B. Notice this cycle is an induced subgraph of B.

Convention 1.11. *If $x_0, x_1, \ldots, x_{k-1}$ are the distinct vertices in a cycle, then x_{i-1} and x_i are adjacent for $i \in \{1, \ldots, k-1\}$ as well as x_{k-1} and x_0.*

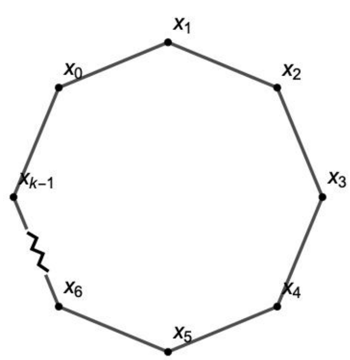
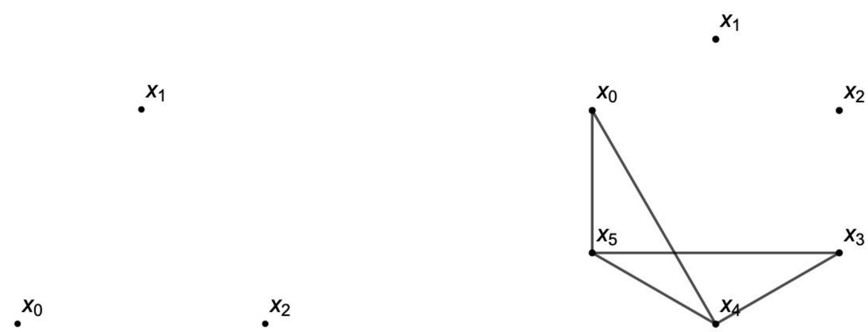


Figure 1.6: Cycle C_k

Observe in fig. 1.6, x_{i-1} and x_i are adjacent for $i \in \{1, \ldots, k-1\}$ as well as x_{k-1} and x_0.

The line break between x_6 and x_{k-1} denotes the continuance of the cycle/walk from vertex x_6 to vertex x_{k-1}.

Definition 1.12. *A graph of order n is called **independent**, provided no two of its vertices are adjacent. We denote such a graph by I_n.*

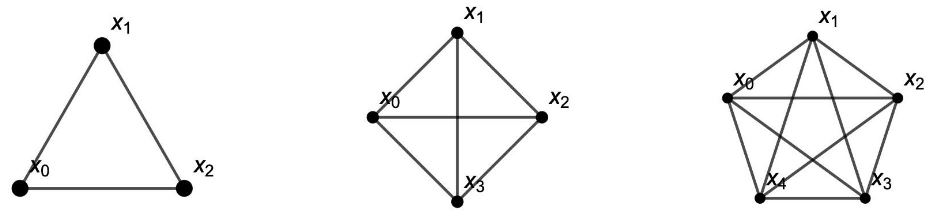


I_3 Graph B

Figure 1.7

In fig. 1.7, the left graph is an independent graph of order 3. In B, x_0, x_1, x_2, and x_3 are the vertices of an independent graph of order 4. Notice this independent graph is an induced subgraph of B.

Definition 1.13. *A graph of order n is called **complete**, provided each pair of distinct vertices is adjacent. Thus in a complete graph each vertex is adjacent to every other vertex. A complete graph of order n is denoted by K_n.*

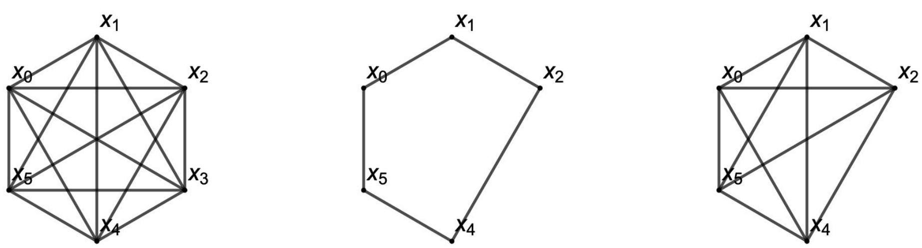


K_3 K_4 K_5

Figure 1.8

In fig. 1.8, complete graphs of order $3, 4$, and 5 are shown.

Example 1.14.



Graph A Graph B Graph C

Figure 1.9

Later on, we focus on cycles and complete graphs. Motivated by definition 1.17 and definition 1.18, we look for cycles and complete graphs that are subgraphs and induced subgraphs. In fig. 1.9, B is a cycle of order 5, and it is a subgraph of A. Alternatively, C is a complete graph of order 5, and it is an induced subgraph of A.

Convention 1.15. *There are two styles of graphs used in the figures throughout this* *paper, and their difference regards how we present our known information. In both styles, solid colored edges denote two adjacent vertices. The styles are:*

1. *If a graph does not use dotted edges, then vertices not connected by an edge are known to be not adjacent.*

2. *If a graph uses dotted edges, then the vertices connected by a dotted edge are not adjacent, and if vertices are not connected by an edge, we make no assumptions about their adjacency.*

We employ the use of both styles in this paper because they help us communicate in two types of ways. Style 1 is used in the scenario when we know all relationships between each pair of vertices. Style 2 is used when we do not know about the adjacency of each pair of vertices; this style is used more often when we know a property of some graph and construct it more fully through deduction.

Definition 1.16. *A graph G^C is said to be the* ***complement*** *of graph G if:*

1. $V(G^C) = V(G)$.

2. *Two vertices of G^C are adjacent if and only if they are not adjacent in G.*

Definition 1.17 (Ramsey's Theorem/numbers)**.** *Let $k, l \in \mathbb{N}$ such that $k \geq 2$ and $l \geq 2$. Let $r \in \mathbb{N}$ be the least integer such that if G is a graph with $|V(G)| = r$, then G has an induced subgraph K_k or an induced subgraph I_l. We call r a Ramsey number and denote it as a function of k and l by $f(k,l) = r$.*

We recognize that we have not shown the existence of such a number r, but its existence is well known [33]. We will prove there exists a finite upper bound on r in theorem 2.5.

Ramsey proved both an infinite and finite version of his theorem. In this paper, we focus on the finite version. In fact, since Ramsey's proof of the existence of such a number r in 1928, only nine Ramsey numbers have been discovered. The most recent discovery was from McKay and Radziszowski [31], proving $f(4,5) = 25$. Even though many Ramsey numbers are not explicitly known, progress has been made in providing upper and lower bounds for many Ramsey numbers.

Instead of continuing to hunt down Ramsey numbers, we introduce a related number which will be the main topic of this thesis.

Definition 1.18. *Let $k, l \in \mathbb{N}$ such that $k \geq 2$ and $l \geq 2$. Let $s \in \mathbb{N}$ be the least integer such that if G is a graph with $|V(G)| = s$, then G has a subgraph C_k or an induced subgraph I_l. We denote s as a function of k and l by $h(k,l) = s$.*

We would like to point out that this related definition sets the stage for the main theorems of this thesis. Moving forward, our main goal is to determine $h(k,l)$. We are successful in identifying the value of $h(k,l)$ for $l = 3$ and $k \geq 3$. We find a lower bound for the value of $h(k,3)$ for $k \geq 3$ by constructing a sequence of counterexamples. We show that this lower bound is indeed an optimal lower bound by an induction proof. Thus, we have determined the value for $h(k,3)$.

Besides focusing on $h(k,l)$, we explore the relationships between the values of $h(k,l)$ and $f(k,l)$. It is important for us to connect our Ramsey-like question, regarding the value of $h(k,l)$, to the original inquiry; as one can imagine, knowing the value of $f(k,l)$ or $h(k,l)$ assists in finding the other. Let us formally begin our exploration.

CHAPTER 2

Main Results

2.1 Initial Values of $f(k, l)$

Let us get our feet wet, so to speak, within Ramsey Theory. We will begin by finding the explicit values of two Ramsey numbers: $f(3,3)$ and $f(4,3)$. These types of proofs require two parts. The first part provides an optimal graph such that the required criteria is not satisfied. The second part proves that the required criteria is satisfied for graphs with exactly one more vertex than the previously provided graph.

Lemma 2.1. $f(3,3) > 5$.

Proof. Let G be a graph as in fig. 2.1. Observe, G does not have an induced K_3. (If G did, a black triangle would be visible.)

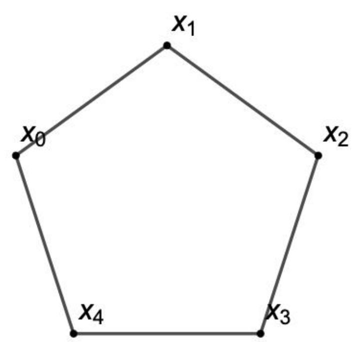


Figure 2.1: $f(3,3)$ counterexample

Without loss of generality, let us consider x_0. If x_0 were a vertex of an I_3, x_2 and x_3 would have to be the remaining two vertices in the I_3 as x_0 is adjacent to x_1 and x_4. However, x_2 and x_3 are adjacent; thus no vertex of G is a vertex of an I_3.

Thus G to has neither an induced K_3 nor I_3. $\square$

Lemma 2.2. $f(3,3) = 6$.

Proof. Let G be a graph such that $|V(G)| = 6$ and $V(G) = \{x_0, \dots, x_5\}$. Suppose G does not have an induced I_3. We will show G has an induced K_3.

Let us consider x_0.

Case 1: Suppose x_0 is disjoint from S where S is an induced subgraph of G and $|V(S)| \geq 3$. Let S be such a graph. Since G does not have an induced I_3 and x_0 is disjoint from S, every pair of vertices of S must be adjacent. Thus S is a complete graph of order at least 3. Thus S has an induced K_3; hence G has an induced K_3.

Case 2: Suppose x_0 is not adjacent with at most 2 vertices of G. Thus, x_0 is adjacent with at least 3 vertices of G. Let x_0 be adjacent with the vertices of S where $|V(S)| \geq 3$ and S is an induced subgraph of G. Since G does not have an induced I_3, without loss of generality let $x_1, x_2 \in V(S)$ be adjacent. Notice $\{x_0, x_1, x_2\}$ are the vertices of an induced K_3. Thus G has an induced K_3.

Thus $f(3,3) \leq 6$. From lemma 2.1, we know $f(3,3) > 5$. Using both of these facts, we have $f(3,3) = 6$. □

Lemma 2.3. $f(3,4) > 8$.

Proof. Let G be a graph as in fig. 2.2. We provide an argument for why G has neither an induced K_3 nor an induced I_4. First, observe G does not have an induced K_3.

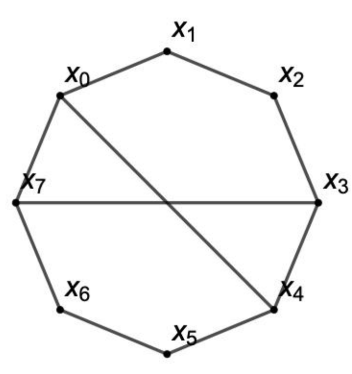


Figure 2.2: $f(4,3)$ counterexample

We will now show G does not have an induced I_4. We shall consider vertices of different degree separately.

Case 1: Without loss of generality, let us consider vertex x_0. By way of contradiction, suppose x_0 is a vertex of an I_4 called I. Then $x_1, x_4, x_7 \notin V(I)$. Since x_2 and x_3 are adjacent, no more than one of them is in $V(I)$. Similarly, x_5 and x_6 are not both in $V(I)$. Since three more vertices are in I, either x_5 and x_6 are in $V(I)$ or x_1 and x_2 are in $V(I)$. $\lightning$

Case 2: Without loss of generality, let us consider vertex x_1. By way of contradiction, suppose x_1 is a vertex of an I_4 called I. Then $x_0, x_2 \notin V(I)$. Since $x_3 - x_4 - x_5 - x_6 - x_7$ is a walk, no two consecutive vertices are in $V(I)$. Thus, $x_3, x_5, x_7 \in V(I)$. However, x_3 and x_7 are adjacent. $\lightning$

Thus G does not have an induced I_4. □

Lemma 2.4. $f(3, 4) = 9$.

Proof. Let G be a graph such that $|V(G)| = 9$ and $V(G) = \{x_0, \ldots, x_8\}$. Suppose G does not have an induced K_3. We will show G has an induced I_4.

Let us consider x_0.

Case 1: Suppose x_0 is not adjacent to at least six other vertices of G. By lemma 2.2, within that subgraph, S, of at least six vertices, there exists an induced K_3 or an induced I_3. Since G does not have an induced K_3, S does not have an induced K_3. Thus, S has an induced I_3. Since x_0 is not adjacent to each vertex of S and S has an induced I_3, observe $V(I_3) \cup \{x_0\}$ are the vertices of an induced I_4. Thus G has an induced I_4.

So x_0 is not adjacent to at most five other vertices of G. This means that x_0 is adjacent to at least three other vertices of G.

Case 2: Suppose x_0 is not adjacent to at most four vertices of G. Then, x_0 is adjacent to at least four vertices of G. Thus we can let S be an induced subgraph of G such that x_0 is adjacent to each vertex of S and $|V(S)| \geq 4$. Since G does not have an induced K_3 and x_0 is adjacent to each vertex of S, every pair of vertices of S must not be adjacent. Thus S has an induced I_4; hence G has an induced I_4.

Case 3: Suppose x_0 is not adjacent to exactly five vertices of G. Since x_0 is any generic vertex of G, this is the case in which each vertex of G is not adjacent to exactly five vertices of G. Thus each vertex of G is adjacent to exactly three vertices of G. Then, the total degree of G is 27. However, for each edge, two is added to the total degree of the graph (one to each of the degrees of its two vertices). Thus a graph with an odd total degree does not exist. $\lightning$ $\square$

We begin with proofs of some of the smallest Ramsey numbers because they help us get a feeling for the type of arguments that will be made later on. As well, these two proofs will be referenced again as they act as building blocks for proving structure in other graphs.

2.2 The $f(k,l)$ and $h(k,l)$ Relationships

Now that we have shown that Ramsey's Theorem holds for two (k,l) pairs, we will provide a constructive proof of Ramsey's Theorem that is true for all $k,l \in \mathbb{N}$ such that $k,l \geq 2$. As well, we will show our Ramsey-like number, $h(k,l)$, also exists as a consequence of Ramsey's Theorem.

Theorem 2.5 (Ramsey's Theorem)**.** *Let $k,l \in \mathbb{N}$ such that $k,l \geq 2$. Then, $f(k,l) < \infty$. In other words, $f(k,l)$ exists.*

Proof. Let $k,l \in \mathbb{N}$ such that $k,l \geq 2$. We will show $f(k,l)$ exists by double induction on k and l. Not only will we prove such a number exists, but we will provide an upper bound for $f(k,l)$ with $k,l \geq 3$.

Base Case: We will show $f(2,l) < \infty$ and $f(k,2) < \infty$ for $k,l \in \mathbb{N} \setminus \{1\}$.

Let G be a graph with $|V(G)| = l$. Suppose G does not have an induced I_l. We will show G has an induced K_2. Since G does not have an induced I_l, there exists two vertices of G that are adjacent. Thus G has an induced K_2.

Thus $f(2,l) \leq l$.

Similarly, $f(k,2) \leq k$.

Let us proceed with our induction on k and l. Now, suppose $k, l \geq 2$. As our inductive hypothesis, let $f(k, l+1) < \infty$ and $f(k+1, l) < \infty$. We will show $f(k+1, l+1) < \infty$. Even better, we will establish $f(k+1, l+1) \leq f(k, l+1) + f(k+1, l)$.

Let G be a graph such that $|V(G)| = f(k, l+1) + f(k+1, l)$. Fix x_0 and x_1 in $V(G)$. Let

$$V(S) = \{y \in V(G) : y \text{ is adjacent to } x_0, y \neq x_0, x_1\}$$

where S is the induced subgraph of G with this vertex set. Similarly, let

$$V(T) = \{z \in V(G) : z \text{ is not adjacent to } x_0, z \neq x_0, x_1\}$$

where T is the induced subgraph of G with this vertex set.

If $|V(S)| \geq f(k, l+1)$, then by the inductive hypothesis, S has an induced K_k or an induced I_{l+1}. If S has an induced I_{l+1}, then we are finished as G would have an induced I_{l+1}. Suppose S has an induced K_k. Since x_0 is adjacent to every vertex of S, $V(K_k) \cup \{x_0\}$ are the vertices of a K_{k+1}. Thus G would have an induced K_{k+1}. So, suppose $|V(S)| \leq f(k, l+1) - 1$.

If $|V(T)| \geq f(k+1, l)$, then by the inductive hypothesis, T has an induced K_{k+1} or an induced I_l. If T has an induced K_{k+1}, then we are finished as G would have an induced K_{k+1}. Suppose T has an induced I_l. Since x_0 is not adjacent to every vertex of T, $V(I_l) \cup \{x_0\}$ are the vertices of an I_{l+1}. Thus G would have an induced I_{l+1}. So, suppose $|V(T)| \leq f(k+1, l) - 1$.

Now, notice

$$\begin{aligned} f(k, l+1) + f(k+1, l) &= |V(G)| \\ &= |V(S)| + |V(T)| + |\{x_0, x_1\}| \\ &\leq [f(k, l+1) - 1] + [f(k+1, l) - 1] + 2 \\ &= f(k, l+1) + f(k+1, l). \end{aligned}$$

Thus we have $|V(S)| = f(k, l+1) - 1$ and $|V(T)| = f(k+1, l) - 1$ as shown in fig. 2.3. Now, x_0 can be adjacent or not adjacent with x_1.

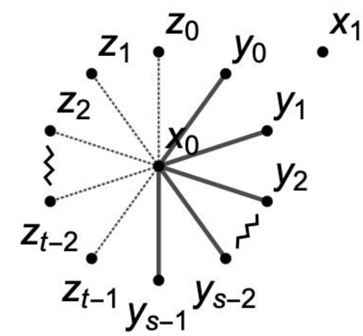


Figure 2.3: Note, $|V(S)| = s$, and $|V(T)| = t$.

Case 1: Let x_1 be adjacent to x_0. Consider the graph S' where $V(S') = V(S) \cup \{x_1\}$. So, $|V(S')| = f(k, l+1)$. Thus, by the inductive hypothesis, S' has an induced K_k or an induced I_{l+1}. If S' has an induced I_{l+1}, then so does G. Suppose S' has an induced K_k called K. Since all vertices of S' are adjacent with x_0, $V(K) \cup \{x_0\}$ are the vertices of an induced K_{k+1}. Hence G has an induced K_{k+1}.

Case 2: Let x_1 not be adjacent to x_0. Consider the graph T' where $V(T') = V(T) \cup \{x_1\}$. So, $|V(T')| = f(k+1, l)$. Thus, by the inductive hypothesis, T' has an induced K_{k+1} or an induced I_l. If T' has an induced K_{k+1}, then so does G. Suppose T' has an induced I_l called I. Since all vertices of T' are not adjacent to x_0, $V(I) \cup \{x_0\}$ are the vertices of an induced I_{l+1}. Hence G has an induced I_{l+1}.

Thus we have shown that G is guaranteed to have an induced K_{k+1} or an induced I_{l+1}. Hence, $f(k+1, l+1) \leq f(k, l+1) + f(k+1, l) < \infty$. $\square$

Lemma 2.6. *Let $k, l \in \mathbb{N}$ such that $k, l \geq 2$. Then $h(k, l) \leq f(k, l)$.*

Proof. Let $k, l \in \mathbb{N}$. Let G be a graph such that $|V(G)| = f(k, l)$. Then, G has an induced K_k or an induced I_l. If G has an induced K_k, then clearly G has a C_k subgraph. Otherwise, G has an induced I_l. Thus whenever $|V(G)| = f(k, l)$, G is guaranteed to have a C_k subgraph or an induced I_l.

Thus $h(k,l) \leq f(k,l)$. □

Corollary 2.7. *Let $k,l \in \mathbb{N}$ such that $k,l \geq 2$. Then, $h(k,l) < \infty$.*

Proof. Let $k,l \in \mathbb{N}$ such that $k,l \geq 2$. From theorem 2.5, we know $f(k,l) < \infty$. From lemma 2.6, we know $h(k,l) \leq f(k,l)$. Using both of these facts, we have $h(k,l) < \infty$. □

Now that we have proven that $f(k,l)$ and $h(k,l)$ exist for $k,l \geq 2$, we can proceed to talk about other characteristics that describe how the two functions are related.

This next relationship is a natural question, and even though we only speak about $f(k,l)$ here, we later ask this question for $h(k,l)$. Since the complement of a complete graph is an independent graph, we derive our next equality regarding $f(k,l)$. This equality becomes insightful to us in lemma 3.1 when trying to determine an unknown $h(k,l)$ value.

Theorem 2.8. *Let $k,l \in \mathbb{N}$. Then $f(k,l) = f(l,k)$.*

Proof. Let $k,l \in \mathbb{N}$. By Ramsey's Theorem (theorem 2.5), $f(k,l), f(l,k) < \infty$. Without loss of generality, let $f(k,l) \leq f(l,k)$. We will show $f(k,l) = f(l,k)$ by showing $f(l,k) \leq f(k,l)$.

Let G be a graph such that $|V(G)| = r = f(k,l)$. We will show G has an induced K_l or an induced I_k. Consider G^C where G^C is the complement of G. Since $|V(G^C)| = r = f(k,l)$, G^C has an induced K_k or an induced I_l.

Case 1: Suppose G^C has an induced K_k. Then G has an induced I_k. Thus G has an induced K_l or an induced I_k.

Case 2: Suppose G^C has an induced I_l. Then G has an induced K_l. Thus G has an induced K_l or an induced I_k.

So, $f(l,k) \leq f(k,l)$. Thus, $f(k,l) = f(l,k)$. □

Lemma 2.9. *Let $l \in \mathbb{N}$. Then $h(3,l) = f(3,l)$.*

Proof. Let $l \in \mathbb{N}$. Let G be a graph. Observe that a complete graph of order 3 is the same as a cycle of order 3. Thus, G is guaranteed to have an a C_3 subgraph or an induced I_l if and only if G is guaranteed to have an induced K_3 or induced I_l. □

2.3 Results of $h(k, l)$

In this section, we focus on Ramsey-like numbers. In our case, this means we looked for the number of vertices required to guarantee a cycle subgraph of order k or an induced complete graph of order l. We decompose the problem of finding Ramsey numbers into smaller manageable pieces. Mathematicians and effective problem solvers use this strategy often. Sometimes after solving enough pieces, a solution for the original problem appears.

Notice how we break the main claim of this section, $h(k + 1, 3) = 2k + 1$, into three separate parts (theorem 2.10, lemma 2.11, and theorem 2.12).

Let us begin to inspect and understand the pieces of our main claim. We begin by establishing a lower bound for the values $h(k, l)$.

Theorem 2.10. *Let $k, l \in \mathbb{N}$. Then $h(k + 1, l + 1) > kl$.*

Proof. We will prove that $h(k + 1, l + 1) > kl$ by providing a counterexample. Let $K_1, K_2, \ldots, K_l$ be complete graphs of order k such that $K_1, K_2, \ldots, K_l$ are pairwise disjoint. Consider $G = \bigcup_{i=1}^{l} K_i$. We will show that G does not have a C_{k+1} subgraph nor an induced I_{l+1}.

Case 1: Suppose G has a C_{k+1} subgraph. Since $G = \bigcup_{i=1}^{l} K_i$ and $K_1, K_2, \ldots, K_l$ are pairwise disjoint, $V(C_{k+1}) \subset V(K_i)$ for some i such that $1 \leq i \leq l$. Without loss of generality, let $V(C_{k+1}) \subset V(K_1)$. Then $k + 1 = |V(C_{k+1})| \leq |V(K_1)| = k$. ↯

Case 2: Suppose G has an induced I_{l+1}. Since $G = \bigcup_{i=1}^{l} K_i$ and $K_1, K_2, \ldots, K_l$ are all complete, $V(I_{l+1})$ includes at most one vertex from K_i for all $i \leq l$. Thus $|V(I_{l+1})| \leq l$. However, $|V(I_{l+1})| = l + 1$. ↯

Thus G has neither a C_{k+1} subgraph nor an induced I_{l+1}. □

In fig. 2.4 and **??**, we show what the graph G from theorem 2.10 would look like for two different k, l pairs.

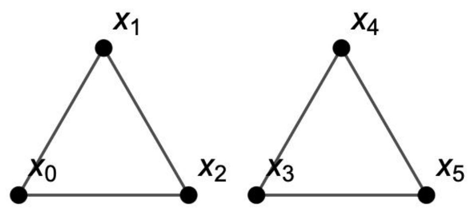


Figure 2.4: $h(4,3)$ counterexample

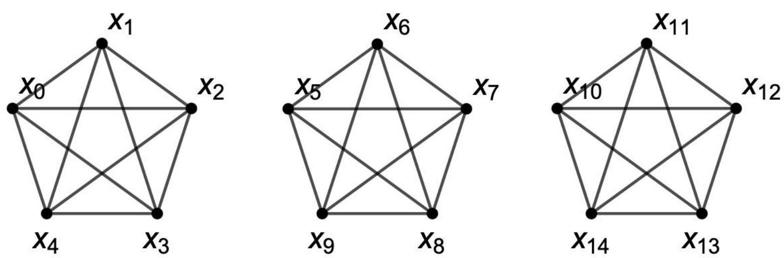


Figure 2.5: $h(6,4)$ counterexample

If an additional vertex is invited into a cycle of length k, then a new cycle of length $k+1$ is formed. This fact is used frequently in theorem 2.12.

Lemma 2.11 (Adjacency Lemma)**.** *Let x_0 and x_1 be two adjacent vertices of a C_k. Suppose x_0 and x_1 are both adjacent to some vertex $y \notin V(C_k)$. Then,*

$$x_0 - y - x_1 - x_2 - \cdots - x_{k-1} - x_0$$

forms a C_{k+1}.

The proof of lemma 2.11 is omitted; the lemma is clear by a simple observation.

We are about to prove the main result of this thesis; we determine a whole class of $h(k,l)$ values (when $l = 3$). The main tactics used in this proof can be found in lemma 2.6

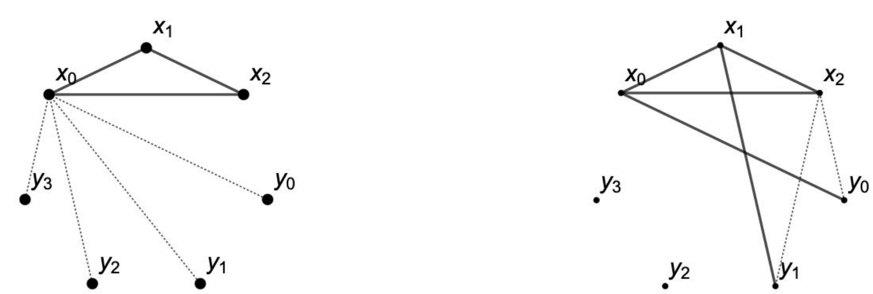
and lemma 2.11. After the base case is provided, we take a somewhat constructive approach to proving the induction.

Theorem 2.12. *Let $k \in \mathbb{N}$ such that $k \geq 3$. Then $h(k+1, 3) = 2k + 1$.*

Proof. Let $k \in \mathbb{N}$. We will prove $h(k+1, 3) = 2k+1$ for $k \geq 3$ by induction on k.

Base Case: Let $k = 3$. We will show $h(4, 3) = 7$. Let G be a graph such that $|V(G)| = 7$. Suppose that G does not have an induced I_3. We will show G has a C_4 subgraph.

By lemma 2.2, we know that $f(3, 3) = 6$. Since $|V(G)| = 7$, $h(3, 3) = f(3, 3) = 6$, and G does not have an induced I_3, G has a C_3 subgraph. Let $V(C_3) = \{x_0, x_1, x_2\}$. Let H be the induced subgraph of G such that $V(H) = V(G) \setminus V(C_3)$. Notice $|V(H)| = 7 - 3 = 4$. Let $V(H) = \{y_0, y_1, y_2, y_3\}$.



Subcase 1 Subcase 2

Figure 2.6

Subcase 1: Suppose some element in $V(C_3)$, namely x_0, is not adjacent to any vertex in $V(H)$. So, x_0 is not adjacent to y_i for $i \in \{0, 1, 2, 3\}$ (see fig. 2.6 Subcase 1). Since G does not have an induced I_3, for $i \neq j$ with $i, j \in \{0, 1, 2, 3\}$, y_i and y_j are adjacent in G and thus in H. Thus, subgraph H is a complete graph of order 4. Thus, H has a C_4 subgraph. Hence, G has a C_4 subgraph.

Subcase 2: Suppose that every element in $V(C_3)$ is adjacent to some element in $V(H)$. By lemma 2.11, we suppose that no two adjacent vertices of C_3 are adjacent to the same vertex of H.

So, let x_0 be adjacent to y_0 and x_1 be adjacent to y_1 (see fig. 2.6 Subcase 2). Note in fig. 2.6 Subcase 2, we omit the edge that joins x_2 with some vertex of H as it does not aid in the proof of the theorem. Notice, x_2 is neither adjacent to y_0 nor y_1 by lemma 2.11. Since x_2 is not adjacent with y_0 nor y_1 and G does not have an induced I_3, y_0 and y_1 are adjacent. Observe, $\{x_0, x_1, y_1, y_0\}$ are the vertices of a C_4. Thus G has a C_4 subgraph.

Thus $h(4,3) \leq 7$. From theorem 2.10, we know $h(4,3) > 6$. Using both of these facts, we have $h(4,3) = 7$.

Let us proceed with our induction on k. Suppose that $h(k+1,3) = 2k+1$. We will show that $h(k+2,3) = h((k+1)+1,3) = 2(k+1)+1 = 2k+3$. Let G be a graph such that $|V(G)| = 2k+3$. Suppose that G does not have an induced I_3. We will show G has a C_{k+2} subgraph.

Since $|V(G)| = 2k+3$, $h(k+1,3) = 2k+1$, and G does not have an induced I_3, G has a C_{k+1} subgraph. Let $V(C_{k+1}) = \{x_0, x_1, \ldots, x_k\}$. Let H be the induced subgraph of G such that $V(H) = V(G) \setminus V(C_{k+1})$. Notice $|V(H)| = k+2$. Let
$V(H) = \{y_0, y_1, \ldots, y_{k+1}\}$.

Subcase 1:

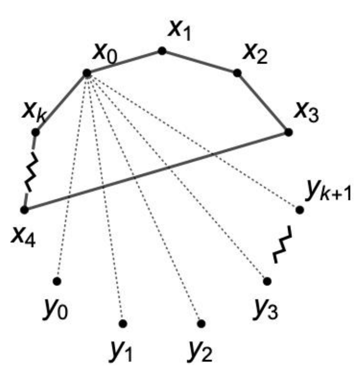


Figure 2.7

Suppose some element in $V(C_{k+1})$, namely x_0, is not adjacent to any vertex in $V(H)$. So, x_0 is not adjacent to y_i for $i \in \{0, \ldots, k+1\}$ (see fig. 2.7). Since G does not have an induced I_3, for $i \neq j$ with $i, j \in \{0, \ldots, k+1\}$, y_i and y_j are adjacent in G and thus in H. Thus, subgraph H is a complete graph of order $k+2$. Thus, H has a C_{k+2} subgraph. Hence, G has a C_{k+2} subgraph.

Subcase 2: Suppose that every element in $V(C_{k+1})$ is adjacent to some element in $V(H)$. By lemma 2.11, we suppose that no two adjacent vertices of a C_{k+1}, disjoint from H, are adjacent to the same vertex of H. Let x_0 and x_1 be adjacent to y_0 and y_1, respectively.

Note in fig. 2.8; fig. 2.9; fig. 2.10; and fig. 2.11, we omit the edges that join x_i for $i \in \{3, 4, \ldots, k\}$ with some vertex of H as they do not aid in the proof of the theorem. Also, we later discern the edge that joins x_2 with its vertex in $V(H)$ as shown in fig. 2.10.

Since no two adjacent vertices of

$$x_0 - x_1 - x_2 - \cdots - x_k - x_0 \tag{2.1}$$

are adjacent to the same vertex in $V(H)$, x_0 and y_1 are not adjacent. Similarly, x_2 and y_1 are not adjacent (see fig. 2.8 (a)).

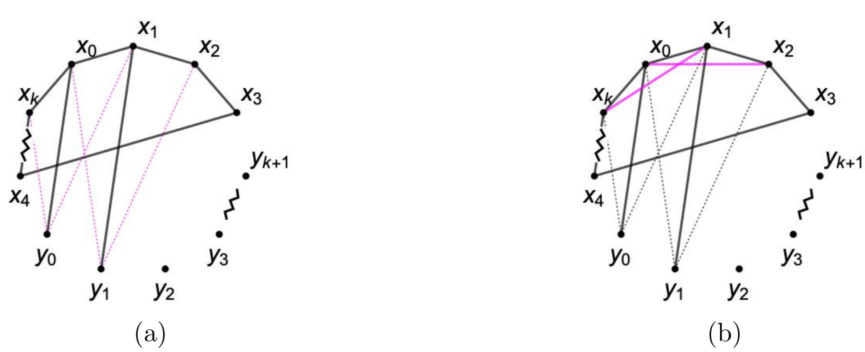


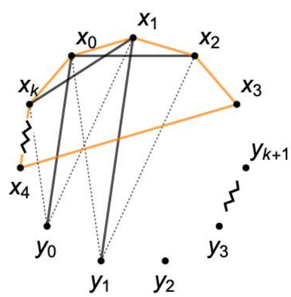
Figure 2.8

Since x_0 and x_2 are not adjacent to y_1 and G does not have an induced I_3, x_0 and x_2 are adjacent. Using no two adjacent vertices of a C_{k+1} are adjacent to the same vertex in $V(H)$, by a similar argument, x_k and x_1 are adjacent (see fig. 2.8 (b)).

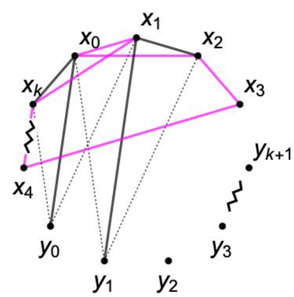
Notice,

$$x_0 - x_2 - x_3 - x_4 - \cdots - x_{k-1} - x_k - x_1 - x_0 \tag{2.2}$$

is a different C_{k+1} than the one in eq. (2.1) that is also disjoint from H.



(a) Cycle of equation (2.1)

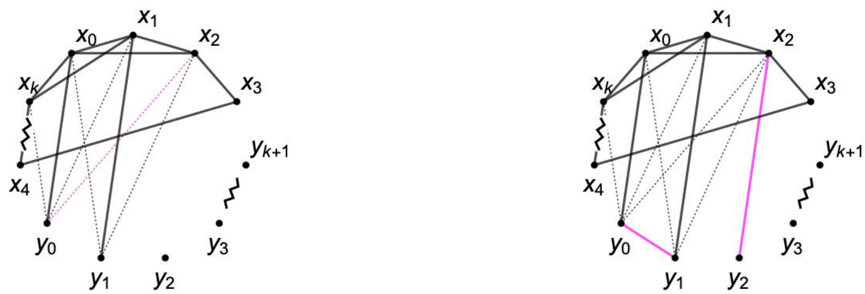


(b) Cycle of equation (2.2)

Figure 2.9

In fig. 2.9 (a) and (b), we draw attention to the two unique cycles of equation 2.1 and equation 2.2, respectively. In the C_{k+1} of equation 2.2, x_0 and x_2 are adjacent. Thus, x_2 is not adjacent to y_0 (see fig. 2.10 (a)). Since x_2 is adjacent to both x_0 and x_1, x_2 is not adjacent to y_0 nor y_1 by lemma 2.11. Let x_2 be adjacent to y_2 as in fig. 2.10 (b).

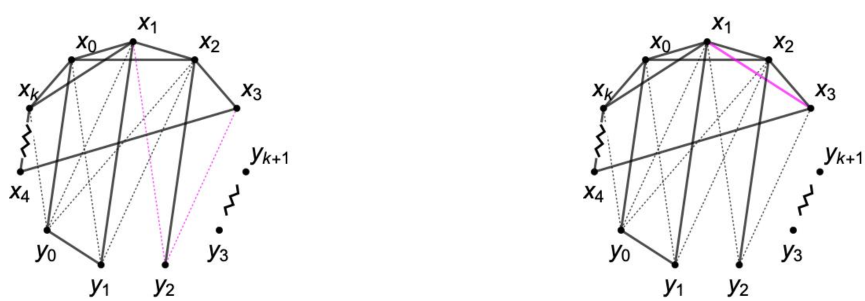
Now that x_2 is not adjacent to y_0 nor y_1, since G does not have an induced I_3, y_0 and y_1 are adjacent (see fig. 2.10 (b)).



(a) (b)

Figure 2.10

Again, since no two adjacent vertices of a C_{k+1} are adjacent to the same vertex in $V(H)$, x_1 and y_2 are not adjacent. Similarly, x_3 and y_2 are not adjacent (see fig. 2.11 (a)). Since G does not have an induced I_3, x_1 and x_3 are adjacent (see fig. 2.11 (b)).



(a) (b)

Figure 2.11

Notice,

$$x_0 - y_0 - y_1 - x_1 - x_3 - x_4 - \cdots - x_{k-1} - x_k - x_0 \tag{2.3}$$

is a C_{k+2}. The cycle mentioned in equation 2.3 is shown in fig. 2.12.

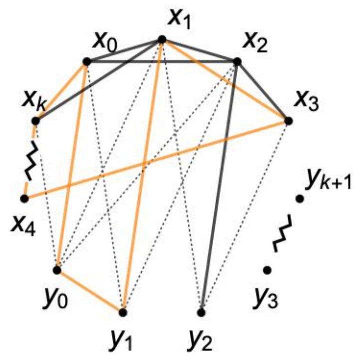


Figure 2.12

Thus G has a C_{k+2} subgraph.

Thus $h(k+2,3) \leq 2k+3$. From theorem 2.10, we know $h(k+2,3) > (k+1) \cdot 2 = 2k+2$. Using both of these facts, we have $h(k+2,3) = 2k+3$. $\square$

We already know by lemma 2.6, $h(k,3) \leq f(k,3)$, but how much more is $f(k,3)$ than $h(k,3)$? Now that we know the value for $h(k,3)$ for $k \geq 3$, we have more information to answer such a question as how $h(k,l)$ and $f(k,l)$ are related. Perhaps, by taking $h(k,3)$ up one notch to $h(k+1,3)$, requiring more, we can get a closer lower bound for $f(k,3)$. We work to uncover this relationship by proving theorem 2.13.

Theorem 2.13. *Let $k \in \mathbb{N}$ such that $k \geq 4$. Then, $f(k,3) > 2k$.*

Proof. Let $k \in \mathbb{N}$ such that $k \geq 4$. We shall proceed by induction on k.

Base Case: Let $k = 4$. It is known that $f(4,3) = 9$. Thus, $f(4,3) = 9 > 8 = 2(4)$.

Let $f(k,3) > 2k$. We will show $f(k+1,3) > 2(k+1)$. Since $f(k,3) > 2k$, let G be a graph such that $|V(G)| = 2k$ and G fails to have both an induced K_k and an induced I_3. Let $V(G) = \{x_0, x_1, \ldots, x_{2k-1}\}$. Let G' be a graph such that G is an induced subgraph of G' and $V(G') = V(G) \cup \{y_0, y_1\}$ such that y_0 and y_1 are not adjacent and y_i and x_j are adjacent for all $i \in \{0, 1, \ldots, 2k-1\}$ and $j \in \{0, 1\}$.

Notice, y_0 and y_1 do not participate in an I_3 subgraph of G'. Since y_0 and y_1 do not participate in an I_3 subgraph of G' and G does not have an induced I_3, G' does not have an induced I_3.

Using a contradiction argument, we will now show that G' does not have an induced K_{k+1}. Suppose G' has an induced K_{k+1}. Since y_0 and y_1 are not adjacent, the K_{k+1} of G' does not contain both y_0 and y_1. However, since G does not have an induced K_k, y_0 or y_1 must be in the K_{k+1}.

Without loss of generality, suppose $y_0 \in V(K_{k+1})$. Then, $V(K_{k+1}) \setminus \{y_0\}$ are the vertices of a K_k. Notice $V(K_{k+1}) \setminus \{y_0\} \subset V(G)$. Thus G has an induced K_k. ↯

So G' fails to have an induced K_{k+1}. Hence, G' is a graph such that $|V(G')| = 2k + 2$ and G' fails to have an induced K_{k+1} and an induced I_3.

Thus $f(k + 1, 3) > 2(k + 1)$. □

Corollary 2.14. *Let $k \in \mathbb{N}$ such that $k \geq 4$. Then $h(k + 1, 3) \leq f(k, 3)$.*

Proof. Let $k \in \mathbb{N}$ such that $k \geq 4$. By theorem 2.13, $f(k, 3) > 2k$. So, $f(k, 3) \geq 2k + 1$. By theorem 2.12, $h(k + 1, 3) = 2k + 1 \leq f(k, 3)$. □

CHAPTER 3

Forward

3.1 Additional Work

Now that we have found the value of $h(k,3)$ for $k \geq 3$, we attempt to extend our knowledge and find values for $h(k,l)$ for $l \geq 4$. We start this discussion by first realizing that $h(k,l)$ does not maintain a property that $f(k,l)$ holds. Specifically, $h(k,l) \neq h(l,k)$ in general.

Lemma 3.1. *There exists $k, l \in \mathbb{N}$ such that $h(k,l) \neq h(l,k)$.*

Proof. Consider $h(4,3)$ and $h(3,4)$. By theorem 2.12, we know $h(4,3) = 7$. By lemma 2.9 and lemma 2.4, we know $h(3,4) = f(3,4) = 9$. Thus,

$$h(4,3) = 7 \neq 9 = h(3,4).$$

□

In pursuit of determining new $h(k,l)$ values, we conclude this thesis with a counterexample for $h(3,5)$. Thus, we have found a lower bound for $h(3,5)$. It is also worth mentioning by lemma 2.9, this counterexample is also a counterexample for $f(3,5)$.

Example 3.2. *The graph in fig. 3.1 below is an example of a graph that does not have a C_3 subgraph nor an induced I_5.*

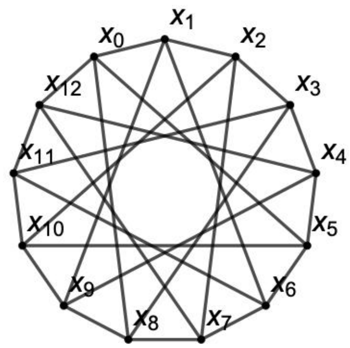


Figure 3.1: $h(3,5), f(3,5)$ counterexample

Proof. Let G be the graph as in fig. 3.1. We first characterize the graph G. Then we prove that G fails to have a C_3 subgraph. Finally, we will show that G does not have an induced I_5.

If arithmetic is done in $\mathbb{Z}_{13}$, x_i is adjacent to $x_{i-1}, x_{i+1}, x_{i-5}$, and x_{i+5} for $i \in \{0, 1, \ldots, 12\}$.

By way of contradiction, suppose G has a cycle subgraph of order 3 called C_3. Without loss of generality, let $x_0 \in V(C_3)$.

Let $x_j, x_k \in V(C_3)$ for some $j, k \in \{1, 2, \ldots, 12\}$ with $j, k \neq 0$. Since $x_0 \in C_3$, x_0 is adjacent to x_j and x_k. Similarly, x_j is adjacent to x_k. Since x_j and x_k are adjacent to x_0, $j, k \in \{1, 5, 8, 12\}$. However, notice x_1, x_5, x_8, and x_{12} are all pairwise disjoint. Thus x_j and x_k are not adjacent. $\lightning$

Thus G does not have a cycle subgraph of order 3.

We now prove that G does not have an induced I_5. By way of contradiction, suppose G has an induced I_5. Since each vertex is indistinguishable, without loss of generality, let $x_0 \in I_5$.

Then, $x_1, x_5, x_8, x_{12} \notin I_5$. We partition the remaining vertices as such: $\{x_2, x_3, x_4\}$, $\{x_6, x_7\}$, and $\{x_9, x_{10}, x_{11}\}$. Since we have four more points in our I_5, and there are three groups in our partition, by the pigeonhole principle, at least two vertices of the I_5 will come from the same set of vertices.

Two vertices of the I_5 cannot be elements of $\{x_6, x_7\}$ as x_6 and x_7 are adjacent.

Since $\{x_2, x_3, x_4\}$ and $\{x_9, x_{10}, x_{11}\}$ are indistinguishable, without loss of generality, suppose at least two vertices of the I_5 are elements of $\{x_2, x_3, x_4\}$. Notice there is only one way to select at least two vertices from $\{x_2, x_3, x_4\}$ where the selected vertices are pairwise disjoint. Thus, $x_2, x_4 \in I_5$. Hence $x_3, x_7, x_9, x_{10} \notin I_5$.

Thus the only other vertices available for the I_5 are x_6 and x_{11}. So $x_6, x_{11} \in I_5$. However, x_6 and x_{11} are adjacent. $\lightning$

Thus G does not have an induced I_5.

We have shown that G does not have a C_3 subgraph nor an induced I_5. Thus, $f(3, 5) \geq 14$.

$\square$

What is in store next? In the next section,we aim to give a brief summary of where the field of Ramsey Theory currently is, and what questions are still open.

3.2 Future Work

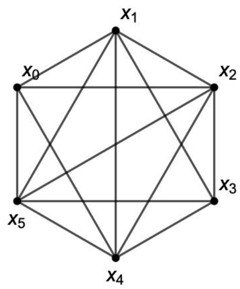
Now, we were not the first ones to pose Ramsey-like questions, and we certainly will not be the last. Our main result was actually first found in 1971 [ChaS]. When it comes to the function $h(k, l)$, many more values are known. In fact, if we were to make a new discovery, we would have to uncover $h(k, 8)$ for $k \geq 10$. This value is conjectured to be $7k - 6$. A table of known $h(k, l)$ values can be found in table 3.1 [30].

Some notes on table 3.1 in regards to keeping consistent with Radziszowski [30]: citations with "-" describe joint credit; the first reference is for the lower bound and the second the upper bound. As well, "/" denotes joint credit. Some papers contain results found with the help of computer algorithms. A list of papers that use some computers where results are easily verifiable with some computations are: [9],[10],[11],[26],[32]. A list of papers where cpu intensive algorithms have to be used to verify or replicate the results are: [1],[12], [16], [29].

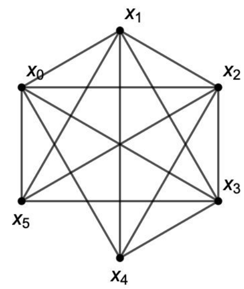
Table 3.1: Known $h(k,l)$ values denoted $f(C_k, K_l)$

	C_3	C_4	C_5	C_6	C_7	C_8	C_9	$\ldots C_k$ for $k \geq l$
K_3	6	7	9	11	13	15	17	$\ldots$ $2k-1$
	[15]-[4]	[5]	$\ldots$					$\ldots$[5]
K_4	9	10	13	16	19	22	25	$\ldots$ $3k-2$
	[15]	[6]	[18]/[25]	[23]	[40]	$\ldots$		$\ldots$[40]
K_5	14	14	17	21	25	29	33	$\ldots$ $4k-3$
	[15]	[7]	[17]/[25]	[23]	[41]	[2]	$\ldots$	$\ldots$[2]
K_6	18	18	21	26	31	36	41	$\ldots$ $5k-4$
	[27]	[9]-[34]	[24]	[35]	$\ldots$			$\ldots$[35]
K_7	23	22	25	31	37	43	49	$\ldots$ $6k-5$
	[26]-[14]	[32]-[22]	[36]	[37]	[37]	[21]/[39]	[39]	$\ldots$[39]
K_8	28	26	29-33	36	43	50	57	$\ldots$ $7k-6$
	[16]-[29]	[32]	[20]	[38]	[39]	[19]/[42]	[28]	$\ldots$ conj.
K_9	36	30						$\ldots$ $8k-7$
	[26]-[16]	[32]-[1]						$\ldots$ conj.
K_{10}	40-42	36						$\ldots$ $9k-8$
	[10]-[12]	[1]						$\ldots$ conj.
K_{11}	47-50	39-44						$\ldots$ $10k-9$
	[11]-[12]	[1]						$\ldots$ conj.

Interestingly enough, there are other Ramsey-like questions that can also be explored; we focused on cycles, but we could have looked at almost complete graphs versus almost complete graphs, cycles versus cycles, cycles versus stars, cycles versus wheels, cycles versus books, etc. Note that almost complete graphs, stars, and wheels are all well defined graphs with examples shown in fig. 3.2; fig. 3.3; and fig. 3.4, respectively.

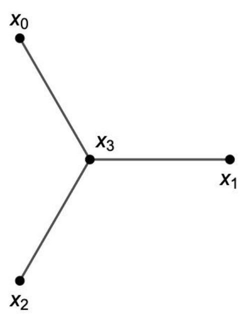


(a) $K_6 - e$ where $e = \{x_0, x_3\}$

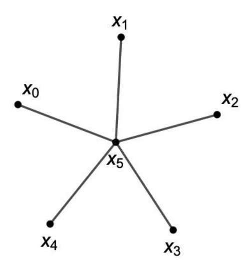


(b) $K_6 - e$ where $e = \{x_4, x_5\}$

Figure 3.2

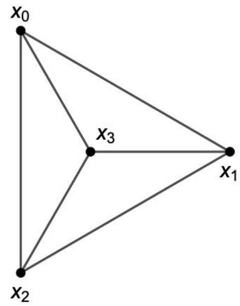


(a) Star of order 4

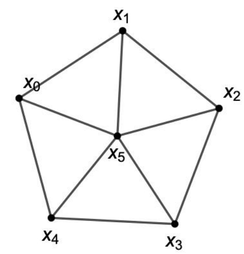


(b) Star of order 6

Figure 3.3



(a) Wheel of order 4



(b) Wheel of order 6

Figure 3.4

Besides looking into other types of induced subgraphs with different structure, we can also look into multicolor Ramsey numbers. In this thesis, edges of graphs either existed or did not exist; hence, all proofs worked with Ramsey numbers of two colors. For reference, twenty three Ramsey numbers are known for the three color case, and no exact Ramsey numbers are known for the four color case. Then, again, we could look into multicolor Ramsey-like numbers that correspond to guaranteeing cycles, stars, wheels, etc.

If looking to learn more about Ramsey's Theorem and its related questions, a great place to start would be the Mathematical Review completed by Stanisław P. Radziszowski [30]. Radziszowski has cited over 700 references. From the review and cited papers, in depth knowledge on the topic can be collected.

If looking to get fairly young students interested in the subject, I recommend looking at Ramsey Theory [13]. It introduces the topic in an interesting and friendly way with plenty of visuals for the reader to investigate.

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