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The Reversed Upper Central Series in the Polarization Process in Groups

DONALD HARRY PILGRIM [1]

Let G be a group and $\{Z_n\}$ be the upper central series of G. If $Z_n = G$ for some positive integer n and if c is the least such positive integer, we define a chain, $\{K_i\}$, of subgroups of G by

(a) $K_i = Z_{c-i+1}$ if $1 \leqslant i \leqslant c$,

(b) $K_i = \{1\}$ if $i \geqslant c+1$.

We shall call the chain $\{K_i\}$ the reversed upper central series of G.

A FUNDAMENTAL LEMMA

Putting together some of the known properties of the chain $\{z_n\}$ we have

Lemma. If G is a nilpotent group, then G has a reversed upper central series $\{K_i\}$ such that

(a) If $n \geqslant 1$, p is prime, and G has no elements of order p, then G/K_n has no elements of order p;

(b) If $\{G_n\}$ is the lower central series of G, then $G_n \subset K_n$;

(c) There exists a positive integer n such that $G_n \subset K_{n+1}$;

(d) If $G_n \subset K_{n+1}$, then $G_n = 1$;

(e) K_n/K_{n+1} is abelian for all $n \geqslant 1$.

Properties (b)-(e) may be found in Hall [1]. Although property (a) is a known result, the author is unable to give a reference for its proof. The reader who does not wish to supply his own proof may consult the author's Ph.D. Thesis [2].

APPLICATION OF THE LEMMA

We may illustrate the use of the lemma as follows: Suppose we are considering a class, S, of groups defined by one or more identical relations involving commutators; for example, by the fourth Engel condition

$$(y,x,x,x,x) = 1.$$

(in particular then, the members of S are not assumed to be periodic.) Suppose, further, that our objective is to determine a bound for the nilpotency class of those groups in S which are nilpotent and which have given number (finite) of generators. If G is in S, it is natural to examine the abelian quotient groups $A_{n+1} = G_{n+1}/G_{n+2}$ and to seek the least n for which the corresponding quotient group has order one. It may turn out, however, that A_{n+1} is finite for some n, but not necessarily of order one. In this case we are prevented from the desired conclusion by the existence, in A_{n+1}, of elements of prime order p, where where p ranges over a finite set, Π, of primes.

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Suppose now we make the assumptions that G is not only in S but is nilpotent and has no elements of order p, p Π. (We need not assume that G is periodic.) Then, if $\{K_i\}$ is a descending central series of G satisfying the conditions of the Lemma, we may conclude that the group

$$B_{n+1} = G_{n+1} / (G_{n+1} \cap K_{n+2})$$

is not only finite (as a homomorphic image of A_{n+1}), but also has no elements of order p, p Π. This means that $B_{n+1} = 1$ and hence that $G_{n+1} = 1$.

In practice we pay no attention to the groups A_{n+1} but, instead, determine the groups B_{n+1} for successive values of n. That is, for each successive n, we work modulo K_{n+2} with the commutators which generate G_{n+1}. Assuming that G has no elements of specified prime order, we seek the least n such that $G_{n+1} \subset K_{n+2}$. Then n is the nilpotency class of G. This process seems to work more smoothly than that devised by Heineken [3]. (See in particular, Lemma 11 of Heineken [3].)

An Example

Suppose that G is nilpotent and has no elements of orders 2 and 3. Suppose, further, that G satisfies the fourth Engel condition

$$(y,x,x,x,x) = 1 \qquad x,y \quad G.$$

Then using the identities

$$(x,yz) = (x,z)(x,y)(x,y,z)$$
$$(xy,z) = (x,z)(x,z,y)(y,z)$$

and working modulo K_6 in the reversed upper central series of G, we can write (y,xz,xz,xz,xz) as a product of sixteen commutators of length 5 in which each argument is either x,y or z. Now for $O \leqslant i \leqslant 4$, let $F_i(x)$ denote the product of all such factors in which x occurs exactly i times. We note that since K_5/K_6 is abelian, it is not necessary to specify the order of the factors in $F_i(x)$. We then have the identical relation

(1) $1 \equiv (y,xz,xz,xz,xz) \equiv F_O(x)F_1(x)F_2(x)F_3(x)F_4(x) \mod K_6$, where for example

$$F_1(x) = (y,x,z,z,z)(y,z,x,z,z)(y,z,z,x,z)(y,z,z,z,x).$$

Since G satisfies the fourth Engel, we can write (1) in the form

(2) $F_1(x)F_2(x)F_3(x) \equiv 1 \mod K_6.$

Replacing x by x^2 in (2), we get

(3) $F_1(x)^2F_2(x)^4F_3(x)^8 \equiv 1 \mod K_6.$

Since K_5/K_6 is abelian, we conclude from (3) that

$$\left[F_1(x)F_2(x)^2F_3(x)^4\right]^2 \equiv 1 \mod K_6.$$

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Now G has no elements order 2 and hence neither does G/K_6, whence

(4) $F_1(x)F_2(x)^2F_3(x)^4 \equiv 1 \bmod K_6$.

Replacing x by x^3 in (2) we get

$$1 \equiv F_1(x^3)F_2(x^3)F_3(x^3) \equiv F_1(x)^3F_2(x)^9F_3(x)^{27} \bmod K_6$$
$$\equiv \left[F_1(x)F_2(x)^3F_3(x)^9\right]^3 \bmod K_6.$$

Since G/K_6 has no elements of order 3, we conclude that

(5) $F_1(x)F_2(x)^3F_3(x)^9 \equiv 1 \bmod K_6$.

From (2) ,(4), (5) and the fact that G/K_6 has no elements of order 2, we easily deduce that

$$F_1(x) \equiv F_2(x) \equiv F_3(x) \equiv 1 \bmod K_6.$$

And so we arrive at the polarized identical relations:

$$(y,x,z,z,z)(y,z,x,z,z,)(y,z,z,x,z)(y,z,z,z,x) \equiv 1 \bmod K_6$$
$$(y,x,x,z,z)(y,x,z,x,z)(y,x,z,z,x)(y,z,z,x,x)$$
$$\cdot(y,z,x,z,x)(y,z,x,x,z) \equiv 1 \bmod K_6$$
$$(y,z,x,x,x)(y,x,z,x,x)(y,x,x,z,x)(y,x,x,x,z) \equiv 1 \bmod K_6.$$

Heineken [3], obtained similar polarized identical relations, though not in the same manner, in groups satisfying the third Engel condition, and thus was able to show that any such group without elements of orders two and five is nilpotent of class at most four. At present, it is not known if finitely generated groups satisfying the fourth Engel condition are nilpotent, nor have any low bounds been given for the nilpotency class of those which are nilpotent.

Literature Cited

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2. Pilgrim, Donald H. Engel Conditions on Groups. Ph.D. Thesis. University of Wisconsin, Madison, Wisconsin. 1963.
3. Heineken, Hermann. 1961. Illinois J. Math. Vol. 5, No. 4, pp. 681-707.

Functions "Close" To Continuous Functions

Let X, Y denote topological spaces, F the set of all functions on X to Y and Ω the family of all open coverings for the space Y. *Definitions*: (1) Let f, g F and $\propto$ Ω. Then g is said to be "$\propto$-close" or "$\propto$-near" to f if and only if for every x X, f(x), g(x) U for some U $\propto$. (2) A function f F is said to be (weakly) nearly continuous if and only if for every $\propto$ Ω, there exists a (nearly) continuous function g F such that g is $\propto$-near to f. (3) A subset $A \subset Y$ is said to be nearly connected (nearly compact) if and only if for every $\propto$ Ω, there exists a connected (compact) set $B \subset Y$ such that $A \subset \propto^*(B)$ and $B \subset \propto^*(A)$. $[\propto^*(A) = \bigcup \{U \quad \propto : U \cap A \neq \emptyset\}]$. The main *results* are: (1) Every continuous function is nearly continuous; but the converse is not true even if Y is T_0, T_1, T_2 or fully normal. However,

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if Y is regular (i.e., satisfies the T_3-axiom), any nearly continuous function $f:X \rightarrow Y$ is continuous. (2) The composition of a continuous and a nearly continuous function (in any order) is nearly continuous; the composition of two nearly continuous functions is weakly nearly continuous. (3) The nearly continuous image of a connected (compact) set is nearly connected (nearly compact). (4) Nearly connectedness and nearly compactness are both continuous invariants. (5) Suppose X is a set and V is a topology for X such that (X, V) is a regular space with the fixed point property. If U be a larger topology for X such that if R is open in U, its closure is the same in both topologies, then (X, U) has the nearly continuous fixed point property (i.e., every nearly continuous function of X into itself maps a point onto itself).