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On the zeta Kirchhoff index of several graph transformations

Danny Cheuk
University of Northern Iowa

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An Abstract of a Thesis

Submitted

in Partial Fulfillment

of the Requirement for the Degree

Master of Arts

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ABSTRACT

In this paper, we first derived the Ihara zeta function, complexity and zeta Kirchhoff index of the k-th semitotal point graph (of regular graphs), a construction by Cui and Hou [5], where we create triangles for every edge in the original graph. Then, we extend the construction to the creation of equilaterals and polygons.

We also derived the zeta Kirchhoff indices for numerous graph transformations on regular graphs, and some selected families of graphs.

At the end, a data summary is provided for enumeration computed on simple connected md2 graphs up to degree 10.

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Date	Dr. Marius Somodi, Chair, Thesis Committee
Date	Dr. Douglas Mupasiri, Thesis Committee Member
Date	Dr. Bill Wood, Thesis Committee Member
Date	Dr. Jennifer Waldron, Dean, Graduate College

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CHAPTER 1

INTRODUCTION

Let $G = (V_G, E_G)$ be a *graph*, where V_G is the set of *vertices* in G and E_G is the set of *edges* in E. The elements in V_G consist of 'points' in the graph G, whereas the elements in E_G consist of 'connections' that join pairs of vertices in V_G. For example, the graph G_1 below has $V_{G_1} = \{a, b, c, d\}$ and $E_{G_1} = \{ab, ac, ad, bc, cd\}$. The graphs in this paper will have edges that are free of weight and direction, in other words, $ab = ba$ for each $ab \in E_G$. Such graphs are called *undirected graphs*. A *subgraph* $H = (V_H, E_H)$ of G is a graph where $V_H \subset V_G$ and $E_H \subset E_G$ such that for each $v = ab \in E_H$, $a \in V_H$ and $b \in V_H$.

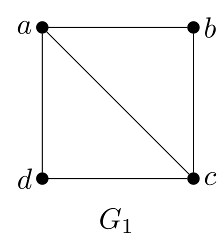


Figure 1.1: Example of a simple graph

We define the *order* and *size* of G to be the number of vertices and edges in G, correspondingly. We also define the degree of any vertex $v \in V_G$ to be the number of edges $e \in E_G$ such that $e = va$ for some $a \in V_G$. The order of the graph G_1 above is 4 and its size is 5. The degree of a, b, c and d are 3, 2, 3 and 2.

A graph with all vertices of degree 2 or higher is called a *minimal degree* 2 *graph*, or md2 graph. Note that it is possible to have an edge that connects the same vertex on both ends; such an edge is called a *loop*. It is also possible for E_G to have multiple edges connecting the same vertices. We define a *simple graph* to be a graph free of loops and multiple edges, and a *multigraph* to be a graph that contains loops and/or multiple edges. Examples of multigraphs are given below as G_2 and G_3. We shall assume all graphs in this paper are simple and undirected beyond this point.

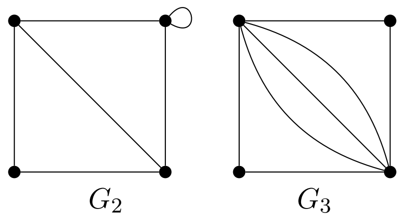


Figure 1.2: Examples of multigraphs

Given $v_1, v_n \in V_G$, a *walk* from v_1 to v_n in G is a sequence of edges in the form of $(v_1v_2, v_2v_3, \ldots, v_{n-1}v_n)$ where $v_1v_2, \ldots, v_{n-1}v_n \in E_G$. A *trail* is a walk without repeated edges, and a *cycle* is a walk where $v_1 = v_n$. For example, (ab, bc, cd) in G_1 is a walk but not a cycle, and (ab, bc, ca) in G_1 is a cycle.

A cycle $(v_1v_2, \ldots, v_{n-1}v_n)$ has *backtracking* if $v_{i+1}v_{i+2} = v_{i+1}v_i$ for some i. If $v_{n-1}v_n = v_2v_1$ then we say the cycle has a *tail.* If a cycle cannot be represented as a power of another cycle (repeating the walk), then we say the cycle is *primitive.* A primitive cycle that has no backtracks or tails is called a *prime cycle.*

Two cycles $(v_1v_2, \ldots, v_{n-1}v_n)$ and $(w_1w_2, \ldots, w_{n-1}w_n)$ are *equivalent* if $v_1v_2 \ldots v_n$ is a cyclic permutation of $w_1w_2 \ldots w_n$. This defines an equivalence relation on the set of all cycles of G. Let $[C]$ be the equivalence class of all cycles that are equivalent to a prime cycle C. Then, the *Ihara zeta function* of a graph G is a function of complex argument u defined, for sufficiently small values of u, by

$$Z_G(u) = \prod_{[C]} (1 - u^{v[C]})^{-1},$$

where $v[C]$ is the number of edges of a representative C of class $[C]$.

For each pair of vertices $v_i, v_j \in V_G$ of a graph G, we call G a *connected graph* if there exist a walk from v_i to v_j. Otherwise, G is called a *disconnected graph.* The number of *components* of a graph is defined as the minimum number of partitions of V_G such that each subgraph in a partition is connected. For example, G_1 above is a connected graph and G_4 below is a disconnected graph with 2 components.

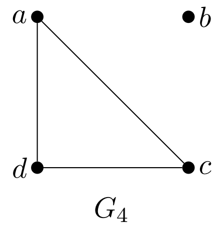


Figure 1.3: Example of a disconnected graph

In this paper, we are mainly working with connected *regular graphs*, that is, connected graphs with vertices of equal degree. We call a graph G an *r-regular graph* (or graph of *regularity r*) if, for each $v \in V_G$, $|v| = r$. Note that, for a connected r-regular graph G, where $r \geq 2$ and $n \geq 3$, G is an md2 graph.

For instance, as shown below, G_5 is a 2-regular graph and G_6 is a 3-regular graph. Note that, if G is an r-regular graph with order n and size m, then $m = \frac{nr}{2}$.

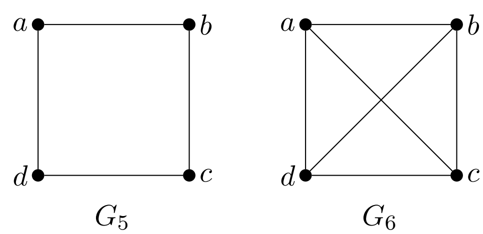


Figure 1.4: Examples of regular graphs

Two graphs G and H are *isomorphic*, if they have the same number of vertices, and there exists a permutation ρ such that for each $ij \in E(G)$, $\rho(i)\rho(j) \in E(H)$.

Note that two graphs of equal regularity r are not necessarily isomorphic. For instance, as shown below, both G_7 and G_8 are 3-regular graphs, but they are not isomorphic.

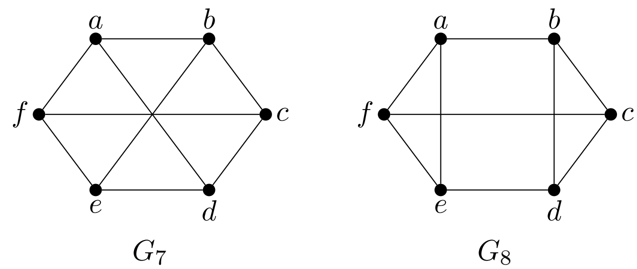


Figure 1.5: Examples of non-isomorphic 3-regular graphs

Since G_8 contains prime cycles of length 3 while G_7 does not, G_7 and G_8 are not isomorphic to one another.

An alternative way of representing a graph is to utilize matrices. If $G = (V_G, E_G)$ where $|V_G| = n$, and let $f : [n] \to V_G$ be any bijective map. Then G can be represented by an *adjacency matrix* A_G, where a_{ij} is the count of $f(i)f(j) \in E_G$. For example, the adjacency matrices for G_3, G_5 and G_6 above can be represented as

$$A_{G_3} = \begin{pmatrix} 0 & 1 & 3 & 1 \\ 1 & 0 & 1 & 0 \\ 3 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}, A_{G_5} = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}, A_{G_6} = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}.$$

It should be noted that there are $n!$ ways to enumerate such a bijective map f, hence for $n > 1$ the adjacency matrix is not unique. From now on, we will fix a bijective map f (which means we will label the vertices $v_1, v_2, \ldots, v_n$, in some order and edges of G as $e_1, e_2, \ldots, e_m$). For undirected simple graphs, all adjacency matrices are (i) symmetric, (ii) $a_{ii} = 0$ for each $i \in [n]$, and (iii) all entries must be either 0 or 1. Observe that two graphs G and H are isomorphic if and only if there exists some permutation matrix P such that $A_G = PA_HP^T$.

Throughout this paper, we shall refer to the i-row, j-column entry of a matrix A, as $(A)_{ij}$.

An $n \times n$ *degree matrix* D_G of a graph G is defined as $d_{ii} = \sum_{k=1}^{n} a_{ik} = \sum_{k=1}^{n} a_{ki}$ and $d_{ij} = 0$ whenever $i \neq j$. A *Laplacian matrix* L_G of a graph G is defined as $L_G = D_G - A_G$. We now give an important theorem proven by Bass.

Theorem 1.1 ([2])**.** *The Ihara zeta function of graph G with order n and size m, satisfies*

$$Z_G(u)^{-1} = (1-u^2)^{m-n} \det\Big(I_n - uA_G + u^2(D_G - I_n)\Big),$$

where I_n is the identity matrix of dimension n.

An undirected simple graph $G = (V_G, E_G)$ where $|V_G| = n$ and $|E_G| = m$, can also be represented as an $n \times m$ *incidence matrix* B_G, where the rows and columns correspond to the vertices and edges of G. We let $b_{ij} = 1$ if i is one of the vertices of edge j. For instance, the incidence matrix for G_6 above can be represented as

$$B_{G_6} = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{pmatrix}.$$

(It should be made aware, that some authors define the incidence matrix B_G of any given graph G differently.)

We define the *adjacency spectrum* of graph G, denoted $\mathrm{Spec}(A_G)$, to be the set of eigenvalues of A_G. Similarly, we define the *Laplacian spectrum* of G, denoted $\mathrm{Spec}(L_G)$, to be the set of eigenvalues of L_G. The *spectral radius* of a matrix A is defined as $\sup\big\{|\lambda_1|, \ldots, |\lambda_n|\big\}$, where $\{\lambda_1, \ldots, \lambda_n\}$ is the set of eigenvalues of A. Now, we present a couple of propositions.

Proposition 1.2 (Perron–Frobenius)**.** *Let G be a connected graph. Then the spectral radius of A_G is bounded above by the greatest degree of its vertices.*

Proposition 1.3 ([6])**.** *Let G be an r-regular graph of order n, then the spectral radius of*

A_G is r. Furthermore, r is the greatest eigenvalue of A_G, with multiplicity equal to the number of components in G.

Notice that if $\mathrm{Spec}(A_G) = \{\lambda_1, \ldots, \lambda_n\}$ and G is r-regular, then the $\mathrm{Spec}(L_G) = \{r - \lambda_1, \ldots, r - \lambda_n\}$. Hence we have the following corollary.

Corollary 1.4. *Let G be r-regular of order n. Then the smallest eigenvalue of L_G is 0, of multiplicity equal to the number of components in G. Furthermore, the spectral radius of L_G is bounded above by $2r$.*

If G is connected and contains no cycles, then G is called a *tree*. For a connected graph G, a *minimal spanning tree* is a subgraph that is a tree and contains the minimal possible number of edges that connects all vertices in G. The total number of such minimal spanning trees of G is called the *complexity* of G, denoted $\tau(G)$ throughout this paper. For example, $\tau(G_5) = 4$.

Theorem 1.5 (Kirchhoff's/Matrix-Tree)**.** *If G is a connected graph of order n, then the complexity of G is given by*

$$\tau(G) = \frac{1}{n} \prod_{i=2}^{n} \mu_i,$$

where Spec$(L_G) = \{\mu_1 = 0, \mu_2, \ldots, \mu_n\}$.

The *resistance distance* between two vertices v_i and v_j is defined as

$$r_{ij} = \frac{\det L_G^{ij}}{\tau(G)},$$

where L_G^{ij} is the matrix obtained from Laplacian matrix of G by removing the i-th, j-th columns and rows. Also, let $r_{ii} = 0$, $\forall i$.

The *Kirchhoff index* of G is defined as

$$Kf(G) = \sum_{1 \le i < j \le n} r_{ij}.$$

By Chen and Gutman,

Theorem 1.6 ([4, 7])**.** *If G is a graph of order n, then*

$$Kf(G) = n \sum_{i=2}^{n} \frac{1}{\mu_i},$$

where $Spec(L_G) = \{\mu_1 = 0, \mu_2, \ldots, \mu_n\}$.

Three other graph invariants were introduced recently, that are also based on resistance distances: (i) the zeta Kirchhoff index [14], given by (for md2 graphs)

$$Kf^z(G) = \sum_{1 \le i < j \le n} (d_{ii} - 2)(d_{jj} - 2) r_{ij};$$

(ii) the additive Kirchhoff index [7], given by

$$Kf^+(G) = \sum_{1 \le i < j \le n} (d_{ii} + d_{jj}) r_{ij};$$

and (iii) the multiplicative Kirchhoff index [4], given by

$$Kf^\times(G) = \sum_{1 \le i < j \le n} d_{ii} d_{jj} r_{ij}.$$

In this paper, formulas for the zeta Kirchhoff index of various graphs and graph transformations will be derived. Therefore, we will assume from here on that all graphs are md2. It should be noted that, for r-regular graphs, $Kf^z(G) = (r-2)^2 Kf(G)$ for $r \ge 2$.

Recall the formula for the Ihara zeta function proven by Bass. By Northshield, the determinant part of that formula satisfies the following.

Theorem 1.7 ([12])**.** *Let G be a graph of order n, size m and $f(u) = \det\Big(I_n - uA_G + u^2(D_G - I_n)\Big)$. Then,*

$$f'(1) = 2(m-n)\tau(G),$$

where $\tau(G)$ is the complexity of G.

The zeta Kirchhoff index can be derived by taking this formula a step further:

Theorem 1.8 ([14])**.** *Let G be a graph of order n, size m and $f(u) = \det\Big(I_n - uA_G + u^2(D_G - I_n)\Big)$. Then,*

$$f''(1) = 2\Big(Kf^z(G) + 2mn - 2n^2 + n\Big)\tau(G),$$

where $\tau(G)$ is the complexity of G.

CHAPTER 2
MAIN RESULTS

2.1 k-th semitotal point graph of regular graphs

The k-th semitotal point graph (also called triangulation of a graph) is a graph transformation that generates a new graph $R^k(G)$ by adding k vertices for every edge in G, such that the vertices are adjacent to both of the vertices incident to that edge in G.

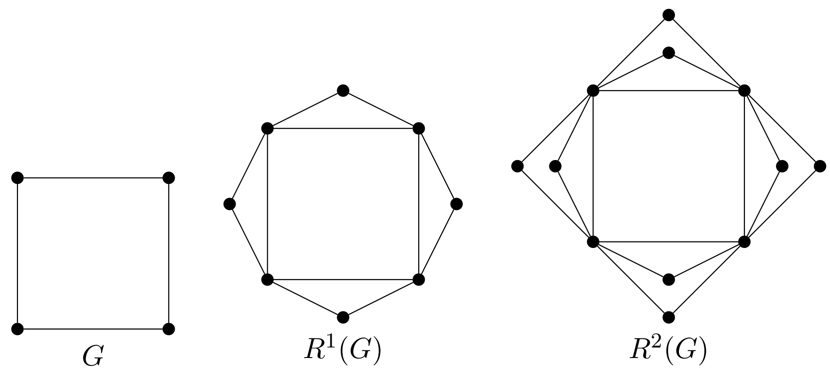


Figure 2.1: Example of k-th semitotal point graph $R^k(G)$ of cycle graph C_4, where $k = 1, 2$

We shall look at the Ihara zeta function, complexity and zeta Kirchhoff index of this graph transformation on a regular graph G.

Theorem 2.1. *If G is an r-regular graph of order n, then the Ihara zeta function of the k-th semitotal point graph $R^k(G)$ satisfies*

$$Z_{R^k(G)}(u)^{-1} = (1-u^2)^{\frac{n(kr+r-2)}{2}}(1+u^2)^{\frac{n(kr-2)}{2}} \cdot \\ \cdot \prod_{i=1}^{n}\Big(1 + ru^2 + kru^4 + ru^4 - u^4 - \lambda_i u\big(1 + ku + u^2\big)\Big),$$

where $Spec(A_G) = \{\lambda_1 = r, , \lambda_2, \ldots, \lambda_n\}$.

Proof. Note that $n_{R^k(G)} = \frac{n(2+kr)}{2}$ and $m_{R^k(G)} = \frac{nr(1+2k)}{2}$ since G is a r-regular graph. By

Theorem 1.1, the Ihara zeta function of $R^k(G)$ satisfies

$$\begin{aligned} Z_{R^k(G)}(u)^{-1} &= (1-u)^{m_{R^k(G)}-n_{R^k(G)}} \cdot \det\Big[I_{n_{R^k(G)}} - uA_{R^k(G)} + u^2(D_{R^k(G)} - I_{n_{R^k(G)}})\Big] = \\ &= (1-u)^{n(kr+r-2)/2} \cdot \det\Big[I_{n+km} - uA_{R^k(G)} + u^2(D_{R^k(G)} - I_{n+km})\Big]. \end{aligned} \tag{2.1}$$

Let us denote $M = I_{n+km} - uA_{R^k(G)} + u^2(D_{R^k(G)} - I_{n+km})$. We proceed similar to the approach used in [5], and write

$$A_{R^k(G)} = \begin{pmatrix} 0_{km} & \Gamma^T \\ \Gamma & A_G \end{pmatrix}, D_{R^k(G)} = \begin{pmatrix} 2I_{km} & 0 \\ 0 & (k+1)D_G \end{pmatrix},$$

where $\Gamma = (B_G, B_G, \ldots, B_G)$, B_G being the $n \times m$ incidence matrix of G defined in the previous chapter. Note that Γ is a $n \times km$ matrix, Γ^T is a $km \times n$ matrix and $\Gamma\Gamma^T = k(D_G + A_G)$.

Hence, we have

$$\begin{aligned} M &= \begin{pmatrix} I_{km} & 0 \\ 0 & I_n \end{pmatrix} - u \begin{pmatrix} 0 & \Gamma^T \\ \Gamma & A_G \end{pmatrix} + u^2 \begin{pmatrix} I_{km} & 0 \\ 0 & (k+1)D_G - I_n \end{pmatrix} = \\ &= \begin{pmatrix} (1+u^2)I_{km} & -u\Gamma^T \\ -u\Gamma & I_n - uA_G + u^2\Big[(k+1)D_G - I_n\Big] \end{pmatrix}. \end{aligned}$$

The Schur complement of $(1+u^2)I_{km}$ is

$$\begin{aligned} & I_n - uA_G + u^2\Big[(k+1)D_G - I_n\Big] - (-u\Gamma)\left(\frac{1}{1+u^2}I_{km}\right)(-u\Gamma^T) = \\ &= (1-u^2)I_n + \left(u^2(k+1) - \frac{ku^2}{1+u^2}\right)D_G - \left(u + \frac{ku^2}{1+u^2}\right)A_G = \\ &= (1-u^2)I_n + r \cdot \left(\frac{u^2 + ku^4 + u^4}{1+u^2}\right)I_n - \left(\frac{u + ku^2 + u^3}{1+u^2}\right)A_G = \\ &= \frac{1}{1+u^2}\Big((1 + ru^2 + kru^4 + ru^4 - u^4)I_n - (u + ku^2 + u^3)A_G\Big). \end{aligned}$$

Now we can calculate $\det(M)$ by using the Schur complement:

$$\begin{aligned}\det(M) &= \det[(1+u^2)I_{km}]\cdot\\ &\cdot \det\left[\frac{1}{1+u^2}\Big((1+ru^2+kru^4+ru^4-u^4)I_n-(u+ku^2+u^3)A_G\Big)\right]=\\ &=(1+u^2)^{km-n}\det\Big[(1+ru^2+kru^4+ru^4-u^4)I_n-(u+ku^2+u^3)A_G\Big]. \quad (2.2)\end{aligned}$$

Note that if $\mathrm{Spec}(A)=\{\lambda_1,\dots,\lambda_n\}$, then $\det(I_n+cA)=\prod_{i=1}^n(1+c\lambda_i)$ for some nonzero real number c.

Therefore, (2.2) is equal to

$$=(1+u^2)^{km-n}\prod_{i=1}^{n}\Big[1+ru^2+kru^4+ru^4-u^4-\lambda_i u(1+ku+u^2)\Big], \qquad (2.3)$$

where $\mathrm{Spec}(A_G)=\{\lambda_1=r,\lambda_2,\dots,\lambda_n\}$.

Therefore, by substituting (2.3) into (2.1), we obtain

$$\begin{aligned}Z_{R^k(G)}(u)^{-1} &= (1-u^2)^{\frac{n(kr+r-2)}{2}}(1+u^2)^{\frac{n(kr-2)}{2}}\cdot\\ &\cdot\prod_{i=1}^{n}\Big(1+ru^2+kru^4+ru^4-u^4-\lambda_i u\big(1+ku+u^2\big)\Big).\end{aligned}$$

□

Theorem 2.2. *Let G be an r-regular graph of order n. Then the complexity of its k-th semitotal point graph $R^k(G)$ satisfies*

$$\tau\big(R^k(G)\big)=2^{n(kr-2)/2+1}(2+k)^{n-1}\cdot\tau(G),$$

where $\tau(G)$ is the complexity of G.

Proof. By Theorem 1.7, $f(u)=\det\Big[I_{n+km}-uA_{R^k(G)}+u^2(D_{R^k(G)}-I_{n+km})\Big]$ satisfies

$$f'(1)=2(m_{R^k(G)}-n_{R^k(G)})\tau\big(R^k(G)\big)=$$

$$= n(kr + r - 2)\tau\big(R^k(G)\big), \tag{2.4}$$

where $\tau\big(R^k(G)\big)$ is the complexity of $R^k(G)$.

Let $a(u) = (1+u^2)^{n(kr-2)/2}$ and $p(u) = \prod_{i=1}^{n} p_i(u)$ where $p_i(u) = 1 + ru^2 + kru^4 + ru^4 - u^4 - \lambda_i u(1 + ku + u^2)$. Thus, by Theorem 2.1

$$f(u) = a(u)\prod_{i=1}^{n} p_i(u).$$

By differentiating $f(u)$ in respect to u and substituting $u = 1$, we obtain

$$\begin{aligned} f'(1) &= a'(1)p(1) + a(1)p'(1) = \\ &= a'(1)p_1(1)p_2(1)\dots p_n(1) + a(1)\Big[p_1'(1)p_2(1)\dots p_n(1) + \\ &+ p_1(1)p_2'(1)p_3(1)\dots p_n(1) + \dots + p_1(1)\dots p_{n-1}(1)p_n'(1)\Big]. \end{aligned} \tag{2.5}$$

Note that $p_1(u) = (u-1)\big[(kr+r-1)u^3 + (kr-1)u^2 + (r-1)u - 1\big]$, so $p_1(1) = 0$ (where $\lambda_i = r$). Also, $p_i(1) = (2+k)(r - \lambda_i)$. Hence (2.5) can be reduced to

$$\begin{aligned} f'(1) &= a(1)p_1'(1)p_2(1)\dots p_n(1) = \\ &= 2^{n(kr-2)/2} \cdot \Big[2(kr+r-2)\Big] \cdot \prod_{i=2}^{n} \Big[(2+k)(r-\lambda_i)\Big] = \\ &= 2^{n(kr-2)/2+1}(kr+r-2)(2+k)^{n-1}\prod_{i=2}^{n}\Big(r - \lambda_i\Big). \end{aligned} \tag{2.6}$$

Finally we substitute (2.6) into $f'(1)$ in (2.4) and solve for $\tau\big(R^k(G)\big)$, in which we obtain

$$\begin{aligned} \tau\big(R^k(G)\big) &= 2^{n(kr-2)/2+1}\frac{(2+k)^{n-1}}{n}\prod_{i=2}^{n}\Big(r-\lambda_i\Big) = \\ &= 2^{n(kr-2)/2+1}(2+k)^{n-1}\cdot\tau(G), \end{aligned}$$

by Theorem 1.5, as desired. $\square$

Theorem 2.3. *Let G be an r-regular graph of order n and let $Kf(G)$ be the Kirchhoff index of G. Then the zeta Kirchhoff index of its k-th semitotal point graph $R^k(G)$ satisfies*

$$Kf^z\big(R^k(G)\big) = \frac{2(kr+r-2)^2}{(2+k)} Kf(G).$$

Proof. By Theorem 1.8, $f(u) = \det\Big[I_{n+km} - uA_{R^k(G)} + u^2(D_{R^k(G)} - I_{n+km})\Big]$, (2.3) satisfies

$$\begin{aligned} f''(1) &= 2\Big(Kf^z\big(R^k(G)\big) + 2m_{R^k(G)}n_{R^k(G)} - 2n^2_{R^k(G)} + n_{R^k(G)}\Big)\tau\big(R^k(G)\big) = \\ &= 2\Big(Kf^z\big(R^k(G)\big) + \frac{n(2+kr)(1+knr+nr-2n)}{2}\Big)\tau\big(R^k(G)\big). \end{aligned} \tag{2.7}$$

Let $f(u) = a(u)p(u) = a(u)p_1(u)\dots p_n(u)$ where $a(u) = (1+u^2)^{n(kr-2)/2}$ and $p_i(u) = 1 + ru^2 + kru^4 + ru^4 - u^4 - \lambda_i u(1+ku+u^2)$. By differentiating $f(u)$ twice with respect to u and substitute $u = 1$, we have

$$\begin{aligned} f''(1) &= a''(1)p(1) + a(1)p''(1) + 2a'(1)p'(1) = \\ &= 2a'(1)p'(1) + a(1)p''(1) + \\ &+ 2a(1)p_1'(1)\Big(p_2'(1)p_3(1)\dots p_n(1) + \dots + p_2(1)\dots p_{n-1}(1)p_n'(1)\Big) = \\ &= \left(\prod_{i=2}^{n} p_i\right)\left(2a'(1)p_1'(1) + a(1)p_1''(1) + 2a(1)p_1'(1)\sum_{i=2}^{n}\frac{p_i'(1)}{p_i(1)}\right) = \\ &= (2+k)^{n-1}\left(\prod_{i=2}^{n}\mu_i\right)2^{n(kr-2)/2+1}\Bigg[(-6+4n+4r+5kr-2nr-4knr+ \\ &\quad + knr^2 + k^2nr^2) + 4\left(\frac{kr+r-2}{2+k}\right)\sum_{i=2}^{n}\frac{-2+2\mu_i+k\mu_i+r+kr}{\mu_i}\Bigg], \end{aligned} \tag{2.8}$$

where $\mathrm{Spec}(L_G) = \{\mu_1 = 0, \mu_2, \dots, \mu_n\}$.

By substituting (2.8) into (2.7) and solving for $Kf^z\big(R^k(G)\big)$, we have

$$
\begin{aligned}
Kf^z\big(R^k(G)\big) &= 2n(1-n)(kr+r-2) + \frac{2n(kr+r-2)}{2+k}\sum_{i=2}^{n}\frac{-2+2\mu_i+k\mu_i+r+kr}{\mu_i} = \\
&= 2(kr+r-2)\left(\frac{n(kr+r-2)}{2+k}\sum_{i=2}^{n}\frac{1}{\mu_i}\right) = \\
&= \frac{2Kf(G)(kr+r-2)^2}{2+k},
\end{aligned}
$$

by Theorem 1.6, where $Kf(G)$ is the Kirchhoff index of G.

□

2.2 k-th semitotal 2-point graph of regular graphs

It is only natural to wonder how the Ihara zeta function will change if we are to add two vertices instead of one vertex for every edge in G.

Let us define the k-th semitotal 2-point graph to be a graph transformation that takes graph G and generates a new graph $R_2^k(G)$ by adding two vertices for every edge in G such that each of the vertices incident to that edge in G is adjacent to exactly one of the new vertices, where the new vertices are adjacent to each other. Repeat this process until $2k$ vertices are added for each edge in G.

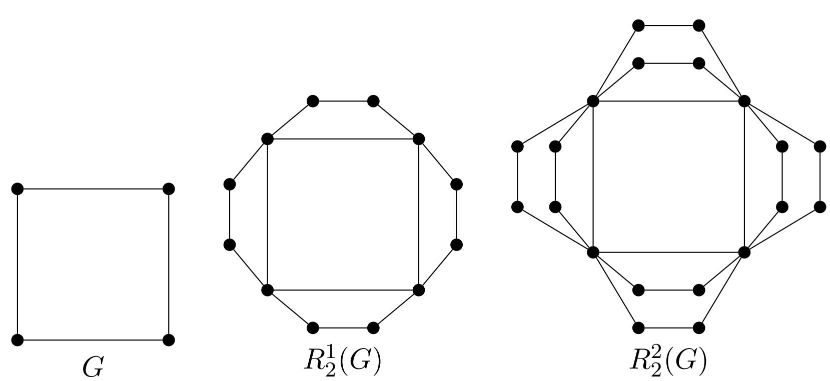


Figure 2.2: Example of k-th semitotal 2-point graph $R_2^k(G)$ of cycle graph C_4, where $k = 1, 2$

Theorem 2.4. *Let G be an r-regular graph of order n, then the Ihara zeta function of its k-th semitotal 2-point graph $R_2^k(G)$ satisfies*

$$\begin{aligned} Z_{R_2^k(G)}(u)^{-1} &= (1-u)^{\frac{knr+nr-2n}{2}}(1+u^2+u^4)^{\frac{knr-2n}{2}} \cdot \\ &\cdot \prod_{i=1}^{n}\Big(1+ru^2+ru^4+kru^6+ru^6-u^6-\lambda_i(ku^3+u+u^3+u^5)\Big), \end{aligned}$$

where $Spec(A_G)=\{\lambda_1=r,\ldots,\lambda_n\}$.

Proof. We first note that in $R_2^k(G)$, there are $n+2mk$ vertices and $m+3mk$ edges where n and m are the order and size of the original graph G.

By Theorem 1.1, the Ihara zeta function of $R_2^k(G)$ satisfies

$$\begin{aligned} Z_{R_2^k(G)}(u)^{-1} &= (1-u^2)^{m_{R_2^k(G)}-n_{R_2^k(G)}}\det\Big(I_{n_{R_2^k(G)}}-uA_{R_2^k(G)}+u^2\big(D_{R_2^k(G)}-I_{n_{R_2^k(G)}}\big)\Big) = \\ &= (1-u^2)^{\frac{knr+nr-2n}{2}}\det\Big(I_{n+2km}-uA_{R_2^k(G)}+u^2\big(D_{R_2^k(G)}-I_{n+2km}\big)\Big). \end{aligned} \tag{2.9}$$

Let B_G be the $n\times m$ incidence matrix of G, and define C_G to be the $n\times 2m$ 'stretched incidence matrix' of B_G, where we take B_G, duplicate each column (placing each duplicate immediately after the original column), then remove the second non-zero entry in the original columns and the first non-zero entry in the duplicated columns. For example, let A_G be a complete graph of order 4, then

$$B_G=\begin{pmatrix}1&0&0&1&1&0\\1&1&0&0&0&1\\0&1&1&0&1&0\\0&0&1&1&0&1\end{pmatrix} \text{ and } C_G=\begin{pmatrix}1&0&0&0&0&0&1&0&1&0&0&0\\0&1&1&0&0&0&0&0&0&0&1&0\\0&0&0&1&1&0&0&0&0&1&0&0\\0&0&0&0&0&1&0&1&0&0&0&1\end{pmatrix}.$$

Now we can setup the adjacency and degree matrices of $R_2^k(G)$ as follows

$$A_{R_2^k(G)}=\left(\begin{array}{c|c}A_G&\Gamma\\\hline\Gamma^T&\Phi\end{array}\right),$$

where $\Gamma = (C_G|\cdots|C_G)$ is a $n \times 2mk$ matrix containing k copies of C_G, and Φ is a $2mk \times 2mk$ square matrix such that

$$\Phi = \begin{pmatrix} 0 & 1 & & & & & \\ 1 & 0 & & & & & \\ & & 0 & 1 & & & \\ & & 1 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & 1 \\ & & & & & 1 & 0 \end{pmatrix}.$$

Hence, continuing from (2.9)

$$\begin{aligned} &\det\left[I_{n+2km} - uA_{R_2^k(G)} + u^2\big(D_{R_2^k(G)} - I_{n+2km}\big)\right] = \\ = \; &\det\left[\left(\begin{array}{c|c} I_n & 0 \\ \hline 0 & I_{2km} \end{array}\right) - u\left(\begin{array}{c|c} A_G & \Gamma \\ \hline \Gamma^T & \Phi \end{array}\right) + u^2\left(\begin{array}{c|c} (kr+r-1)I_n & 0 \\ \hline 0 & I_{2km} \end{array}\right)\right] = \\ = \; &\det\left[\left(\begin{array}{c|c} \left[1+u^2(kr+r-1)\right]I_n - uA_G & -u\Gamma \\ \hline -u\Gamma^T & (1+u^2)I_{2km} - u\Phi \end{array}\right)\right]. \end{aligned} \tag{2.10}$$

Let $\Omega = \Big((1+u^2)I_{2km} - u\Phi\Big)$ from the above matrix (2.10). Note that $(1+u^2)\left(\frac{1+u^2}{1+u^2+u^4}\right) - u\left(\frac{u}{1+u^2+u^4}\right) = 1$ and that $u\left(\frac{1+u^2}{1+u^2+u^4}\right) - u\left(\frac{1+u^2}{1+u^2+u^4}\right) = 0$, hence Ω

is invertible and

$$\Omega^{-1} = \begin{pmatrix} 1+u^2 & -u & & & & & \\ -u & 1+u^2 & & & & & \\ & & 1+u^2 & -u & & & \\ & & -u & 1+u^2 & & & \\ & & & & \ddots & & \\ & & & & & 1+u^2 & -u \\ & & & & & -u & 1+u^2 \end{pmatrix}^{-1} =$$

$$= \begin{pmatrix} \frac{1+u^2}{1+u^2+u^4} & \frac{u}{1+u^2+u^4} & & & & & \\ \frac{u}{1+u^2+u^4} & \frac{1+u^2}{1+u^2+u^4} & & & & & \\ & & \frac{1+u^2}{1+u^2+u^4} & \frac{u}{1+u^2+u^4} & & & \\ & & \frac{u}{1+u^2+u^4} & \frac{1+u^2}{1+u^2+u^4} & & & \\ & & & & \ddots & & \\ & & & & & \frac{1+u^2}{1+u^2+u^4} & \frac{u}{1+u^2+u^4} \\ & & & & & \frac{u}{1+u^2+u^4} & \frac{1+u^2}{1+u^2+u^4} \end{pmatrix},$$

and its determinant is

$$\det(\Omega) = \left[\det\begin{pmatrix} 1+u^2 & -u \\ -u & 1+u^2 \end{pmatrix}\right]^{km} =$$
$$= (1+u^2+u^4)^{km}.$$

Therefore, continuing from (2.10), we can calculate the determinant by using the Schur complement of $\Big((1+u^2)I_{2km} - u\Phi\Big)$, where

$$\det\left[\left(\begin{array}{c|c} \Big[1+u^2(kr+r-1)\Big]I_n - uA_G & -u\Gamma \\ \hline -u\Gamma^T & \Omega \end{array}\right)\right] =$$

$$= \det(\Omega)\cdot\det\Big[\Big(1+u^2(kr+r-1)\Big)I_n - uA_G - u^2\Gamma\Omega^{-1}\Gamma^T\Big] =$$

$$= (1+u^2+u^4)^{km} \det \Bigg[\Big(1+u^2(kr+r-1)\Big) I_n - uA_G -$$

$$- \left(\frac{kru^2(1+u^2)}{1+u^2+u^4} I_n + \frac{ku^3}{1+u^2+u^4} A_G \right) \Bigg] =$$

$$= (1+u^2+u^4)^{km} \det \left[\left(\frac{1+r(u^2+u^4+u^6+ku^6)-u^6}{1+u^2+u^4} \right) I_n - A_G \left(\frac{u+ku^3+u^3+u^5}{1+u^2+u^4} \right) \right] =$$

$$= (1+u^2+u^4)^{km-n} \cdot \prod_{i=1}^{n} \Big(1+r(u^2+u^4+u^6+ku^6) - \lambda_i(u+u^3+u^5+ku^3) - u^6\Big). \tag{2.11}$$

Now combine (2.9) with (2.11), we have

$$Z_{R_2^k(G)}(u)^{-1} = (1-u)^{\frac{knr+nr-2n}{2}} (1+u^2+u^4)^{\frac{knr-2n}{2}} \cdot$$

$$\cdot \prod_{i=1}^{n} \Big(1+ru^2+ru^4+kru^6+ru^6-u^6-\lambda_i(ku^3+u+u^3+u^5)\Big).$$

□

Theorem 2.5. *Let G be an r-regular graph of order n, then the complexity of its k-th semitotal 2-point graph $R_2^k(G)$ satisfies*

$$\tau\big(R_2^k(G)\big) = 3^{\frac{knr-2n}{2}+1} (3+k)^{n-1} \tau(G),$$

where $\tau(G)$ is the complexity of G.

Proof. By Theorem 1.7, we have

$$f'(1) = 2(m_{R_2^k(G)} - n_{R_2^k(G)}) \tau\big(R_2^k(G)\big), \tag{2.12}$$

where $f(u) = \det\Big(I_{n_{R_2^k(G)}} - uA_{R_2^k(G)} + u^2\big(D_{R_2^k(G)} - I_{n_{R_2^k(G)}}\big)\Big)$.

By (2.4), $f(u) = a(u)p_1(u)p_2(u)\ldots p_n(u)$ where $a(u) = (1+u^2+u^4)^{\frac{knr-2n}{2}}$, $p_i(u) = 1+ru^2+ru^4+kru^6+ru^6-u^6-\lambda_i(ku^3+u+u^3+u^5)$, and $\mathrm{Spec}(A_G) = \{\lambda_1 = r, \lambda_2, \ldots, \lambda_n\}$. Note that $p_1(1) = 0$, $p_1'(1) = 3(kr+r-2)$ and

$p_i(1) = (3+k)(r-\lambda_i)$. Hence we have

$$
\begin{aligned}
f'(1) &= a(1)p_1'(1)p_2(1)\cdots p_n(1) = \\
&= 3^{(knr-2n)/2}\Big(3(kr+r-2)\Big)\prod_{i=2}^{n}\Big[(3+k)(r-\lambda_i)\Big] = \\
&= 3^{(knr-2n)/2+1}(kr+r-2)(3+k)^{n-1}\prod_{i=2}^{n}(r-\lambda_i). \qquad (2.13)
\end{aligned}
$$

Now by substituting (2.13) into (2.12) and solving for $\tau\big(R_2^k(G)\big)$, we have

$$
\tau\big(R_2^k(G)\big) = 3^{(knr-2n)/2+1}(3+k)^{n-1}\frac{1}{n}\prod_{i=2}^{n}(r-\lambda_i) = 3^{(knr-2n)/2+1}(3+k)^{n-1}\tau(G),
$$

by Theorem 1.5. □

Theorem 2.6. *Let G be an r-regular graph of order n and let $Kf(G)$ be the Kirchhoff index of G. Then the zeta Kirchhoff index of its k-th semitotal 2-point graph $R_2^k(G)$ satisfies*

$$
Kf^z\big(R_2^k(G)\big) = \frac{3Kf(G)(kr+r-2)^2}{3+k}.
$$

Proof. By Theorem 1.8, we have

$$
f''(1) = 2\Big(Kf^z\big(R_2^k(G)\big) + 2m_{R_2^k(G)}n_{R_2^k(G)} - 2n_{R_2^k(G)}^2 + n_{R_2^k(G)}\Big)\tau\big(R_2^k(G)\big), \qquad (2.14)
$$

where $f(u) = a(u)p_1(u)p_2(u)\dots p_n(u)$ and $a(u) = (1+u^2+u^4)^{\frac{knr-2n}{2}}$ and $p_i(u) = 1 + ru^2 + ru^4 + kru^6 + ru^6 - u^6 - \lambda_i(ku^3 + u + u^3 + u^5)$ as the previous theorem.

Since $p_1(1) = 0$, we have

$$
\begin{aligned}
f''(1) &= 2a'(1)p_1'(1)p_2(1)\dots p_n(1) + a(1)p_1''(1)p_2(1)\dots p_n(1) + \\
&+ 2a(1)p_1'(1)\cdot\sum_{j=2}^{n}\frac{p_j'(1)}{p_j(1)}\prod_{i=2}^{n}p_i(1). \qquad (2.15)
\end{aligned}
$$

Note that $p_1'(1) = 3(kr + r - 2)$, $p_1''(1) = 6\big(r(4k+3) - 5\big)$, $p_i(1) = (3+k)(r - \lambda_i)$ and $p_i'(1) = -6 - 3\lambda_i(3+k) + 6(2+k)r$, so

$$f''(1) = 3^{\frac{knr-2n}{2}+1}\left(\prod_{i=2}^{n} p_i(1)\right)\cdot\Bigg[2(-5+4n+3r+4kr-2nr-4knr+knr^2+k^2nr^2)+ \\ + (kr+r-2)\sum_{j=2}^{n}\frac{6+9\lambda_j+3k\lambda_j-12r-6kr}{(3+k)(\lambda_j - r)}\Bigg]. \tag{2.16}$$

By substituting (2.16) into (2.14) and solving for $Kf^z\big(R_2^k(G)\big)$, we get

$$\begin{aligned} Kf^z\big(R_2^k(G)\big) &= 3n(1-n)(kr+r-2)+ \\ &\quad + \frac{knr+nr-2n}{3+k}\sum_{j=2}^{n}\frac{6+9\lambda_j+3k\lambda_j-12r-6kr}{\lambda_j - r} = \\ &= 3n(1-n)(kr+r-2)+ \\ &\quad + \frac{knr+nr-2n}{3+k}\sum_{j=2}^{n}\left[\frac{9(\lambda_j - r)}{\lambda_j - r} + \frac{3k(\lambda_j - r)}{\lambda_j - r} + \frac{6-3kr-3r}{\lambda_j - r}\right] = \\ &= 3n(1-n)(kr+r-2) + \frac{n(n-1)(kr+r-2)(9+3k)}{3+k} + \\ &\quad + \frac{n(kr+r-2)(3kr+3r-6)}{3+k}\sum_{j=2}^{n}\frac{1}{r-\lambda_j} = \\ &= \frac{3Kf(G)(kr+r-2)^2}{3+k}, \end{aligned}$$

by Theorem 1.6.

□

2.3 k-th semitotal multipoint graph of regular graphs

We will now generalize the semitotal point graph to cases beyond two vertices. Define the k-th semitotal s-point graph to be a graph transformation that takes graph G and generates a new graph $R_s^k(G)$ by adding exactly s vertices and $s+1$ edges for every edge in G, such that all the new vertices has a degree of 2. One can think of it as adding a path graph of order s for each edge in G, and connect the endpoints of the path graph to the two vertices adjacent to that edge. Repeat this process until ks vertices are added for each edge.

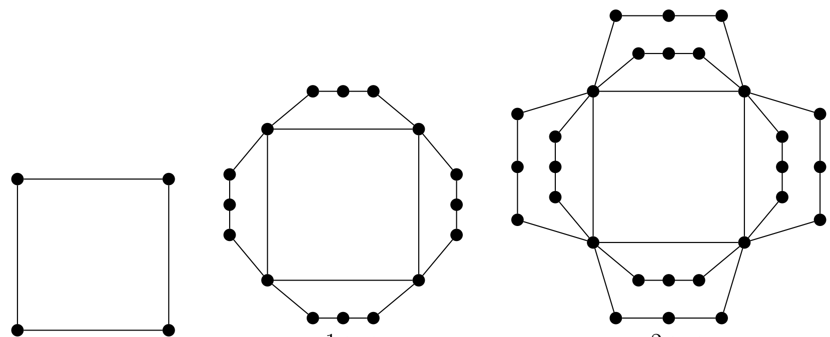


Figure 2.3: Example of k-th semitotal 3-point graph $R_3^k(G)$ of cycle graph C_4, where $k = 1, 2$

Theorem 2.7. *If G is an r-regular graph of order n, then the Ihara zeta function of its k-th semitotal s-point graph $R_s^k(G)$, where $s \geq 3$, satisfies*

$$
\begin{aligned}
Z_{R_s^k(G)}(u)^{-1} = (1-u^2)^{\frac{nr+knr-2n}{2}} \cdot \left(\frac{u^{2+2s}-1}{u^2-1}\right)^{\frac{knr}{2}} \left(\frac{1}{u^{2+2s}-1}\right)^{n} \cdot \\
\cdot \prod_{i=1}^{n} \Big[-1 + u^2 - ru^2 + u^{2+2s} - kru^{2+2s} - u^{4+2s} + ru^{4+2s} + kru^{4+2s} + \\
+ \lambda_i (u + ku^{1+s} - ku^{3+s} - u^{3+2s}) \Big],
\end{aligned}
$$

where $Spec(A_G) = \{\lambda_1 = r, \lambda_2, \ldots, \lambda_n\}$.

Proof. We first make the observation that there are $n + kms$ vertices and $m + km(s+1)$ edges in the graph $R_s^k(G)$. Hence by Theorem 1.1, the Ihara zeta function of $R_s^k(G)$ satisfies

$$
Z_{R_s^k(G)}(u)^{-1} = (1-u^2)^{\frac{nr+knr-2n}{2}} \det \Big[I_{n+kms} - uA_{R_s^k(G)} + u^2 \big(D_{R_s^k(G)} + I_{n+kms} \big) \Big]. \quad (2.17)
$$

Let B_G be the $n \times m$ incidence matrix of G, and defined C_G^s to be the $n \times ms$

'stretched incidence matrix' of B_G, similar to the construction in our previous construction for $R_2^k(G)$. We take B_G, duplicate each column, remove the lower entry in the first column and the upper entry in the second duplicated column, then insert $s-2$ empty columns in between them. For example, let A_G be the adjacency matrix of the complete graph of order 3, then

$$B_G = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \text{ and } C_G = \begin{pmatrix} 1 & 0 \cdots 0 & 0 & 0 & 0 \cdots 0 & 0 & 1 & 0 \cdots 0 & 0 \\ 0 & 0 \cdots 0 & 1 & 1 & 0 \cdots 0 & 0 & 0 & 0 \cdots 0 & 0 \\ 0 & \underbrace{0 \cdots 0}_{s-2 \text{ zeros}} & 0 & 0 & \underbrace{0 \cdots 0}_{s-2 \text{ zeros}} & 1 & 0 & \underbrace{0 \cdots 0}_{s-2 \text{ zeros}} & 1 \end{pmatrix}.$$

Thus, we have our adjacency matrix

$$A_{R_s^k(G)} = \left(\begin{array}{c|c} A_G & \Gamma \\ \hline \Gamma^T & \Psi \end{array} \right),$$

where $\Gamma = \big(C_G|\dots|C_G\big)$ is a $n \times kms$ matrix containing exactly k copies of C_G and Ψ is a $kms \times kms$ block diagonal matrix such that

$$\Psi = \underbrace{\begin{pmatrix} \psi_s & & & & \\ & \psi_s & & & \\ & & \ddots & & \\ & & & \psi_s & \\ & & & & \psi_s \end{pmatrix}}_{km \text{ copies of } \psi_s}, \text{ where } \psi_s = \underbrace{\begin{pmatrix} 0 & 1 & 0 & & & \\ 1 & 0 & 1 & & & \\ 0 & 1 & 0 & \ddots & & \\ & & \ddots & 0 & 1 & 0 \\ & & & 1 & 0 & 1 \\ & & & 0 & 1 & 0 \end{pmatrix}}_{s \text{ columns}}.$$

More precisely, ψ_s is an $s \times s$ matrix where $(\psi_s)_{ij} = 1$ for $j = i \pm 1$, and $(\psi_s)_{ij} = 0$ otherwise.

Let $M = I_{n+kms} - uA_{R_s^k(G)} + u^2\big(D_{R_s^k(G)} - I_{n+kms}\big)$ in (2.17), then

$$M = \left(\begin{array}{c|c} I_n & 0 \\ \hline 0 & I_{kms} \end{array}\right) - u\left(\begin{array}{c|c} A_G & \Gamma \\ \hline \Gamma^T & \Psi \end{array}\right) + u^2\left(\begin{array}{c|c} (k+1)D_G - I_n & 0 \\ \hline 0 & I_{kms} \end{array}\right) =$$

$$= \left(\begin{array}{c|c} \big[1+(kr+r-1)u^2\big]I_n - uA_G & -u\Gamma \\ \hline -u\Gamma^T & \Omega \end{array}\right),$$

where $\Omega = (1+u^2)I_{kms} - u\Psi$.

Note that Ω is a diagonal block matrix where

$$\Omega = \underbrace{\begin{pmatrix} \omega_s & & \\ & \ddots & \\ & & \omega_s \end{pmatrix}}_{km \text{ copies of } \omega_s}, \text{ where } \omega_s = \underbrace{\begin{pmatrix} 1+u^2 & -u & 0 & & & \\ -u & 1+u^2 & -u & & & \\ 0 & -u & 1+u^2 & \ddots & & \\ & & \ddots & 1+u^2 & -u & 0 \\ & & & -u & 1+u^2 & -u \\ & & & 0 & -u & 1+u^2 \end{pmatrix}}_{s \text{ columns}},$$

hence

$$\Omega^{-1} = \underbrace{\begin{pmatrix} \omega_s^{-1} & & \\ & \ddots & \\ & & \omega_s^{-1} \end{pmatrix}}_{km \text{ copies of } \omega_s^{-1}}.$$

More precisely, ω_s is an $s \times s$ matrix such that

$$(\omega_s)_{ij} = \begin{cases} 1+u^2 & , \text{ for } i=j; \\ -u & , \text{ for } j = i \pm 1; \\ 0 & , \text{ elsewhere.} \end{cases}$$

By [17], since ω_s is a tridiagonal matrix, we have

$$\left(\omega_s^{-1}\right)_{ij} = \begin{cases} \dfrac{(-1)^{i+j}(-u)^{j-i}\theta_{i-1}\phi_{j+1}}{\theta_s} & \text{if } i < j, \\ \dfrac{\theta_{i-1}\phi_{j+1}}{\theta_s} & \text{if } i = j, \\ \dfrac{(-1)^{i+j}(-u)^{i-j}\theta_{j-1}\phi_{i+1}}{\theta_s} & \text{if } i > j, \end{cases}$$

where $(\theta_i)_{i=0}^s = \theta_0, \theta_1, \ldots, \theta_s$ and $(\phi_i)_{i=1}^{s+1} = \phi_1, \phi_2, \ldots, \phi_{s+1}$ are sequences defined as

$$(\theta_i)_{i=0}^s = 1, 1+u^2, 1+u^2+u^4, \ldots, \sum_{k=0}^{s} u^{2k}, \text{ and}$$
$$(\phi_i)_{i=1}^{s+1} = \sum_{k=0}^{s} u^{2k}, \sum_{k=0}^{s-1} u^{2k}, \ldots, 1+u^2, 1.$$

Hence $\left(\omega_s^{-1}\right)_{ij}$ can be written as

$$\left(\omega_s^{-1}\right)_{ij} = \begin{cases} \dfrac{u^{j-i}\left(\sum_{k=0}^{i-1} u^{2k}\right)\left(\sum_{k=0}^{s-j} u^{2k}\right)}{\left(\sum_{k=0}^{s} u^{2k}\right)} & \text{if } i < j, \\ \dfrac{\left(\sum_{k=0}^{i-1} u^{2k}\right)\left(\sum_{k=0}^{s-j} u^{2k}\right)}{\left(\sum_{k=0}^{s} u^{2k}\right)} & \text{if } i = j, \\ \dfrac{u^{i-j}\left(\sum_{k=0}^{j-1} u^{2k}\right)\left(\sum_{k=0}^{s-i} u^{2k}\right)}{\left(\sum_{k=0}^{s} u^{2k}\right)} & \text{if } i > j, \end{cases} \tag{2.18}$$

and the Schur's complement of Ω is

$$\left[1+(kr+r-1)u^2\right]I_n - uA_G - (-u\Gamma)\Omega^{-1}(-u\Gamma^T) = \tag{2.19}$$
$$= \left[1+(kr+r-1)u^2\right]I_n - uA_G - u^2\left(\Gamma\Omega^{-1}\Gamma^T\right) =$$
$$= \left[1+(kr+r-1)u^2\right]I_n - uA_G - u^2\Big(\underbrace{C_G\big|\cdots\big|C_G}_{k\text{ copies}}\Big)\underbrace{\begin{pmatrix} \omega_s^{-1} & & \\ & \ddots & \\ & & \omega_s^{-1} \end{pmatrix}}_{km\text{ copies}}\left.\begin{pmatrix} C_G^T \\ \hline \vdots \\ \hline C_G^T \end{pmatrix}\right\}\text{k copies} =$$

$$= \big[1 + (kr + r - 1)u^2\big] I_n - uA_G - ku^2 \cdot C_G \underbrace{\begin{pmatrix} \omega_s^{-1} & & \\ & \ddots & \\ & & \omega_s^{-1} \end{pmatrix}}_{m \text{ copies}} C_G^T. \tag{2.20}$$

Now, let $\rho_1, \ldots, \rho_m$ be $n \times s$ matrices such that $C_G = \big(\rho_1 | \cdots | \rho_m\big)$, where each ρ_ℓ has only two nonzero entries of value 1, located on the first and last column on rows corresponding to the incident vertices α_ℓ and β_ℓ. Hence, we have

$$C_G \underbrace{\begin{pmatrix} \omega_s^{-1} & & \\ & \ddots & \\ & & \omega_s^{-1} \end{pmatrix}}_{m \text{ copies}} C_G^T = \sum_{\ell=1}^{m} \rho_\ell \omega_s^{-1} \rho_\ell^T =$$

$$= \sum_{\ell=1}^{m} \begin{pmatrix} \vdots & 0 \ \cdots \ 0 & \vdots \\ \rho_{\alpha_\ell 1} = 1 & \vdots & 0 \\ \vdots & & \vdots \\ 0 & \vdots & \rho_{\beta_\ell s} = 1 \\ \vdots & \underbrace{0 \ \cdots \ 0}_{s-2 \text{ zeros}} & \vdots \end{pmatrix} \omega_s^{-1} \begin{pmatrix} \cdots & \rho^T_{1\alpha_\ell} = 1 & \cdots & 0 & \cdots \\ 0 & & & & 0 \\ \vdots & & & & \vdots \\ 0 & & & & 0 \\ \cdots & 0 & \cdots & \rho^T_{s\beta_\ell} = 1 & \cdots \end{pmatrix} =$$

$$= \sum_{\ell=1}^{m} \begin{pmatrix} 0 & \cdots & 0 \\ (\rho_\ell \omega_s^{-1})_{\alpha_\ell 1} = (\omega_s^{-1})_{11} & \cdots & (\rho_\ell \omega_s^{-1})_{\alpha_\ell s} = (\omega_s^{-1})_{1s} \\ 0 & \cdots & 0 \\ (\rho_\ell \omega_s^{-1})_{\beta_\ell 1} = (\omega_s^{-1})_{s1} & \cdots & (\rho_\ell \omega_s^{-1})_{\beta_\ell s} = (\omega_s^{-1})_{ss} \\ 0 & \cdots & 0 \end{pmatrix} \cdot$$

$$\cdot \begin{pmatrix} \cdots & \rho^T_{1\alpha_\ell} = 1 & \cdots & 0 & \cdots \\ 0 & & & & 0 \\ \vdots & & & & \vdots \\ 0 & & & & 0 \\ \cdots & 0 & \cdots & \rho^T_{s\beta_\ell} = 1 & \cdots \end{pmatrix} =$$

$$= \sum_{\ell=1}^{m} \begin{pmatrix} 0 & & \cdots & & 0 \\ \vdots & \left(\rho_\ell \omega_s^{-1} \rho_\ell^T\right)_{\alpha_\ell \alpha_\ell} = (\omega_s^{-1})_{11} & 0 & \left(\rho_\ell \omega_s^{-1} \rho_\ell^T\right)_{\alpha_\ell \beta_\ell} = (\omega_s^{-1})_{1s} & \vdots \\ 0 & & \vdots & & 0 \\ \vdots & \left(\rho_\ell \omega_s^{-1} \rho_\ell^T\right)_{\beta_\ell \alpha_\ell} = (\omega_s^{-1})_{s1} & 0 & \left(\rho_\ell \omega_s^{-1} \rho_\ell^T\right)_{\beta_\ell \beta_\ell} = (\omega_s^{-1})_{ss} & \vdots \\ 0 & & \cdots & & 0 \end{pmatrix}, \text{ and by (2.18),}$$

$$= \sum_{\ell=1}^{m} \begin{pmatrix} 0 & & \cdots & & 0 \\ \vdots & \left(\rho_\ell \omega_s^{-1} \rho_\ell^T\right)_{\alpha_\ell \alpha_\ell} = \frac{\sum_{k=0}^{s-1} u^{2k}}{\sum_{k=0}^{s} u^{2k}} & 0 & \left(\rho_\ell \omega_s^{-1} \rho_\ell^T\right)_{\alpha_\ell \beta_\ell} = \frac{u^{2(s-1)}}{\sum_{k=0}^{s} u^{2k}} & \vdots \\ 0 & & \vdots & & 0 \\ \vdots & \left(\rho_\ell \omega_s^{-1} \rho_\ell^T\right)_{\beta_\ell \alpha_\ell} = \frac{u^{2(s-1)}}{\sum_{k=0}^{s} u^{2k}} & 0 & \left(\rho_\ell \omega_s^{-1} \rho_\ell^T\right)_{\beta_\ell \beta_\ell} = \frac{\sum_{k=0}^{s-1} u^{2k}}{\sum_{k=0}^{s} u^{2k}} & \vdots \\ 0 & & \cdots & & 0 \end{pmatrix} =$$

$$= \left(\frac{\sum_{k=0}^{s-1} u^{2k}}{\sum_{k=0}^{s} u^{2k}}\right) D_G + \left(\frac{u^{2(s-1)}}{\sum_{k=0}^{s} u^{2k}}\right) A_G,$$

therefore (2.20) becomes

$$\begin{aligned} &\left[1 + (kr + r - 1)u^2\right] I_n - uA_G - \left[ku^2 \left(\frac{\sum_{k=0}^{s-1} u^{2k}}{\sum_{k=0}^{s} u^{2k}}\right) D_G + ku^2 \left(\frac{u^{s-1}}{\sum_{k=0}^{s} u^{2k}}\right) A_G\right] = \\ &= \left[1 + (kr + r - 1)u^2\right] I_n - uA_G - \left(\frac{ku^2(u^{2s} - 1)}{u^{2+2s} - 1}\right) D_G - \left(\frac{ku^2(u^2 - 1)u^{s-1}}{u^{2+2s} - 1}\right) A_G = \\ &= \left(\frac{-1 + u^2 - ru^2 + u^{2+2s} - kru^{2+2s} - u^{4+2s} + ru^{4+2s} + kru^{4+2s}}{u^{2+2s} - 1}\right) I_n + \\ &\quad + \left(\frac{u + ku^{1+s} - ku^{3+s} - u^{3+2s}}{u^{2+2s} - 1}\right) A_G. \end{aligned}$$

Now to find $\det(\Omega)$, we should take advantage of it being a block diagonal matrix, where

$$\det(\Omega) = \det \underbrace{\begin{pmatrix} \omega_s & & \\ & \ddots & \\ & & \omega_s \end{pmatrix}}_{km \text{ copies}} = \det(\omega_s)^{km} =$$

$$= \det \underbrace{\begin{pmatrix} 1+u^2 & -u & 0 & & & \\ -u & 1+u^2 & -u & & & \\ 0 & -u & 1+u^2 & \ddots & & \\ & & \ddots & 1+u^2 & -u & 0 \\ & & & -u & 1+u^2 & -u \\ & & & 0 & -u & 1+u^2 \end{pmatrix}^{km}}_{s \text{ columns}} =$$

$$= \left(\frac{u^{2+2s}-1}{u^2-1}\right)^{km},$$

which allows us to proceed and calculate $\det(M)$,

$$\begin{aligned}
\det(M) = \; & \det(\Omega)\det\Big[\big(1+(kr+r-1)u^2\big)I_n - uA_G - (-u\Gamma)\Omega^{-1}(-u\Gamma^T)\Big] = \\
= \; & \left(\frac{u^{2+2s}-1}{u^2-1}\right)^{\frac{knr}{2}} \left(\frac{1}{u^{2+2s}-1}\right)^n \cdot \\
& \cdot \det\Bigg[\left(-1+u^2-ru^2+u^{2+2s}-kru^{2+2s}-u^{4+2s}+ru^{4+2s}+kru^{4+2s}\right)I_n+ \\
& + \left(u+ku^{1+s}-ku^{3+s}-u^{3+2s}\right)A_G\Bigg] = \\
= \; & \left(\frac{u^{2+2s}-1}{u^2-1}\right)^{\frac{knr}{2}} \left(\frac{1}{u^{2+2s}-1}\right)^n \cdot \\
& \cdot \prod_{i=1}^{n}\Big[-1+u^2-ru^2+u^{2+2s}-kru^{2+2s}-u^{4+2s}+ru^{4+2s}+kru^{4+2s}+ \\
& + \lambda_i\left(u+ku^{1+s}-ku^{3+s}-u^{3+2s}\right)\Big]. \qquad (2.21)
\end{aligned}$$

Therefore, by substituting (2.21) back into (2.17) we obtain the Ihara zeta function

$$\begin{aligned}
Z_{R_s^k(G)}(u)^{-1} = \; & (1-u^2)^{\frac{nr+knr-2n}{2}} \cdot \left(\frac{u^{2+2s}-1}{u^2-1}\right)^{\frac{knr}{2}} \left(\frac{1}{u^{2+2s}-1}\right)^n \cdot \\
& \cdot \prod_{i=1}^{n}\Big[-1+u^2-ru^2+u^{2+2s}-kru^{2+2s}-u^{4+2s}+ru^{4+2s}+kru^{4+2s}+ \\
& + \lambda_i(u+ku^{1+s}-ku^{3+s}-u^{3+2s})\Big],
\end{aligned}$$

where $\mathrm{Spec}(A_G) = \{\lambda_1 = r, , \lambda_2, \ldots, \lambda_n\}$. $\square$

We will now find the complexity of $R_s^k(G)$ combinatorically from any graph G, for $s \geq 1$.

Theorem 2.8. *Let G be an graph of order n, size m and complexity of $\tau(G)$. Then the complexity of its k-th semitotal s-point graph $R_s^k(G)$ where $s \geq 1$ satisfies*

$$\tau\big(R_s^k(G)\big) = (1 + k + s)^{n-1}(1 + s)^{km-n+1}\tau(G).$$

Proof. Let complexity of G be $\tau(G)$, and let the order and size of G be n and m.

We will count the number of spanning trees of $R_s^k(G)$ in two disjoint sets A, B, where the spanning trees in A contains some spanning tree of G as a subgraph while B do not. Recall that in $R_s^k(G)$, we create k unique trails between v_i and v_j for every edge v_iv_j in G. Each trail consist of s vertices and $s + 1$ edges excluding v_i and v_j. Note that there are $1 + s$ ways to disconnect each of these trails.

We know that every element in set A contains some spanning tree of G as a subgraph, therefore all of the k extra trails created must be disconnected. Hence, there are $(1 + s)^{km}$ ways to disconnect the trails and $\tau(G)$ ways to pick a spanning trees of G. Thus, $|A| = \tau(G)(1 + s)^{km}$.

For each element in set B, it can only contain a proper subgraph of any spanning tree of G. Since there are $\tau(G)$ spanning trees of G and each spanning tree contains $n - 1$ edges, there are $\sum_{i=1}^{n-1} \binom{n-1}{i}$ proper subgraphs for each spanning tree of G. Note that i counts the number of edges that was removed from spanning trees of G, hence we must reconnect the two vertices incident to the edges taken out for each of these proper subgraphs. There are k^i ways to pick the paths for the removed edges and $(1 + s)^{(k-1)i}$ ways to generate the remaining 'disconnected trails.' And for the remaining $m - i$ edges, there are $(1 + s)^{(m-i)k}$ ways to generate such 'disconnected trails."

Hence

$$
\begin{aligned}
|B| &= \tau(G)\sum_{i=1}^{n-1}\binom{n-1}{i}k^i(1+s)^{(k-1)i}(1+s)^{k(m-i)} = \\
&= \tau(G)\sum_{i=1}^{n-1}\binom{n-1}{i}k^i(1+s)^{km-i} = \\
&= \tau(G)(1+s)^{km}\sum_{i=1}^{n-1}\binom{n-1}{i}\left(\frac{k}{1+s}\right)^i = \\
&= \tau(G)(1+s)^{km}\left[\left(1+\frac{k}{1+s}\right)^{n-1} - \binom{n-1}{0}\left(1+\frac{k}{1+s}\right)^0\right] = \\
&= \tau(G)(1+s)^{km}\left[\frac{(1+k+s)^{n-1}}{(1+s)^{n-1}} - 1\right],
\end{aligned}
$$

and therefore by adding cardinality of the two sets we have

$$
\begin{aligned}
|A| + |B| &= \tau(G)(1+s)^{km} + \tau(G)(1+s)^{km}\left[\frac{(1+k+s)^{n-1}}{(1+s)^{n-1}} - 1\right] = \\
&= \tau(G)(1+s)^{km-n+1}(1+k+s)^{n-1}.
\end{aligned}
$$

□

We also provide an alternative proof that derives the complexity of $R_s^k(G)$ from the Ihara zeta function of r-regular graphs. Note that this corollary follows immediately from the preceding theorem but is restricted to r-regular graphs.

Corollary 2.9. *Let G be an r-regular graph of order n, then the complexity of its k-th semitotal s-point graph $R_s^k(G)$ where $s \geq 3$ satisfies*

$$
\tau\big(R_s^k(G)\big) = (1+k+s)^{n-1}(1+s)^{\frac{knr}{2}-n+1}\tau(G),
$$

where $\tau(G)$ is the complexity of G.

Proof. By [8, Thm 7.7] and Theorem 1.7, the Ihara zeta function of $R_s^k(G)$ satisfies

$$\lim_{u\to 1^-}\frac{(1-u)^{m_{R_s^k(G)}-n_{R_s^k(G)}+1}}{Z_{R_s^k(G)}(u)^{-1}} = -\frac{1}{2^{m_{R_s^k(G)}-n_{R_s^k(G)}+1}(m_{R_s^k(G)}-n_{R_s^k(G)})\tau\big(R_s^k(G)\big)}, \quad (2.22)$$

where $\tau(G)$ is the complexity of G.

Recall that $m_{R_s^k(G)} - n_{R_s^k(G)} = \frac{knr+nr-2}{2}$, hence the right hand side of (2.22) is equal to

$$-\frac{1}{2^{\frac{knr+nr-2}{2}+1}(knr-nr-2)\tau\big(R_s^k(G)\big)}. \quad (2.23)$$

As for the left hand side of (2.22), we have

$$\begin{aligned}
&\lim_{u\to 1^-}\frac{(1-u)^{m_{R_s^k(G)}-n_{R_s^k(G)}+1}}{Z_{R_s^k(G)}(u)^{-1}} = \\
&= \lim_{u\to 1^-}\frac{(1-u)^{\frac{knr+nr-2}{2}+1}}{(1-u^2)^{\frac{knr+nr-2}{2}}\left(\frac{u^{2+2s}-1}{u^2-1}\right)^{\frac{knr}{2}}\prod_{i=1}^{n}\rho_i} = \\
&= \lim_{u\to 1^-}\frac{1-u}{(1+u)^{\frac{knr+nr-2}{2}}\left(\frac{u^{2+2s}-1}{u^2-1}\right)^{\frac{knr}{2}}\prod_{i=1}^{n}\rho_i}, \quad (2.24)
\end{aligned}$$

where

$$\begin{aligned}
\rho_i = \Bigg[&\frac{-1+u^2-ru^2+u^{2+2s}-kru^{2+2s}-u^{4+2s}+ru^{4+2s}+kru^{4+2s}}{u^{2+2s}-1}+ \\
&+\frac{\lambda_i(u+ku^{1+s}-ku^{3+s}-u^{3+2s})}{u^{2+2s}-1}\Bigg].
\end{aligned}$$

Note that

$$\begin{aligned}
\rho_1 = \Bigg[&\frac{-1+u^2-ru^2+u^{2+2s}-kru^{2+2s}-u^{4+2s}+ru^{4+2s}+kru^{4+2s}}{u^{2+2s}-1}+ \\
&+\frac{r(u+ku^{1+s}-ku^{3+s}-u^{3+2s})}{u^{2+2s}-1}\Bigg] =
\end{aligned}$$

$$= \frac{(u-1)(u^{1+s}-1)\Big(-1-u+ru-u^{1+s}+kru^{1+s}-u^{2+s}+ru^{2+s}+kru^{2+s}\Big)}{u^{2+2s}-1},$$

and

$$\lim_{u\to 1^-} \frac{(u^{1+s}-1)\Big(-1-u+ru-u^{1+s}+kru^{1+s}-u^{2+s}+ru^{2+s}+kru^{2+s}\Big)}{u^{2+2s}-1} = kr+r-2.$$

Hence, (2.24) is reduced to

$$\lim_{u\to 1^-} \frac{-1}{(1+u)^{\frac{knr+nr-2}{2}}\left(\frac{u^{2+2s}-1}{u^2-1}\right)^{\frac{knr}{2}}(kr+r-2)\prod_{i=2}^{n}\rho_i}. \tag{2.25}$$

Then, we find the limit of each of the remaining factors in the denominator of (2.25), where

$$\lim_{u\to 1^-}(1+u)^{\frac{knr+nr-2}{2}} = 2^{\frac{knr+nr-2}{2}};$$
$$\lim_{u\to 1^-}\left(\frac{u^{2+2s}-1}{u^2-1}\right)^{\frac{knr}{2}} = (1+s)^{\frac{knr}{2}}, \text{ and };$$
$$\lim_{u\to 1^-} p_i = \frac{(r-\lambda_i)(1+k+s)}{1+s}.$$

Therefore, (2.25) becomes

$$\frac{-1}{2^{\frac{knr+nr-2}{2}}(1+s)^{\frac{knr}{2}}(kr+r-2)\prod_{i=2}^{n}\left(\frac{(r-\lambda_i)(1+k+s)}{1+s}\right)} =$$
$$= \frac{-1}{2^{\frac{knr+nr-2}{2}}(1+s)^{\frac{knr}{2}-n+1}(kr+r-2)(1+k+s)^{n-1}\cdot n\tau(G)}, \tag{2.26}$$

by Theorem 1.5.

Finally we set (2.26) equal to (2.23) and solve for $\tau\big(R_s^k(G)\big)$, we have

$$\tau\big(R_s^k(G)\big) = \frac{2^{\frac{knr+nr-2}{2}}(1+s)^{\frac{knr}{2}-n+1}(kr+r-2)(1+k+s)^{n-1}\cdot n\tau(G)}{2^{\frac{knr+nr-2}{2}+1}(knr-nr-2)} =$$

$$= (1+k+s)^{n-1}(1+s)^{\frac{knr}{2}-n+1}\tau(G),$$

as desired. $\square$

We will now derive a generalized formula for the zeta Kirchhoff index of $R_s^k(G)$ of a regular graph G.

Theorem 2.10. *Let G be an r-regular graph, then the zeta Kirchhoff index of its k-th semitotal s-point graph $R_s^k(G)$ satisfies*

$$Kf^z\Big(R_s^k(G)\Big) = \frac{Kf(G)(kr+r-2)^2(s+1)}{k+s+1},$$

where $Kf(G)$ is the Kirchhoff index of G.

Proof. We partition the set of vertices of $R_s^k(G)$ into two sets A and B, where A consists of vertices created from the graph transform, and B consists of vertices from the base graph G.

Let $n_R = n + \frac{knrs}{2}$ be the degree of $R_s^k(G)$ and let r_{ij} be the resistance distance between vertices i and j in $R^k(s)$. Then, we have

$$\begin{aligned} Kf^z\Big(R_s^k(G)\Big) &= \sum_{1\le i<j\le n_R} (d_{ii}-2)(d_{jj}-2)r_{ij} = \\ &= \sum_{\substack{1\le i<j\le n_R\\ i\in B, j\in B}} (d_{ii}-2)(d_{jj}-2)r_{ij} + \sum_{\substack{1\le i<j\le n_R\\ i\in A}} (d_{ii}-2)(d_{jj}-2)r_{ij}. \end{aligned} \tag{2.27}$$

Note that if $i \in A$, then $d_{ii} = 2$; and if $i \in B$, then $d_{ii} = kr + r$. Hence (2.27) is reduced to

$$Kf^z\Big(R_s^k(G)\Big) = \sum_{\substack{1\le i<j\le n_R\\ i\in B, j\in B}} (kr+r-2)^2 r_{ij}. \tag{2.28}$$

If $i, j \in B$ and are incident in $R_s^k(G)$, then

$$r_{ij} = \frac{1}{1+\frac{k}{s+1}} \cdot \overline{r_{ij}} = \frac{s+1}{k+s+1} \cdot \overline{r_{ij}},$$

by the law of resistance of circuits in parallel and series, where $\overline{r_{ij}}$ is the resistance between vertices i, j in G.

Therefore, (2.28) becomes

$$\begin{aligned} Kf^z\left(R_s^k(G)\right) &= \sum_{\substack{1\le i<j\le n_R \\ i\in B, j\in B}} (kr+r-2)^2\left(\frac{s+1}{k+s+1}\cdot\overline{r_{ij}}\right) = \\ &= \frac{(kr+r-2)^2(s+1)}{k+s+1} \cdot \sum_{\substack{1\le i<j\le n_R \\ i\in B, j\in B}} \overline{r_{ij}} = \\ &= \frac{Kf(G)(kr+r-2)^2(s+1)}{k+s+1}, \end{aligned}$$

where $Kf(G)$ is the Kirchhoff index of G. □

2.4 Middle graph of regular graphs

The middle graph of a graph G is the graph $M(G)$, where the vertices correspond to the edges and vertices of G; two vertices in $M(G)$ are adjacent if the corresponding vertices or edges are incident in G.

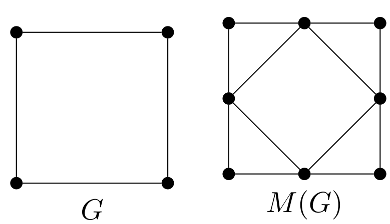


Figure 2.4: Example of the middle graph of cycle graph C_4

The Ihara zeta function and complexity of the middle graph of a regular graph are given as follows by Kwak and Sato:

Theorem 2.11 ([10]). *Let G be an r-regular graph. Then the Ihara zeta function of its middle graph $M(G)$ satisfies*

$$Z_{M(G)}(u)^{-1} = (1-u^2)^{\frac{n}{2}(r^2-2)} a(u) \prod_{i=1}^{n} p_i(u),$$

where $a(u) = \Big(1+2u+(2r-1)u^2\Big)^{m-n}$ *and*
$p_i(u) = \Big(1-(r-2)u+(2r-2)u^2-(r-1)(r-2)u^3-u(1+u+(r-1)u^2)\lambda_i+(r-1)(2r-1)u^4\Big).$

Theorem 2.12 ([10]). *Let G be an r-regular graph. Then the complexity of its middle graph $M(G)$ satisfies*

$$\tau\big(M(G)\big) = \frac{1}{n} 2^{m-n+1}(r+1)^{m-1} \prod_{i=2}^{n} (r-\lambda_i),$$

where $Spec(A_G) = \{\lambda_1 = r, \lambda_2, \ldots, \lambda_n\}$.

We can now prove:

Theorem 2.13. *Let G be an r-regular graph of order n with Kirchhoff index $Kf(G)$. Then the zeta Kirchhoff index of its middle graph $M(G)$ satisfies*

$$\begin{aligned} Kf^z\big(M(G)\big) &= \frac{4Kf(G)(r^2-2)^2 + n(r-2)\Big(2+n(r^2-2)(r+1)\Big)}{2(1+r)} = \\ &= \frac{2(r^2-2)^2}{r+1} Kf(G) + \frac{2(m-n)}{r+1} + (m-n)n(r^2-2). \end{aligned}$$

Proof. Let $f(u) = \det\Big(I_{n+m} - uA_{M(G)} + u^2\big(D_{M(G)} - I_{n+m}\big)\Big)$. Note that the order of $M(G)$ is $n+m$ and the size of $M(G)$ is $2m+2m(r-1)$. By Theorems 1.8 and 2.12, we have

$$f''(1) = 2\Big(Kf^z\big(M(G)\big) + 2m_{M(G)}n_{M(G)} - 2n_{M(G)}^2 + n_{M(G)}\Big)\tau\big(M(G)\big) =$$

$$= \frac{2^{m-n+2}(1+r)^{m-1}}{n} \left(\prod_{i=2}^{n}(r-\lambda_i)\right) \Big(Kf^z\big(M(G)\big) + (m+n)(2mr-2n+1)\Big), \tag{2.29}$$

where $\mathrm{Spec}(A_G) = \{\lambda_1 = r, \lambda_2, \dots, \lambda_n\}$.

By Theorem 2.11,

$$f(u) = a(u)\prod_{i=1}^{n} p_i(u). \tag{2.30}$$

Note that we have $a(1) = (2+2r)^{m-n}$, $a'(1) = \frac{2r(m-n)(2+2r)^{m-n}}{1+r}$, $p_1(1) = 0$, $p_1'(1) = 2r^2-4$, $p_1''(1) = 2(6r^2-5r-2)$, $p_i(1) = (1+r)(r-\lambda_i)$ and $p_i'(1) = 5r^2-3r\lambda_i-4$. Therefore by differentiating (2.30) twice we obtain,

$$\begin{aligned}
f''(1) &= a'(1)p_1'(1)p_2(1)\dots p_n(1) + \Big(a'(1)p_1'(1)p_2(1)\dots p_n(1) + a(1)p_1''(1)p_2(1)\dots p_n(1) + \\
&\quad + 2a(1)p'(1)\Big(p_2'(1)p_3(1)\dots p_n(1) + \dots + p_2(1)\dots p_{n-1}(1)p_n'(1)\Big) = \\
&= 2\left(\frac{2r(m-n)(2+2r)^{m-n}}{1+r}\right)(2r^2-4)\prod_{i=2}^{n}\Big((1+r)(r-\lambda_i)\Big) + \\
&\quad + 2(2+2r)^{m-n}(6r^2-5r-2)\prod_{i=2}^{n}\Big((1+r)(r-\lambda_i)\Big) + \\
&\quad + 2(2+2r)^{m-n}(2r^2-4)\sum_{j=2}^{n}\left(\frac{5r^2-3\lambda_j r-4}{(1+r)(r-\lambda_j)}\prod_{i=2}^{n}\Big((1+r)(r-\lambda_i)\Big)\right) = \\
&= 2^{m-n+1}(1+r)^{m-n}\prod_{i=2}^{n}\Big((1+r)(r-\lambda_i)\Big)\left[\frac{2r(2r^2-4)(m-n)}{1+r} + (6r^2-5r-2) + \right. \\
&\quad \left. + \frac{2r^2-4}{1+r}\sum_{j=1}^{n}\frac{5r^2-3\lambda_j r-4}{(1+r)(r-\lambda_j)}\right] = \\
&= 2^{m-n+1}(1+r)^{m-1}\prod_{i=2}^{n}\mu_i\left[\frac{2r(2r^2-4)(m-n)}{1+r} + (6r^2-5r-2) + \right. \\
&\quad \left. + \frac{2r^2-4}{1+r}\sum_{j=2}^{n}\frac{3r\mu_j+2r^2-4}{\mu_j}\right] = \\
&= 2^{m-n+1}(1+r)^{m-1}\prod_{i=2}^{n}\mu_i\left[\frac{2r(2r^2-4)(m-n)}{1+r} + (6r^2-5r-2) + \right.
\end{aligned}$$

$$+\frac{2r^2-4}{1+r}\left(3r(n-1)+(2r^2-4)\sum_{j=2}^{n}\frac{1}{\mu_j}\right)\Bigg] =$$

$$= 2^{m-n+1}(1+r)^{m-1}\prod_{i=2}^{n}\mu_i\left[\frac{2r^3(2m+n)+r^2+r(5-4n-8m)-2}{1+r}+\right.$$

$$\left.+\frac{(2r^2-4)^2}{1+r}\sum_{j=2}^{n}\frac{1}{\mu_j}\right], \tag{2.31}$$

where $\mathrm{Spec}(L_G) = \{\mu_1 = 0, \mu_2, \ldots, \mu_n\}$.

By substituting (2.31) into (2.29) and solving for $Kf^z\big(M(G)\big)$, we obtain

$$Kf^z\big(M(G)\big) = \frac{n}{2}\left[\frac{2r^3(2m+n)+r^2+r(5-4n-8m)-2}{1+r}+\frac{(2r^2-4)^2}{1+r}\sum_{j=2}^{n}\frac{1}{\mu_j}\right]-$$

$$-(m+n)(2mr-2n+1) =$$

$$= \frac{n\big[2r^3(2m+n)+r^2+r(5-4n-8m)-2\big]+Kf(G)(2r^2-4)^2}{2(1+r)}-$$

$$-(m+n)(2mr-2n+1) =$$

$$= \frac{4Kf(G)(r^2-2)^2+n(r-2)\big(2+n(r^3+r^2-2r-2)\big)}{2(1+r)},$$

by Theorem 1.6.

□

2.5 Quasitotal graph of regular graphs

The quasitotal graph of a graph G is the graph $QT(G)$, where the vertices correspond to the edges and vertices of G; two vertices in $QT(G)$ are adjacent if (i) the corresponding vertices or edges are incident in G, or (ii) the corresponding vertices are not adjacent in G.

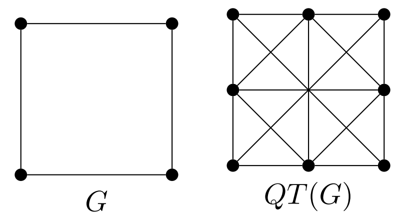


Figure 2.5: Example of the quasitotal graph of cycle graph C_4

The Ihara zeta function and complexity of the quasitotal graph of a regular graph are given as follows:

Theorem 2.14 ([16])**.** *Let G be an r-regular graph with $Spec(A_G) = \{\lambda_1 = r, \lambda_2, \ldots, \lambda_n\}$. Then the Ihara zeta function of its quasitotal graph $QT(G)$ satisfies*

$$Z_{QT(G)}(u)^{-1} = (1-u^2)^{\frac{n}{2}(r^2+n-r-3)}a(u)b(u)p(u)$$

where

$$\begin{aligned}
a(u) &= \Big(1+2u+u^2(2r-1)\Big)^{m-n},\\
b(u) &= (u-1)\Big(-1+(-4+n+r)u+(-3+2n+r-2nr+2r^2)u^2+\\
&\quad +(2-n-4r+2nr)u^3\Big),\\
p(u) &= \prod_{i=2}^{n}\Big[1+(3-r)u+(-1+n-\lambda_i r-\lambda_i^2)u^2+\\
&\quad +(-5+2n-nr+4r+\lambda_i-\lambda_i n++2\lambda_i r)u^3+(n-2)(2r-1)u^4\Big].
\end{aligned}$$

We shall denote each factor in the product $p(u)$ as $p_i(u)$. So,

$$p(u) = \prod_{i=2}^{n} p_i(u), \text{ where}$$

$$p_i(u) = \Big[1+(3-r)u+(-1+n-\lambda_i r-\lambda_i^2)u^2+$$

$$+(-5+2n-nr+4r+\lambda_i-\lambda_i n++2\lambda_i r)u^3+(n-2)(2r-1)u^4\Big].$$

Theorem 2.15 ([16]). *Let G be an r-regular graph. Then the complexity of its quasitotal graph $QT(G)$ satisfies*

$$\tau\big(QT(G)\big)=\frac{2}{n}(2+2r)^{\frac{n}{2}(r-2)}\left(\prod_{i=2}^{n}[\lambda_i-\lambda_i^2+2n-\lambda_i n-r+\lambda_i r+nr]\right),$$

where $Spec(A_G)=\{\lambda_1=r,\lambda_2,\ldots,\lambda_n\}$.

We can now prove:

Theorem 2.16. *Let G be an r-regular graph of order n with $Spec(A_G)=\{\lambda_1=r,\lambda_2,\ldots,\lambda_n\}$. Then the zeta Kirchhoff index of its quasitotal graph $QT(G)$ satisfies*

$$Kf^z\big(QT(G)\big)=\frac{n(1-r)(6-8n+2n^2+6r-nr-n^2r+3nr^2-nr^3)}{2(1+r)}+$$
$$+n(r^2-r+n-3)\left[\sum_{i=2}^{n}\frac{-6+\lambda_i-\lambda_i n-3r+2\lambda_i r+3nr}{\lambda_i-\lambda_i^2+2n-\lambda_i n-r+\lambda_i r+nr}\right].$$

Proof. Let $f(u)=\det\Big(I_{n+m}-uA_{QT(G)}+u^2\big(D_{QT(G)}-I_{n+m}\big)\Big)$.

By Theorems 1.8, we have

$$f''(1)=\Big(Kf^z\big(QT(G)\big)+(m+n)(1-2m-3n+n^2+nr^2)\Big)\tau\big(QT(G)\big). \tag{2.32}$$

By Theorem 2.14,

$$f(u)=a(u)b(u)\prod_{i=2}^{n}p_i(u). \tag{2.33}$$

Note that $a(1)=(2+2r)^{m-n}$, $a'(1)=\frac{2r(m-n)(2+2r)^{m-n}}{1+r}$, $b(1)=0$, $b'(1)=2(r^2-r+n-3)$, $b''(1)=2(4r^2+2nr-9r+2n-4)$, $p_i(1)=\lambda_i-\lambda_i^2+2n-\lambda_i n-r+\lambda_i r+nr$ and

$p_i'(1) = 3\lambda_i - 2\lambda_i^2 + 4n - 3\lambda_i n - 5r + 4\lambda_i r + 5nr - 6.$

Therefore, by differentiating (2.33) twice, we got

$$
\begin{aligned}
f''(1) &= a(1)b''(1)p(1) + 2a'(1)b'(1)p(1) + 2a(1)b'(1)p'(1) = \\
&= 8p(1)\frac{r(m-n)(2+2r)^{m-n}(r^2-r+n-3)}{1+r} + 2p(1)(2+2r)^{m-n}\cdot \\
&\quad \cdot (4r^2+2nr-9r+2n-4) + 4(2+2r)^{m-n}(r^2-r+n-3)p'(1) = \\
&= 2\left(\prod_{j=2}^{n}[\lambda_j - \lambda_j^2 + 2n - \lambda_j n - r + \lambda_j r + nr]\right)(2+2r)^{\frac{nr}{2}-n}\cdot \\
&\quad \cdot \left[\frac{4r(m-n)(r^2-r+n-3)}{1+r} + 4r^2 + 2nr - 9r + 2n - 4 + \right. \\
&\quad \left. + 2(r^2-r+n-3)\cdot\left(\sum_{i=2}^{n}\frac{p_i'(1)}{p_i(1)}\right)\right]. \qquad (2.34)
\end{aligned}
$$

Now, by setting (2.34) equal to (2.32), we obtain

$$
\begin{aligned}
&Kf^z\big(QT(G)\big) = \\
&= \frac{6n - 12n^2 + 2n^3 + n^2r - 3n^3r - 6nr^2 + 2n^2r^2 + n^3r^2 - 4n^2r^3 + n^2r^4}{2(1+r)} + \\
&\quad + n(r^2-r+n-3)\left(\sum_{i=2}^{n}\frac{p_i'(1)}{p_i(1)}\right) = \\
&= \frac{6n - 12n^2 + 2n^3 + n^2r - 3n^3r - 6nr^2 + 2n^2r^2 + n^3r^2 - 4n^2r^3 + n^2r^4}{2(1+r)} + \\
&\quad + n(r^2-r+n-3)\left(\sum_{i=2}^{n}\frac{-6 + 3\lambda_i - 2\lambda_i^2 + 4n - 3\lambda_i n - 5r + 4\lambda_i r + 5nr}{\lambda_i - \lambda_i^2 + 2n - \lambda_i n - r + \lambda_i r + nr}\right) = \\
&= \frac{n(1-r)(6 - 8n + 2n^2 + 6r - nr - n^2r + 3nr^2 - nr^3)}{2(1+r)} + \\
&\quad + n(r^2-r+n-3)\left(\sum_{i=2}^{n}\frac{-6 + \lambda_i - \lambda_i n - 3r + 2\lambda_i r + 3nr}{\lambda_i - \lambda_i^2 + 2n - \lambda_i n - r + \lambda_i r + nr}\right).
\end{aligned}
$$

□

Corollary 2.17. *Let G be an r-regular graph of order n with $Spec(L_G) = \{\mu_1 = r, \mu_2, \ldots, \mu_n\}$. Then the zeta Kirchhoff index of its quasitotal graph*

QT(G) satisfies

$$Kf^z\big(QT(G)\big) = \frac{n(1-r)(6-8n+2n^2+6r-nr-n^2r+3nr^2-nr^3)}{2(1+r)} +$$
$$+\, n(r^2-r+n-3)\left[\sum_{i=2}^{n} \frac{(n-2r-1)\mu_i + 2r^2+2nr-2r-6}{\mu_i(n+r-1-\mu_i)+2n}\right].$$

2.6 Corona of two regular graphs

A corona $G \circ H$ of two graphs G (order n) and H (order k) is constructed by taking G, making a copy of H for each vertex of G, and connecting each vertex in G to all vertices of the corresponding copy of H.

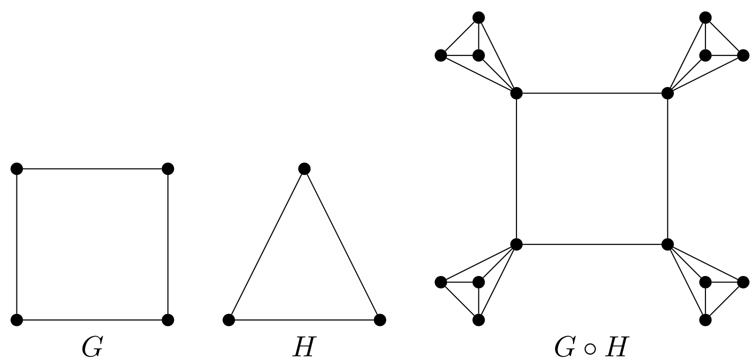


Figure 2.6: Example of the corona $G \circ H$ of $G = C_4, H = C_3$

The Ihara zeta function and complexity of $G \circ H$ are given as follows:

Theorem 2.18 ([15]). *Let G be an r-regular graph of order n and let H be an s-regular graph of order k. Then the Ihara zeta function of the corona $G \circ H$ satisfies*

$$Z_{G\circ H}(u)^{-1} = (1-u^2)^{\frac{n}{2}(ks+r-2)} a(u) \left(\prod_{i=2}^{n} p_i(u)\right)\left(\prod_{j=2}^{k} q_j(u)\right),$$

where $a(u)=(1-u)\big(1-(r+s-1)u+rsu^2-s(k+r-1)u^3\big)$,

$$p_i(u)=1-(s+\lambda_i)u+(s+r+s\lambda_i-1)u^2-s(\lambda_i+k+r-1)u^3+s(k+r-1)u^4,$$

$$q_j(u)=(1-\theta_j u+su^2)^n,$$

and $Spec(A_G)=\{\lambda_1=r,\lambda_2,\ldots,\lambda_n\}$, $Spec(A_H)=\{\theta_1=s,\theta_2,\ldots,\theta_k\}$.

Theorem 2.19 ([15])**.** *Let G be an r-regular graph of order n and let H be an s-regular graph of order k. Then the complexity of the corona $G\circ H$ satisfies*

$$\tau(G\circ H)=\tau(G)\prod_{j=2}^{k}(1+\gamma_j)^n,$$

where $Spec(L_H)=\{\gamma_1=0,\gamma_2,\ldots,\gamma_k\}$.

Theorem 2.20. *Let G be an r-regular graph of order n and let H be an s-regular graph of order k. Then the zeta Kirchhoff index of the corona $G\circ H$ satisfies*

$$Kf^z\big(G\circ H\big)=n(s-1)(k+kns-ks+nr-2n)+Kf(G)(ks+r-2)^2+$$
$$+n^2(ks+r-2)(s-1)\sum_{j=2}^{k}\frac{1}{1+\gamma_j},$$

where $Kf(G)$ is the Kirchhoff index of G and $Spec(L_H)=\{\gamma_1=0,\gamma_2,\ldots,\gamma_k\}$.

Proof. Let $f(u)=\det\Big(I_{n_{G\circ H}}-uA_{G\circ H}+u^2(D_{G\circ H}-I_{n_{G\circ H}})\Big)$. By Theorem 1.8, we have

$$f''(1)=2\Big(Kf^z(G\circ H)+n(1+k)\big(1+n(2ks+r-2)\big)\Big)\tau\big(G\circ H\big), \tag{2.35}$$

and by Theorem 2.18,

$$f(u)=a(u)\Bigg(\prod_{i=2}^{n}p_i(u)\Bigg)\Bigg(\prod_{j=2}^{k}q_j(u)\Bigg). \tag{2.36}$$

Note that $a(1)=0$, $a'(1)=-2+r+ks$, $a''(1)=2(-1+r-2s+3ks+rs)$,

$p_i(1) = r - \lambda_i$, $p_i'(1) = -2 - \lambda_i + 2r + ks - \lambda_i s + rs$, $q_j(1) = (1 + s - \theta_j)^n$ and $q_j'(1) = n(2s - \theta_j)(1 + s - \theta_j)^{n-1}$.

Therefore, from (2.36) we have

$$\begin{aligned}
&f''(1) = \\
&= a''(1)\left(\prod_{i=2}^{n} p_i(1)\right)\left(\prod_{j=2}^{k} q_j(1)\right) + \\
&+2a'(1)\left(\prod_{i=2}^{n} p_i(1)\right)\Big(p_2'(1)p_3(1)\dots p_n(1) + \dots + p_2(1)\dots p_{n-1}(1)p_n'(1)\Big) + \\
&+2a'(1)\left(\prod_{j=2}^{k} q_j(1)\right)\Big(q_2'(1)q_3(1)\dots q_k(1) + \dots + q_2(1)\dots q_{k-1}(1)q_k'(1)\Big) = \\
&= \left(\prod_{i=2}^{n} p_i(1)\right)\left(\prod_{j=2}^{k} q_j(1)\right)\left(a''(1) + 2a'(1)\sum_{i=2}^{n}\frac{p_i'(1)}{p_i(1)} + 2a'(1)\sum_{j=2}^{k}\frac{q_j'(1)}{q_j(1)}\right) = \\
&= 2\left(\prod_{i=2}^{n}(r - \lambda_i)\right)\left(\prod_{j=2}^{k}(1 + s - \theta_j)^n\right)\Bigg[3ks + rs + r - 2s - 1 + \\
&+(ks + r - 2)\sum_{i=2}^{n}\frac{2 - 2r - ks - rs + \lambda_i + \lambda_i s}{\lambda_i - r} + (ks + r - 2)\sum_{j=2}^{k}\frac{n(2s - \theta_j)}{1 + s - \theta_j}\Bigg] = \\
&= 2\left(\prod_{i=2}^{n}(r - \lambda_i)\right)\left(\prod_{j=2}^{k}(1 + s - \theta_j)^n\right)\Bigg[1 - 2kn + knr + 2ks - 2ns + k^2ns + \\
&+nrs - ks^2 + kns^2 + (ks + r - 2)^2\sum_{i=2}^{n}\frac{1}{r - \lambda_i} + (ks + r - 2)\sum_{j=2}^{k}\frac{ns - n}{1 + s - \theta_j}\Bigg]. \qquad (2.37)
\end{aligned}$$

Now we substitute (2.37) and $\tau(G \circ H)$ (by applying Theorem 2.19) into (2.35). Then by solving for $Kf^z(G \circ H)$, we obtain

$$\begin{aligned}
Kf^z(G \circ H) &= n(s - 1)(k + kns - ks + nr - 2n) + n(ks + r - 2)^2\sum_{i=2}^{n}\frac{1}{r - \lambda_i} + \\
&+n(ks + r - 2)\sum_{j=2}^{k}\frac{ns - n}{1 + s - \theta_j} = \\
&= n(s - 1)(k + kns - ks + nr - 2n) + Kf(G)(ks + r - 2)^2 +
\end{aligned}$$

$$+n^2(ks+r-2)(s-1)\sum_{j=2}^{k}\frac{1}{1+\gamma_j},$$

by Theorem 1.6. □

2.7 Edge corona of two regular graphs

An edge corona $G \diamond H$ of two graphs G (order n) and H (order k) is constructed by taking G, making n copies of H, then for each edge in G connects both incident vertices of that edge to all vertices in each copy of H.

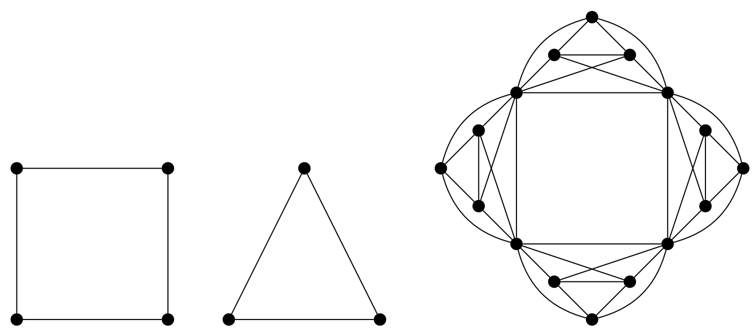


Figure 2.7: Example of the edge corona $G \diamond H$ of $G = C_4, H = C_3$

The Ihara zeta function and complexity of $G \diamond H$ are given as follows:

Theorem 2.21 ([15]). *Let G be an r-regular graph of order n and size m. Let H be an s-regular graph of order k. Then the Ihara zeta function of the edge corona $G \diamond H$ satisfies*

$$Z_{G\diamond H}(u)^{-1} = (1-u^2)^{\frac{n(2kr+krs+2r-4)}{4}} a(u)\left(\prod_{i=2}^{n} p_i(u)\right)\left(\prod_{j=2}^{k} q_j(u)\right),$$

where $a(u) = (1-u)\big[1-(r+s-1)u+(rs-kr+1)u^2-(s+1)(kr+r-1)u^3\big],$

$$p_i(u) = 1 - (s+\lambda_i)u + (r+s-k\lambda_i+s\lambda_i)u^2 - (krs+rs-s+s\lambda_i+\lambda_i)u^3+$$
$$+ (s+1)(kr+r-1)u^4,$$
$$q_j(u) = \Big(1-\theta_j u + (s+1)u^2\Big)^m,$$

where $Spec(A_G) = \{\lambda_1 = r, \lambda_2, \ldots, \lambda_n\}$ *and* $Spec(A_H) = \{\theta_1 = s, \theta_2, \ldots, \theta_k\}$.

Theorem 2.22 ([15]). *Let G be an r-regular graph of order n and size m. Let H be an s-regular graph of order k. Then the complexity of the edge corona $G \diamond H$ satisfies*

$$\tau(G \diamond H) = 2^{m-n+1}(k+2)^{n-1}\prod_{i=2}^{k}(\gamma_i+2)^m \tau(G),$$

where $Spec(L_H) = \{\gamma_1 = 0, \gamma_2, \ldots, \gamma_k\}$.

Now we derive the zeta Kirchhoff index:

Theorem 2.23. *Let G be an r-regular graph of order n and size m. Let H be an s-regular graph of order k. Then the zeta Kirchhoff index of the edge corona $G \diamond H$ satisfies*

$$Kf^z\big(G \diamond H\big) = \frac{1}{(2+k)2^{(4-2n+nr)/2}}\Bigg[(2kr+krs+2r-4)\Bigg(n^2r(k-1)(2+k)+$$
$$+ 2n(n-1)(4+2k+s) + 2Kf(G)(2kr+krs+2r-4)+$$
$$+ n^2sr(2+k)\sum_{j=2}^{k}\frac{1}{2+\gamma_j}\Bigg) + 2n(2+k)(3krs+5kr+4r-2s+rs-6)\Bigg]-$$
$$- \frac{n(2+kr)(2-4n+2nr+2knr+knrs)}{4},$$

where $Kf(G)$ is the Kirchhoff index of G and $Spec(L_H) = \{\gamma_1 = 0, \gamma_2, \ldots, \gamma_k\}$.

Proof. Let $f(u) = \det\Big(I_{n_{G\diamond H}} - uA_{G\diamond H} + u^2(D_{G\diamond H} - I_{n_{G\diamond H}})\Big)$. By Theorem 1.8,

$$f''(1) = 2\Big(Kf^z(G\diamond H) + \frac{1}{4}n(2+kr)(2-4n+2nr+2knr+knrs)\Big)\tau(G\diamond H). \qquad (2.38)$$

By Theorem 2.21, we have

$$f(u) = a(u)\Bigg(\prod_{i=2}^{n} p_i(u)\Bigg)\Bigg(\prod_{j=2}^{k} q_j(u)\Bigg), \tag{2.39}$$

Note that $a(1) = 0$, $a'(1) = -4 + 2r + 2kr + krs$,
$a''(1) = 2(-6 + 4r + 5kr - 2s + rs + 3krs)$, $p_i(1) = (2 + k)(r - \lambda_i)$,
$p_i'(1) = -4 - 4\lambda_i - 2k\lambda_i + 6r + 4kr - \lambda_i s + rs + krs$, $q_j(1) = (2 + s - \theta_j)^m$ and
$q_j'(1) = m(2 + 2s - \theta_j)(2 + s - \theta_j)^{m-1}$.

By differentiating (2.38) twice with respect to u and substituting $u = 1$, we have

$$\begin{aligned}
&f''(1) = \\
&= a''(1)\Bigg(\prod_{i=2}^{n} p_i(1)\Bigg)\Bigg(\prod_{j=2}^{k} q_j(1)\Bigg) + \\
&+2a'(1)\Bigg(\prod_{i=2}^{n} p_i(1)\Bigg)\Big(p_2'(1)p_3(1)\dots p_n(1) + \dots + p_2(1)\dots p_{n-1}(1)p_n'(1)\Big) + \\
&+2a'(1)\Bigg(\prod_{j=2}^{k} q_j(1)\Bigg)\Big(q_2'(1)q_3(1)\dots q_k(1) + \dots + q_2(1)\dots q_{k-1}(1)q_k'(1)\Big) = \\
&= \Bigg(\prod_{i=2}^{n} p_i(1)\Bigg)\Bigg(\prod_{j=2}^{k} q_j(1)\Bigg)\Bigg(a''(1) + 2a'(1)\sum_{i=2}^{n}\frac{p_i'(1)}{p_i(1)} + 2a'(1)\sum_{j=2}^{k}\frac{q_j'(1)}{q_j(1)}\Bigg) = \\
&= 2\Bigg(\prod_{i=2}^{n}(k+2)(r-\lambda_i)\Bigg)\Bigg(\prod_{j=2}^{k}(2+s-\theta_j)^m\Bigg)\cdot\Bigg[-6 + 4r + 5kr - 2s + rs + 3krs - \\
&\quad -(2kr + krs + 2r - 4)\sum_{i=2}^{n}\frac{4 + 4\lambda_i + 2k\lambda_i - 6r - 4kr + \lambda_i s - rs - krs}{(2+k)(r-\lambda_i)} + \\
&\quad +(2kr + krs + 2r - 4)\sum_{j=2}^{k}\frac{m(2 + 2s - \theta_j)}{2 + s - \theta_j}\Bigg].
\end{aligned} \tag{2.40}$$

Now we substitute (2.40), $\tau(G \diamond H)$ (by applying Theorem 2.22) into (2.38) and solve for the zeta Kirchhoff index. We obtain

$$Kf^z(G \diamond H) =$$

$$= \frac{2n(-6+4r+5kr-2s+rs+3krs)}{2^{(4-2n+nr)/2}} -$$
$$- \frac{2n(2kr+krs+2r-4)}{2^{(4-2n+nr)/2}(2+k)} \sum_{i=2}^{n} \frac{4+4\lambda_i+2k\lambda_i-6r-4kr+\lambda_i s-rs-krs}{r-\lambda_i} +$$
$$+ \frac{2kr+krs+2r-4}{2^{(4-2n+nr)/2}} \sum_{j=2}^{k} \frac{m(2+2s-\theta_j)}{2+s-\theta_j} -$$
$$- \frac{n(2+kr)(2-4n+2nr+2knr+knrs)}{4} =$$
$$= \frac{1}{(2+k)2^{(4-2n+nr)/2}} \Bigg[(2kr+krs+2r-4) \Bigg(n^2 r(k-1)(2+k) +$$
$$+ 2n(n-1)(4+2k+s) + 2Kf(G)(2kr+krs+2r-4) + n^2 sr(2+k) \sum_{j=2}^{k} \frac{1}{2+\gamma_j} \Bigg) +$$
$$+ 2n(2+k)(3krs+5kr+4r-2s+rs-6) \Bigg] -$$
$$- \frac{n(2+kr)(2-4n+2nr+2knr+knrs)}{4},$$

by Theorem 1.6. □

2.8 Join of two regular graphs

The join $G+H$ of two graphs G (order n) and H (order k) is constructed by connecting every vertex in G to each vertex in H with t edges.

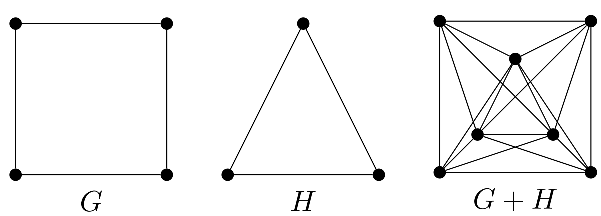


Figure 2.8: Example of the join $G+H$ of $G = C_4, H = C_3$

The Ihara zeta function of $G+H$ is given as follows:

Theorem 2.24 ([3]). *Let G be an r-regular graph of order n and let H be an s-regular graph of order k where $Spec(A_G) = \{\lambda_1 = r, \lambda_2, \ldots, \lambda_n\}$ and $Spec(A_H) = \{\theta_1 = s, \theta_2, \ldots, \theta_k\}$. Then the Ihara zeta function of the join $G + H$ satisfies*

$$Z_{G+H}(u)^{-1} = (1-u^2)^{\frac{1}{2}(-2k-2n+2kn+nr+ks)} a(u) \left(\prod_{i=2}^{n} p_i(u)\right) \left(\prod_{j=2}^{k} q_j(u)\right)$$

where $a(u) = (1-u)\Big[1-(r+s-1)u + \big(rs-(n-1)(k-1)\big)u^2 - (n+s-1)(k+r-1)u^3\Big]$, $p_i(u) = 1 - \lambda_i u + (r-1+k)u^2$ *and* $q_j(u) = 1 - \theta_j u + (s-1+n)u^2$.

Theorem 2.25. *Let G be an r-regular graph of order n and let H be an s-regular graph of order k. Then the complexity of the join $G + H$ satisfies*

$$\tau_{G+H} = \left(\prod_{i=2}^{n} r + k - \lambda_i\right) \left(\prod_{j=2}^{k} s + n - \theta_j\right),$$

where $Spec(A_G) = \{\lambda_1 = r, \lambda_2, \ldots, \lambda_n\}$ *and* $Spec(A_H) = \{\theta_1 = s, \theta_2, \ldots, \theta_k\}$.

Proof. Let $f(u) = \det\Big(I_{n_{G+H}} - uA_{G+H} + u^2\big(D_{G+H} - I_{n_{G+H}}\big)\Big)$. By Theorem 1.7, we have

$$f'(1) = 2(m_{G+H} - n_{G+H})\tau\big(G+H\big), \tag{2.41}$$

and by [3],

$$f(u) = a(u)\left(\prod_{i=2}^{n} p_i(u)\right)\left(\prod_{j=2}^{k} q_j(u)\right). \tag{2.42}$$

Observe that $a(1) = 0$, $a'(1) = -2k - 2n + nr + ks + 2kn$, $p_i(1) = r - \lambda_i + k$ and $q_j(1) = s + n - \theta_j$. Therefore, by differentiating (2.42) once and set $u = 1$, we have

$$f'(1) = a'(1)\left(\prod_{i=2}^{n} p_i(1)\right)\left(\prod_{j=2}^{k} q_j(1)\right) =$$

$$= a'(1)\left(\prod_{i=2}^{n} r+k-\lambda_i\right)\left(\prod_{j=2}^{k} s+n-\theta_j\right). \tag{2.43}$$

Finally, we substitute (2.43) into (2.41) and solve for $\tau\big(G+H\big)$. Note that $n_{G+H} = n+k$ and $m_{G+H} = \frac{nr}{2} + \frac{ks}{2} + kn$. We obtain

$$\begin{aligned}\tau\big(G+H\big) &= \frac{a'(1)\prod_{i=2}^{n} p_i(1)\prod_{j=2}^{k} q_j(1)}{2(m_{G+H}-n_{G+H})} = \\ &= \left(\prod_{i=2}^{n} r+k-\lambda_i\right)\left(\prod_{j=2}^{k} s+n-\theta_j\right).\end{aligned}$$

□

With both the Ihara zeta function and the complexity of $G+H$, we are ready to derive the zeta Kirchhoff index:

Theorem 2.26. *Let G be a r-regular graph of order n and let H be a s-regular graph of order k. Then the zeta Kirchhoff index of the join $G+H$ satisfies*

$$\begin{aligned}Kf^z\big(G+H\big) = (k+r-2)(n+s-2) + (-2k-2n+nr+ks+2kn)\cdot \\ \cdot\left[\sum_{i=2}^{n}\frac{r+k-2}{r-\lambda_i+k} + \sum_{j=2}^{k}\frac{s+n-2}{s-\theta_j+n}\right],\end{aligned}$$

where $Spec(A_G) = \{\lambda_1 = r, \lambda_2, \ldots, \lambda_n\}$ and $Spec(A_H) = \{\theta_1 = s, \theta_2, \ldots, \theta_k\}$.

Proof. Again let $f(u) = \det\Big(I_{n_{G+H}} - uA_{G+H} + u^2\big(D_{G+H} - I_{n_{G+H}}\big)\Big)$. By Theorem 1.8, we have

$$f''(1) = 2\Big(Kf^z\big(G+H\big) + 2m_{G+H}n_{G+H} - 2n_{G+H}^2 + n_{G+H}\Big)\tau\big(G+H\big). \tag{2.44}$$

We now take the second derivative of $f(u)$ from (2.42) and substitute $u = 1$. Note that $a(1) = 0$, $a'(1) = -2k-2n+nr+ks+2kn$,
$a''(1) = 2(4-2r-2s+rs-5k-5n+3nr+3ks+5kn^2)$, $p_i(1) = r-\lambda_i+k$,
$p_i'(1) = -2-\lambda_i+2r+2k$, $q_j(1) = s+n-\theta_j$ and $q_j'(1) = -2+2s+2n-\theta_j$.

Therefore, we have

$$
\begin{aligned}
&f''(1) = \\
&= a''(1)\left(\prod_{i=2}^{n} p_i(1)\right)\left(\prod_{j=2}^{k} q_j(1)\right) + \\
&+2a'(1)\left(\prod_{i=2}^{n} p_i(1)\right)\Big(p_2'(1)p_3(1)\dots p_n(1) + \dots + p_2(1)\dots p_{n-1}(1)p_n'(1)\Big) + \\
&+2a'(1)\left(\prod_{j=2}^{k} q_j(1)\right)\Big(q_2'(1)q_3(1)\dots q_k(1) + \dots + q_2(1)\dots q_{k-1}(1)q_k'(1)\Big) = \\
&= \left(\prod_{i=2}^{n} p_i(1)\right)\left(\prod_{j=2}^{k} q_j(1)\right)\left(a''(1) + 2a'(1)\sum_{i=2}^{n}\frac{p_i'(1)}{p_i(1)} + 2a'(1)\sum_{j=2}^{k}\frac{q_j'(1)}{q_j(1)}\right) = \\
&= 2\left(\prod_{i=2}^{n}(r+k-\lambda_i)\right)\left(\prod_{j=2}^{k}(s+n-\theta_j)\right)\Bigg[4 - 2r - 2s + rs - 5k - 5n + 3nr + 3ks + \\
&\quad + 5kn + (-2k - 2n + nr + ks + 2kn)\sum_{i=2}^{n}\frac{2r + 2k - \lambda_i - 2}{r - \lambda_i + k} + \\
&\quad + (-2k - 2n + nr + ks + 2kn)\sum_{j=2}^{k}\frac{2s + 2n - \theta_j - 2}{s - \theta_j + n}\Bigg].
\end{aligned} \tag{2.45}
$$

Now we substitute (2.45) into (2.44), applying Theorem (2.25) and solve for $Kf^z(G+H)$:

$$
\begin{aligned}
Kf^z(G+H) &= \frac{1}{t}\Big[4 - 2r - 2s + rs - 6k - 6n + 2(k+n)^2 + 3nr + 3ks + 5kn - \\
&\quad - (k+n)\big(ks + n(r+2k)\big)\Big] + (-2k - 2n + nr + ks + 2kn)\cdot \\
&\quad \cdot\left[\sum_{i=2}^{n}\frac{2r + 2k - \lambda_i - 2}{r - \lambda_i + k} + \sum_{j=2}^{k}\frac{2s + 2n - \theta_j - 2}{s - \theta_j + n}\right] = \\
&= (k + r - 2)(n + s - 2) + (-2k - 2n + nr + ks + 2kn)\cdot \\
&\quad \cdot\left[\sum_{i=2}^{n}\frac{r + k - 2}{r - \lambda_i + k} + \sum_{j=2}^{k}\frac{s + n - 2}{s - \theta_j + n}\right].
\end{aligned}
$$

□

Consider the case where $H = K_1$. Then the resulting graph is a 'cone' with G as a base. We have the following corollary.

Corollary 2.27. *Let G be an r-regular graph of order n. Then the zeta Kirchhoff index of the cone on G satisfies*

$$Kf^z\big(G+K\big) = (r-1)\left[n-2+(nr-2)\sum_{i=2}^{n}\frac{1}{\gamma_i+1}\right],$$

where $Spec(L_G) = \{\gamma_1 = r, \gamma_2, \ldots, \gamma_n\}$.

Proof. It follows immediately from (2.25) by letting $k=1$ and $s=0$. □

2.9 Line graph of biregular graphs

In Sato's paper [13], the Ihara zeta function and complexity of the line graph of a biregular graph were derived as follows:

Theorem 2.28 ([13]). *Let $G = (V_1, V_2)$ be an (r,s)-biregular bipartite graph of order n, size m where $|V_1| = a$, $|V_2| = b$ and $a < b$. Let $Spec(A_G) = \{\lambda_1 = \sqrt{rs}, \lambda_2, \lambda_a, 0, \ldots, 0\}$. The Ihara zeta function of the line graph $L(G)$ of G satisfies*

$$Z_{L(G)}(u)^{-1} = (1-u^2)^{ar\cdot(r+s-4)/2}g(u)\prod_{i=1}^{a}p(u),$$

$$\begin{aligned} g(u) &= \Big(1+2u+(r+s-3)u^2\Big)^{m-n}\Big(1+(2-s)u+(r+s-3)u^2\Big)^{b-a}, \\ p_i(u) &= 1+(4-r-s)u+\Big((r-1)(s-1)+r+s-3-\lambda_i^2\Big)u^2+ \\ &\quad +(r+s-3)(4-r-s)u^3+(r+s-3)^2u^4. \end{aligned}$$

Theorem 2.29 ([13]). *Let $G = (V_1, V_2)$ be an (r,s)-biregular bipartite graph of order n, size m where $|V_1| = a$, $|V_2| = b$ and $a < b$. Let $Spec(A_G) = \{\lambda_1 = \sqrt{rs}, \lambda_2, \lambda_a, 0, \ldots, 0\}$.*

Then the complexity of the line graph $L(G)$ of G satisfies

$$\tau\big(L(G)\big) = \frac{1}{a} r^{b-a-1}(r+s)^{m-n+1} \cdot \left(\prod_{i=2}^{a}(rs-\lambda_i^2)\right).$$

We now derive the zeta Kirchhoff index of such a graph:

Theorem 2.30. *Let $G = (V_1, V_2)$ be a (r, s)-biregular graph of order n, size m where $|V_1| = a$, $|V_2| = b$ and $a < b$. Then the zeta Kirchhoff index of the line graph $L(G)$ of G satisfies*

$$\begin{aligned} Kf^z\big(L(G)\big) = &\frac{a(r+s-4)^2(r-ar+ar^2-as)}{r+s} + \\ &+ ar(r+s-4)(r^2+s^2+2rs-4r-4s)\sum_{i=2}^{a}\frac{1}{rs-\lambda_i^2}, \end{aligned}$$

where $Spec(A_G) = \{\lambda_1 = \sqrt{rs}, \lambda_2, \lambda_a, 0, \dots, 0\}$.

Proof. Let $f(u) = \det\Big(I_{n_{L(G)}} - uA_{L(G)} + u^2\big(D_{L(G)} - I_{n_{L(G)}}\big)\Big)$. By Theorem 1.8, we have

$$\begin{aligned} f''(1) &= 2\Big(Kf^z\big(L(G)\big) + 2m_{L(G)}m_{L(G)} - 2n_{L(G)}^2 + n_{L(G)}\Big)\tau\big(L(G)\big) = \\ &= 2\Big(Kf^z\big(L(G)\big) + m + amr(r+s-4)\Big)\tau\big(L(G)\big). \end{aligned} \tag{2.46}$$

From Theorem 2.28, we know that

$$f(u) = g(u)\prod_{i=1}^{a} p(u). \tag{2.47}$$

We want to differentiate $f(u)$ twice and substitute $u = 1$. Note that

$$g(1) = r^{b-a}(r+s)^{m-n},$$

$$g'(1) = r^{b-a}(r+s)^{m-n}\left(\frac{2(m-n)(r+s-2)}{r+s} - \frac{(a-b)(2r+s-4)}{r}\right),$$

$$p_1(1) = 0,$$

$$p_1'(1) = (r+s)(r+s-4),$$

$$p_1''(1) = 2(16 - 15r + 3r^2 - 15s + 6rs + 3s^2),$$

$$p_i(1) = rs - \lambda_i^2, \text{ and}$$

$$p_i'(1) = -2\lambda_i^2 - 4r + r^2 - 4s + 4rs + s^2.$$

Hence (2.48) becomes

$$\begin{aligned}
f''(1) &= 2g'(1)p_1'(1)\prod_{i=2}^{a} p_i(1) + g(1)p_1''(1)\prod_{i=2}^{a} p_i(1) + \\
&\quad + 2g(1)p_1'(1)\Big(p_2'(1)p_3(1)\dots p_a(1) + \dots + p_2(1)\dots p_{a-1}(1)p_a'(1)\Big) = \\
&= p_1'(1)\left(\prod_{i=2}^{a} p_i(1)\right)\left[2g'(1) + g(1)\frac{p_1''(1)}{p_1'(1)} + 2g(1)\sum_{i=2}^{a}\frac{p_i'(1)}{p_i(1)}\right] = \\
&= (r+s-4)(r+s)\left(\prod_{i=2}^{a} rs - \lambda_i^2\right)\Bigg[2r^{b-a}(r+s)^{m-n}\Bigg(\frac{2(m-n)(r+s-2)}{r+s} + \\
&\quad + \frac{(b-a)(2r+s-4)}{r}\Bigg) + r^{b-a}(r+s)^{m-n}\frac{32 + 6r^2 - 30s + 6s^2 + 6r(2s-5)}{(r+s-4)(r+s)} + \\
&\quad + 2r^{b-a}(r+s)^{m-n}\sum_{i=2}^{a}\frac{s^2 - 2\lambda_i^2 - 4r + r^2 - 4s + 4rs}{rs - \lambda_i^2}\Bigg] = \\
&= 2(r+s-4)r^{b-a}(r+s)^{m-n+1}\left(\prod_{i=2}^{a} rs - \lambda_i^2\right)\Bigg[\Bigg(\frac{2(m-n)(r+s-2)}{r+s} + \\
&\quad + \frac{(b-a)(2r+s-4)}{r}\Bigg) + \frac{16 + 3r^2 - 15s + 3s^2 + 3r(2s-5)}{(r+s-4)(r+s)} + \\
&\quad + \sum_{i=2}^{a}\frac{s^2 - 2\lambda_i^2 - 4r + r^2 - 4s + 4rs}{rs - \lambda_i^2}\Bigg].
\end{aligned} \tag{2.48}$$

Now, by we substitute (2.48) into (2.46) and apply Theorem 2.29. Then we solve for $Kf^z\big(L(G)\big)$, and get

$$\begin{aligned}
Kf^z\big(L(G)\big) &= ar(r+s-4)\Bigg[\frac{2(m-n)(r+s-2)}{r+s} + \frac{(b-a)(2r+s-4)}{r} + \\
&\quad + \frac{16 + 3r^2 - 15s + 3s^2 + 3r(2s-5)}{(r+s-4)(r+s)} + \sum_{i=2}^{a}\frac{s^2 - 2\lambda_i^2 - 4r + r^2 - 4s + 4rs}{rs - \lambda_i^2}\Bigg] - \\
&\quad - \Big(m + amr(r+s-4)\Big) =
\end{aligned}$$

$$= \frac{a(r+s-4)^2(r-ar+ar^2-as)}{r+s} +$$
$$+ ar(r+s-4)(r^2+s^2+2rs-4r-4s)\sum_{i=2}^{a}\frac{1}{rs-\lambda_i^2}.$$

□

Observe that the resulting line graph $L(G)$ is a regular graph of regularity $(r+s-2)$, hence we can derive the Kirchhoff index of $L(G)$.

Corollary 2.31. *Let $G = (V_1, V_2)$ be an (r,s)-biregular bipartite graph of order n, size m where $|V_1| = a$, $|V_2| = b$ and $a < b$. Then the Kirchhoff index of the line graph $L(G)$ of G satisfies*

$$Kf\big(L(G)\big) = \frac{a(r-ar+ar^2-as)}{r+s} + \frac{ar(r^2+s^2+2rs-4r-4s)}{r+s-4}\sum_{i=2}^{a}\frac{1}{rs-\lambda_i^2},$$

where $Spec(A_G) = \{\lambda_1 = \sqrt{rs}, \lambda_2, \lambda_a, 0, \ldots, 0\}$.

CHAPTER 3

ZETA KIRCHHOFF INDEX OF SELECTED FAMILIES OF GRAPHS

3.1 Cycle graphs

A cycle graph C_n where $n \geq 3$, is a connected 2-regular graph. By [1, 9], the Kirchhoff index of C_n is given as

$$Kf(C_n) = \frac{n(n-1)(n+1)}{12},$$

hence by Theorems 2.3 and 2.13, we can derive the following corollaries.

Corollary 3.1. *For $n \geq 3$, the zeta Kirchhoff index of cycle graph C_n is 0.*

Proof. Since C_n is 2-regular, $Kf^z(C_n) = 0^2 Kf(C_n) = 0$. □

Corollary 3.2. *Let C_n be a cycle graph with $n \geq 3$. Then the zeta Kirchhoff index of its k-th semitotal point graph $R^k(C_n)$ satisfies*

$$Kf^z\big(R^k(C_n)\big) = \frac{2k^2 n(n-1)(n+1)}{3(2+k)}.$$

Corollary 3.3. *Let C_n be a cycle graph with $n \geq 3$. Then the zeta Kirchhoff index of its middle graph $M(C_n)$ satisfies*

$$Kf^z\big(M(C_n)\big) = \frac{2n(n-1)(n+1)}{9}.$$

3.2 Complete graphs

A complete graph K_n is a connected $(n-1)$-regular graph such that every vertex is adjacent to all the other $(n-1)$ vertices. By [11], the Kirchhoff index of K_n is given as

$$Kf(K_n) = n - 1,$$

therefore we can easily derive the following corollaries by Theorems 2.3 and 2.13,

Corollary 3.4. *Let K_n be a complete graph. Then the zeta Kirchhoff index of K_n satisfies*

$$Kf^z(K_n) = (n-1)(n-3)^2.$$

Corollary 3.5. *Let K_n be a complete graph. Then the zeta Kirchhoff index of its k-th semitotal point graph $R^k(K_n)$ satisfies*

$$Kf^z\big(R^k(K_n)\big) = \frac{2(n-1)\Big(k(n-1)+n-3\Big)^2}{2+k}.$$

Note that for a complete graph K_n, its quasitotal graph $QT(K_n)$ is equivalent to its middle graph $M(K_n)$, hence

Corollary 3.6. *Let K_n be a complete graph. Then the zeta Kirchhoff indices of its middle graph $M(K_n)$ and quasitotal graph $QT(K_n)$ are equal and satisfy*

$$Kf^z\big(M(K_n)\big) = Kf^z\big(QT(K_n)\big) = \frac{n^6 - n^5 - 15n^4 + 27n^3 + 10n^2 - 18n - 4}{2n}.$$

3.3 Biregular graphs

A semiregular bipartite graph, or biregular graph $G = (V_1 \cup V_2, E_G) = (V_1, V_2)$ of order n, size m is a graph such that there is a partition of sets of vertices V_1 and V_2 that satisfies: (i) $|V_1| = a$, $|V_2| = b$ where $a < b$; (ii) $\forall v_i \in V_1$, $\deg v_i = r$ and $\forall v_j \in V_2$, $\deg v_j = s$; (iii) all vertices in V_1 are not incident to each other, similarly for all vertices in V_2.

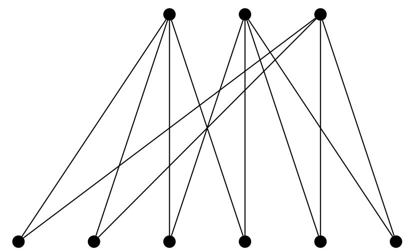


Figure 3.1: Example of the bi-regular graph $G = (V_1, V_2)$ where $|V_1| = 3$, $|V_2| = 6$, $r = 4$, $s = 2$

Hashimoto derived the Ihara zeta function of a biregular bipartite graph is as follows:

Theorem 3.7 ([8]). *Let $G = (V_1, V_2)$ be an (r, s)-biregular bipartite graph of order n, size m where $|V_1| = a$, $|V_2| = b$, $a < b$ and $Spec(A_G) = \{\pm\lambda_1 = \sqrt{rs}, \pm\lambda_2, \ldots, \pm\lambda_a, 0, \ldots, 0\}$. The Ihara zeta function of G satisfies*

$$Z_G(u)^{-1} = (1-u^2)^{m-n} h(u) \cdot \prod_{i=1}^{a} p_i(u),$$

where $h(u) = \big(1 + (s-1)u^2\big)^{b-a}$, $p_i(u) = 1 - \Big(\lambda_i^2 - (r-1) - (s-1)\Big)u^2 + (r-1)(s-1)u^4$.

The complexity of a biregular bipartite graph is given as follows:

Theorem 3.8 ([13]). *Let $G = (V_1, V_2)$ be an (r, s)-biregular bipartite graph of order n, size m where $|V_1| = a$, $|V_2| = b$, $a < b$ and $Spec(A_G) = \{\pm\lambda_1 = \sqrt{rs}, \pm\lambda_2, \ldots, \pm\lambda_a, 0, \ldots, 0\}$. The complexity of G satisfies*

$$\tau(G) = \frac{s^{b-a}}{2\big(a(r-1) - b\big)}(2rs - 2r - 2s)\left(\prod_{i=2}^{a} rs - \lambda_i^2\right).$$

Next, we derive the zeta Kirchhoff index of a biregular bipartite graph:

Theorem 3.9. *Let $G = (V_1, V_2)$ be an (r, s)-biregular graph of order n, size m where $|V_1| = a$, $|V_2| = b$, $a < b$ and $Spec(A_G) = \{\pm\lambda_1 = \sqrt{rs}, \pm\lambda_2, \ldots, \pm\lambda_a, 0, \ldots, 0\}$. Then the* *zeta Kirchhoff index of G satisfies*

$$
\begin{aligned}
Kf^z(G) = {} & (a+b)(2a+2b-ar-bs-1)+ \\
& + \big(a(r-1)-b\big)\left(\frac{4(b-a)(s-1)}{s}+4a+1+\frac{4}{rs-r-s}\right)+ \\
& + 4(a(r-1)-b)(r+s-rs)\left(\sum_{i=2}^{a}\frac{1}{\lambda_i^2-rs}\right).
\end{aligned}
$$

Proof. Let $f(u)=\det\Big(I_{n_G}-uA_G+u^2\big(D_G-I_{n_G}\big)\Big)$. By Theorem 1.8, we have

$$f''(1)=2\Big(Kf^z(G)+2mn-2n^2+n\Big)\tau(G), \tag{3.1}$$

and by Theorem 3.7,

$$f(1)=h(u)\cdot\prod_{i=1}^{a}p_i(u). \tag{3.2}$$

Note that $h(1)=s^{b-a}$, $h'(1)=2(b-a)(s-1)s^{b-a-1}$, $p_1(1)=0$, $p_1'(1)=2(-r-s+rs)$, $p_1''(1)=2(4-5r-5s+5rs)$, $p_i(1)=rs-\lambda_i^2$ and $p_i'(1)=-2(\lambda_i^2+r+s-2rs)$.

Hence by differentiating (3.2) twice and substituting $u=1$, we obtain

$$
\begin{aligned}
f''(1) = {} & h'(1)p_1'(1)p_2(1)\cdots p_a(1)+\Big(h'(1)p_1'(1)p_2(1)\dots p_a(1)+h(1)p_1''(1)p_2(1)\dots p_a(1)+ \\
& + h(1)p_1'(1)p_2'(1)p_3(1)\dots p_a(1)+\cdots+h(1)p_1'(1)p_2(1)\dots p_{a-1}(1)p_a'(1)\Big)+ \\
& + h(1)p_1'(1)p_2'(1)p_3(1)\dots p_a(1)+\cdots+h(1)p_1'(1)p_2(1)\dots p_{a-1}(1)p_a'(1)= \\
= {} & 2h'(1)p_1'(1)\left(\prod_{i=2}^{a}p_i(1)\right)+h(1)\left(\frac{p_1''(1)}{p_1'(1)}\right)p_1'(1)\left(\prod_{i=2}^{a}p_i(1)\right)+ \\
& + 2h(1)p_1'(1)\left(\prod_{i=2}^{a}p_i(1)\right)\sum_{i=2}^{a}\frac{p_i'(1)}{p_i(1)} \\
= {} & p_1'(1)\left(\prod_{i=2}^{a}p_i(1)\right)\left(2h'(1)+h(1)\frac{p_1''(1)}{p_1'(1)}+2h(1)\sum_{i=2}^{a}\frac{p_i'(1)}{p_i(1)}\right)=
\end{aligned}
$$

$$
\begin{aligned}
&= p_1'(1)\left(\prod_{i=2}^{a}(rs-\lambda_i^2)\right)\left(4(b-a)(s-1)s^{b-a-1}+s^{b-a}\left(5+\frac{4}{rs-r-s}\right)+\right.\\
&\quad\left.+2s^{b-a}\sum_{i=2}^{a}\left(2+\frac{2(r+s-rs)}{\lambda_i^2-rs}\right)\right)=\\
&= s^{b-a}(2rs-2r-2s)\left(\prod_{i=2}^{a}(rs-\lambda_i^2)\right)\left(\frac{4(b-a)(s-1)}{s}+\left(5+\frac{4}{rs-r-s}\right)+\right.\\
&\quad\left.+2\sum_{i=2}^{a}\left(2+\frac{2(r+s-rs)}{\lambda_i^2-rs}\right)\right).
\end{aligned}
\tag{3.3}
$$

Finally, we substitute (3.3) and complexity (by applying Theorem 3.8) into (3.1) and solve for $Kf^z(G)$. We obtain:

$$
\begin{aligned}
Kf^z(G) &= (a+b)(2a+2b-ar-bs-1)+\\
&\quad+\big(a(r-1)-b\big)\left(\frac{4(b-a)(s-1)}{s}+5+\frac{4}{rs-r-s}\right)+\\
&\quad+2\big(a(r-1)-b\big)\left(\sum_{i=2}^{a}\left(2+\frac{2(r+s-rs)}{\lambda_i^2-rs}\right)\right)=\\
&= (a+b)(2a+2b-ar-bs-1)+\\
&\quad+\big(a(r-1)-b\big)\left(\frac{4(b-a)(s-1)}{s}+4a+1+\frac{4}{rs-r-s}\right)+\\
&\quad+4(a(r-1)-b)(r+s-rs)\left(\sum_{i=2}^{a}\frac{1}{\lambda_i^2-rs}\right),
\end{aligned}
$$

as desired. □

3.4 Complete bipartite graphs

A complete bipartite graph K_{n_1,n_2} is a graph of order n_1+n_2 where vertices can be divided in two groups: one with n_1 vertices and the other with n_2, such that all vertices in the same group are not adjacent to one another but is adjacent to all vertices in the other group. By [18], the Kirchhoff index of K_{n_1,n_2} satisfies

$$Kf(K_{n_1,n_2}) = \frac{(n_1+n_2-1)(n_1^2+n_2^2)-n_1n_2}{n_1n_2}.$$

Hence, if $n_1 = n_2 = n$, then

$$Kf(K_{n,n}) = 4n-3,$$

and again by Theorems 2.3 and 2.13 we have the following corollaries.

Corollary 3.10. *Let $K_{n,n}$ be a complete bipartite graph of order $2n$. Then the zeta Kirchhoff index of its k-th semitotal point graph $R^k(K_{n,n})$ satisfies*

$$Kf^z\Big(R^k(K_{n,n})\Big) = \frac{2(4n-3)(kn+n-2)^2}{2+k}.$$

Corollary 3.11. *Let $K_{n,n}$ be a complete bipartite graph of order $2n$. Then the zeta Kirchhoff index of its middle graph $M(K_{n,n})$ satisfies*

$$Kf^z\big(M(K_{n,n})\big) = \frac{2(n^6+3n^5-7n^4-14n^3+17n^2+14n-12)}{1+n}.$$

CHAPTER 4
ENUMERATION

For simple connected md2 (minimal degree 2) graphs of degree at most 10, we have computed the Ihara zeta function, Kirchhoff indices, and the characteristic polynomials for their adjacency, Laplacian and normalized Laplacian matrices up to degree 10 by using Wolfram Mathematica. We provide a data summary as shown below.

Order (number of vertices)	4	5	6	7	8	9	10
Count of non-isomorphic simple connected graphs	6	21	112	853	11,117	261,080	11,716,571
Count of non-isomorphic simple connected md2 graphs	3	11	61	507	7,442	197,772	9,808,209
Count of distinct Ihara zeta functions	3	11	61	507	7,441	197,769	9,803,142
Count of distinct and unique Ihara zeta functions	3	11	61	507	7,440	197,766	9,798,079
Count of distinct and nonunique Ihara zeta functions	0	0	0	0	1	3	5,063
Count of distinct Kf^z (zeta Kirchhoff index)	3	10	58	486	7,157	193,411	9,691,697
Count of distinct and unique Kf^z	3	9	55	470	6,959	190,542	9,614,302
Count of distinct and nonunique Kf^z	0	1	3	16	198	2,869	77,395

Order	4	5	6	7	8	9	10
Count of distinct Kf (Kirchhoff index)	3	11	55	457	6,501	176,093	9,021,467
Count of distinct and unique Kf	3	11	50	412	5,816	159,619	8,414,619
Count of distinct and nonunique Kf	0	0	5	45	685	16,474	606,848
Count of distinct Kf^+ (additive Kirchhoff index)	3	11	58	496	7,052	194,135	9,589,128
Count of distinct and unique Kf^+	3	11	55	485	6,757	191,491	9,418,001
Count of distinct and nonunique Kf^+	0	0	3	11	295	2,644	171,127
Count of distinct $Kf^\times$ (multiplicative Kirchhoff index)	3	11	59	483	7,174	193,680	9,701,729
Count of distinct and unique $Kf^\times$	3	11	57	465	6,985	191,031	9,632,048
Count of distinct and nonunique $Kf^\times$	0	0	2	18	189	2,649	69,681
Pairs of graphs with identical $Kf^z, Kf, Kf^+, Kf^\times$ and Ihara zeta function	0	0	0	0	0	1	4,958
Triplets of graphs with identical $Kf^z, Kf, Kf^+, Kf^\times$ and Ihara zeta function	0	0	0	0	0	0	4

Order	4	5	6	7	8	9	10
Pairs of graphs with identical $Kf^z, Kf, Kf^+, Kf^\times$, Ihara zeta function, characteristic polynomials of adjacency, Laplacian and normalized Laplacian matrices	0	0	0	0	0	1	4,311
Triplets of graphs with identical $Kf^z, Kf, Kf^+, Kf^\times$, Ihara zeta function, characteristic polynomials of adjacency, Laplacian and normalized Laplacian matrices	0	0	0	0	0	0	4

We have found a pair of non-isomorphic simple connected md2 graphs of order 9 (see Figure 4.1) where the two graphs have equal Ihara zeta function, all Kirchhoff indices as well as identical characteristic polynomials of the adjacency, Laplacian and normalized Laplacian matrices. We have

$$\begin{aligned}
Z_{A_1}(u)^{-1} = Z_{A_2}(u)^{-1} &= (1-u^2)^9\big(1+9u^2-14u^3+15u^4-184u^5-146u^6-1098u^7 \\
&\quad -957u^8-3870u^9-2482u^{10}-8666u^{11}-2055u^{12}-11744u^{13} \\
&\quad +5595u^{14}-7560u^{15}+17604u^{16}+15552u^{18}\big),\ \tau_{A_1}=\tau_{A_2}=9840, \\
\rho_{A_{A_1}}(x) = \rho_{A_{A_2}}(x) &= -8-12x+46x^2+62x^3-60x^4-67x^5+14x^6+18x^7-x^9, \\
\rho_{L_{A_1}}(x) = \rho_{L_{A_2}}(x) &= -88560x+187812x^2-168450x^3+83826x^4-25393x^5 \\
&\quad +4806x^6-556x^7+36x^8-x^9,
\end{aligned}$$

where $\rho_{A_{A_1}}(x), \rho_{A_{A_2}}(x), \rho_{L_{A_1}}(x), \rho_{L_{A_2}}(x)$ are the characteristic polynomials of the adjacency and Laplacian matrices of A_1 and A_2 in terms of x. For the Kirchhoff indices, we have

$$Kf^z(A_1) = Kf^z(A_2) = \frac{28813}{410}, \qquad Kf(A_1) = Kf(A_2) = \frac{15651}{820},$$
$$Kf^+(A_1) = Kf^+(A_2) = \frac{246217}{1640}, \qquad Kf^\times(A_1) = Kf^\times(A_2) = \frac{241239}{820}.$$

To see that A_1 and A_2 in Figure 4.1 are non-isomorphic, we pick out the two vertices in each graph with degree 3 and name them a_1 and a_2. Since they have degree 3, they are adjacent to exactly 3 vertices; we name the 3 vertices that are adjacent to a_1 as b_1, c_1, d_1, where the degree of b_1 is 5. Similarly, we name the 3 vertices adjacent to a_2 as b_2, c_2, d_2, where the degree of b_2 is 5. Note that in graph A_1, c_1 and d_1 are non-adjacent as well as c_2 and d_2 (in another words, a_1, c_1, d_1 and a_2, c_2, d_2 are not). However, in graph A_2, c_1 and d_1 are adjacent as well as c_2 and d_2. A further evidence to see that A_1 and A_2 are non-isomorphic is to observe that in graph A_1, b_1, b_2 are adjacent to c_1, d_1 and c_2, d_2, while in A_2 this does not hold.

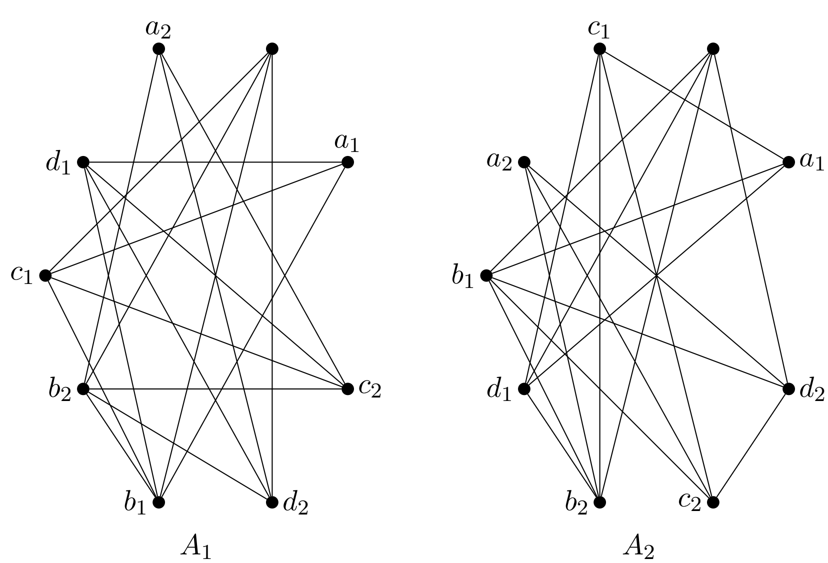


Figure 4.1: Non-isomorphic graphs A_1 and A_2 of degree 9 with identical Ihara zeta function, adjacency/Laplacian spectrum and Kirchhoff indices.

For graphs of order 10, we have found 4311 pairs and 4 triplets (see Figures 4.2, 4.3, 4.4, 4.5) that have the same Ihara zeta function, all Kirchhoff indices as well as identical characteristic polynomials of the adjacency, Laplacian and normalized Laplacian matrices.

For B_1, B_2, B_3 in Figure 4.2, we have

$$
\begin{aligned}
Z_{B_1}(u)^{-1} = Z_{B_2}(u)^{-1} = Z_{B_3}(u)^{-1} &= (1-u^2)^{15}\big(1 + 15u^2 - 28u^3 + 38u^4 - 660u^5 \\
&\quad - 874u^6 - 7000u^7 - 10291u^8 - 43472u^9 - 54345u^{10} - 171072u^{11} \\
&\quad - 156328u^{12} - 427520u^{13} - 194704u^{14} - 629760u^{15} + 184576u^{16} \\
&\quad - 421888u^{17} + 950272u^{18} + 983040u^{20}\big), \\
\rho_{A_{B_1}}(x) = \rho_{A_{B_2}}(x) = \rho_{A_{B_3}}(x) &= 128x^2 - 112x^3 - 204x^4 + 118x^5 + 117x^6 \\
&\quad - 28x^7 - 25x^8 + x^{10}, \\
\rho_{L_{B_1}}(x) = \rho_{L_{B_2}}(x) = \rho_{L_{B_3}}(x) &= -3328200x + 5927305x^2 - 4631838x^3 + 2085186x^4 \\
&\quad - 596152x^5 + 112279x^6 - 13934x^7 + 1099x^8 - 50x^9 + x^{10},
\end{aligned}
$$

and for the Kirchhoff indices, we have

$$
\begin{aligned}
Kf^z(B_1) = Kf^z(B_2) = Kf^z(B_3) &= \frac{291034}{1849}, \\
Kf(B_1) = Kf(B_2) = Kf(B_3) &= \frac{1185461}{66564}, \\
Kf^+(B_1) = Kf^+(B_2) = Kf^+(B_3) &= \frac{5899615}{33282}, \\
Kf^\times(B_1) = Kf^\times(B_2) = Kf^\times(B_3) &= \frac{7333460}{16641}.
\end{aligned}
$$

To see that B_1, B_2 and B_3 are non-isomorphic, we first label the only vertices of degree 4 and 6 as a and b. There is exactly one vertex that is adjacent to both a and b; we label this vertex c. Now, besides vertices a and b, there is one vertex that is adjacent to c and not adjacent to either a or c; we label this vertex d. Finally, among the vertices adjacent to d besides c, there is exactly one vertex that is not adjacent to b; we label this vertex e.

Note that in graph B_2, c and e are adjacent but they are not adjacent in B_1 and B_3. Hence B_2 is not isomorphic to B_1 and B_3. There are exactly 3 vertices that are adjacenct to both c and e in B_3 but there are 4 such vertices in B_1. Hence B_1 and B_3 are not isomorphic.

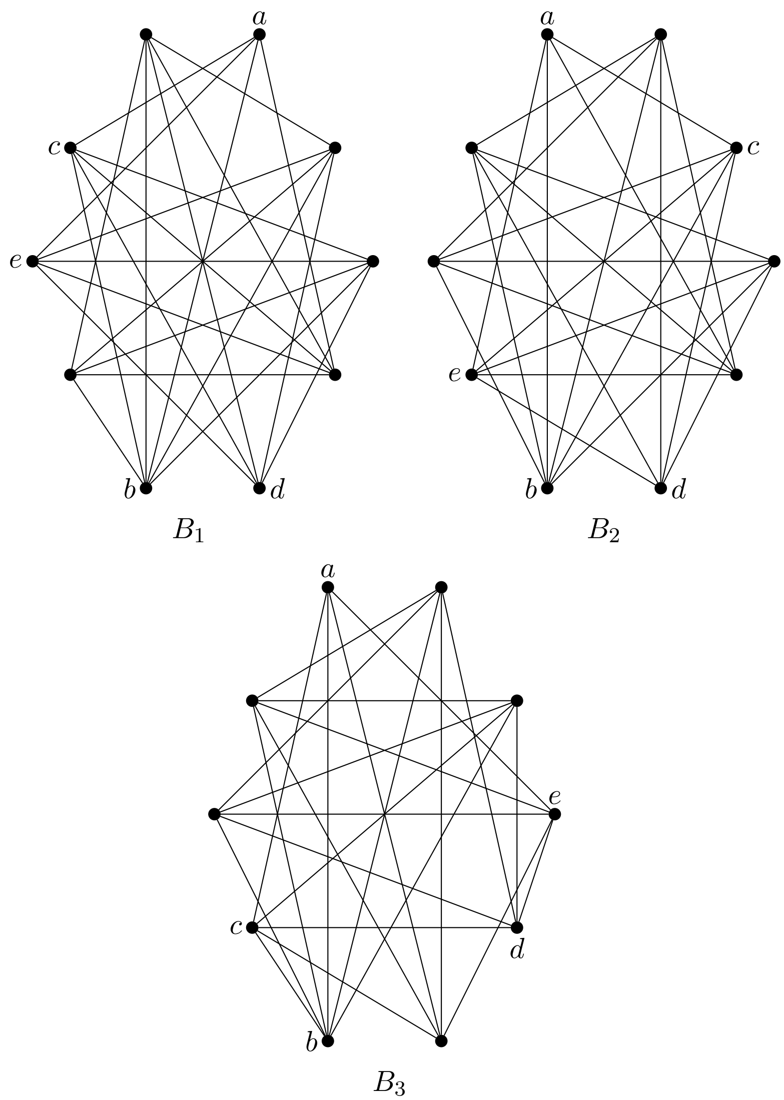


Figure 4.2: First triplet of non-isomorphic graphs B_1, B_2 and B_3 of degree 10 with identical Ihara zeta function, adjacency/Laplacian spectrum and Kirchhoff indices.

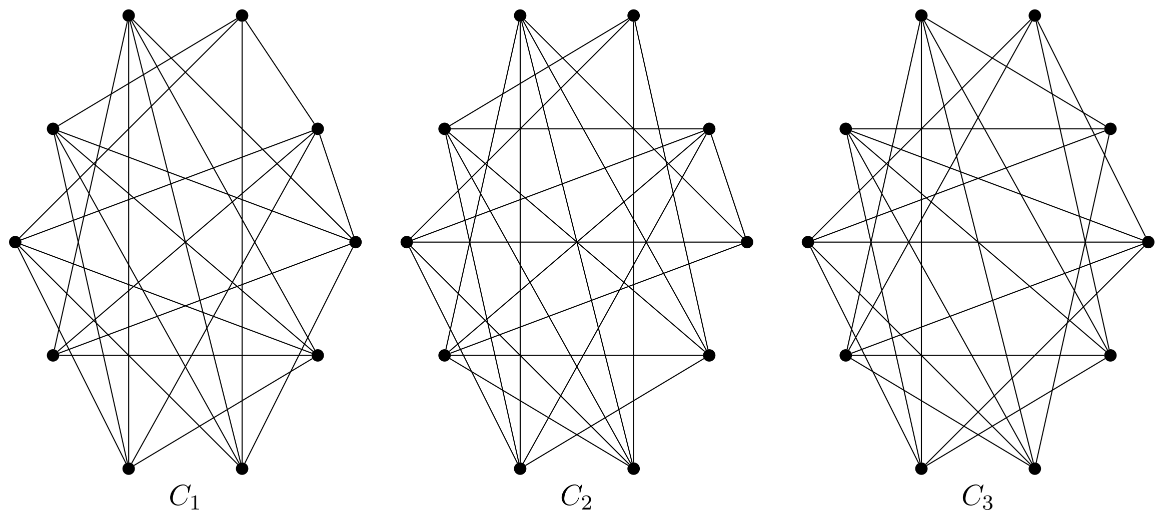


Figure 4.3: Second triplet of non-isomorphic graphs C_1, C_2 and C_3 of degree 10 with identical Ihara zeta function, adjacency/Laplacian spectrum and Kirchhoff indices.

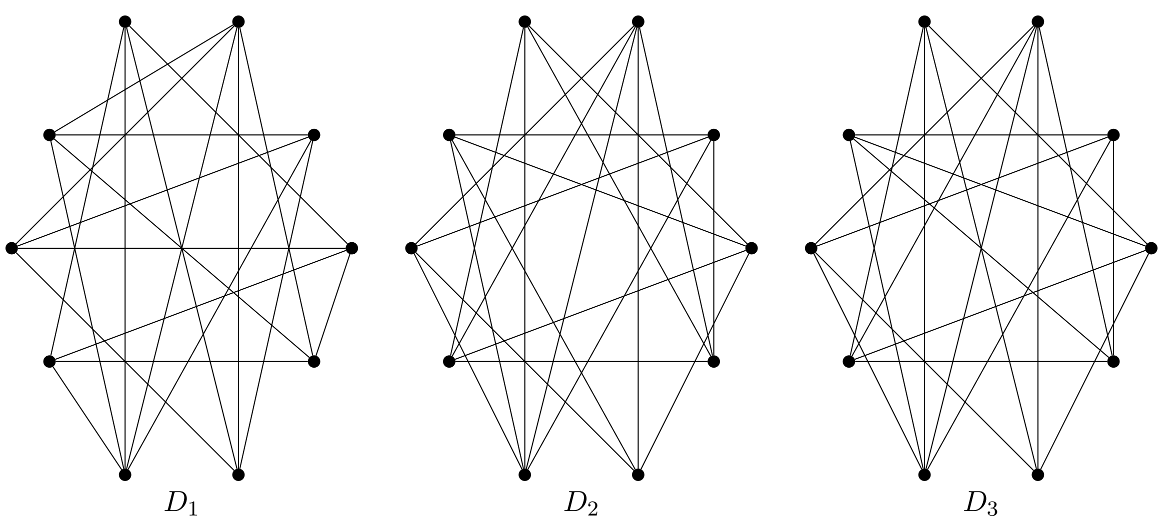
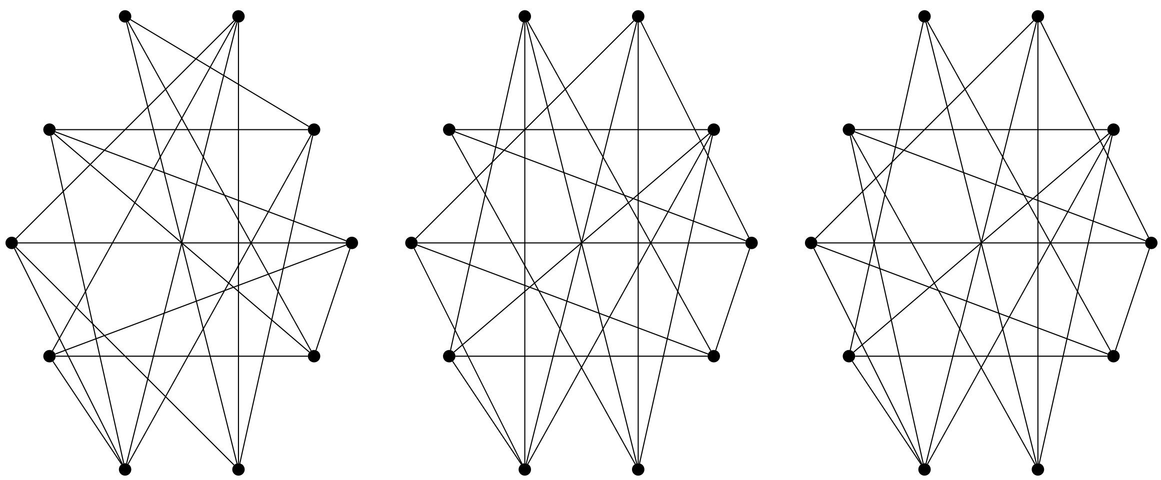


Figure 4.4: Second triplet of non-isomorphic graphs D_1, D_2 and D_3 of degree 10 with identical Ihara zeta function, adjacency/Laplacian spectrum and Kirchhoff indices.



E_1 E_2 E_3

Figure 4.5: Fourth triplet of non-isomorphic graphs E_1, E_2 and E_3 of degree 10 with identical Ihara zeta function, adjacency/Laplacian spectrum and Kirchhoff indices.

APPENDIX A

SELECTED MATHEMATICA CODE

```
(***** MD2 DATA GENERATOR *****)
(* This program imports .g6 file of degree %n, and exports matrices \
that are md2, in parts of every %partsize matrices, in the form of a \
list .mx, starting from part 0*)

(* MODIFY DEGREE n *)
n = 4;

(* MODIFY PART SIZE HERE *)
(* Too allow more efficient processing, lists are processed in parts.*)

partsize =
  1000; (* DEFAULT IS 1000, DO NOT MODIFY UNLESS ABSOLUTELY \
NECESSARY, MANY OTHER PROGRAMS DEPENDS ON THIS *)

(* MODIFY FILE LOCATION HERE *)
data = Import[
   "D:\\Google Drive\\Maff Programs\\data\\graph" <> ToString[n] <>
    "c\\graph" <> ToString[n] <> "c.g6", "AdjacencyMatrix"];

(***** NO MORE MODIFIERS BELOW *****)
Print["Data Imported"];
iilimit = Length[data];
alist = {};
klist = {};
kalist = {};
```

```
kmlist = {};
kzlist = {};
data2 = {};
count = 0;
partcount = 0;

ilimit = Length[data];
For[i = 1, i <= ilimit, i++, {
   current = data[[i]];
   For[j = 1, j <= n, j++, {
     md2check = True;
     If[Total[current[[j]]] < 2, {
       md2check = False;
       j = n;
       }];
     }];
   If[md2check, {
     data2 = Append[data2, current];
     count++;
     If[count == partsize, {
       PrintTemporary["Part ", partcount, " created."];
       Export[
        "graph" <> ToString[n] <> "c_md2_part" <>
         ToString[partcount] <> ".mx", data2];
       data2 = {};
       count = 0;
       partcount = partcount + 1;
       }];
```

```
      }];
    }];
Export["graph" <> ToString[n] <> "c_md2_part" <> ToString[partcount] <>
     ".mx", data2];

(*For[i=1,i<partcount,i++,{
data=Import["graph"<>ToString[n]<>"c_md2_part"<>ToString[i]<>".mx"];
DeleteFile["graph"<>ToString[n]<>"c_md2_part"<>ToString[i]<>".mx"];
data2=Join[data2,data];
}];
Export["graph"<>ToString[n]<>"c_md2.mx",data2];
*)

Print["There are ", partcount*1000 + count,
  " MD2 non-isomorphic graphs of degree ", n, "."];
Print["Data have been separated into ", partcount + 1,
  " part of maximum size ", partsize, "."];
Print["Data Exported."];

(*** Kirchhoff Index Generator ***)
(* This program imports md2 graphs in part of %partsize, expected in \
the format .mx as a list, calculate its kirchhoff index and outputs a \
list (.mx) of the same partsize, where the indices' index and part \
number is the same as the original md2 graphs file *)

(* MODIFY DEGREE n HERE *)
n = 8;
partsize = 1000;
```

```
(* MODIFY IMPORT DIRECTORY HERE *)
directory =
  "C:\\Users\\jedi\\Desktop\\math research\\graph" <> ToString[n] <>
   "c_md2\\";
directory =
  "D:\\Google Drive\\Maff Programs\\data\\graph" <> ToString[n] <>
   "c_md2\\";
(* This the directory of the data location *)

(***** NO MORE MODIFIERS BELOW *****)
(*ParallelNeeds["ComputerArithmetic`"];*) (*To be added if necessary *)

partcount = 0;
total = 0;

While[FileExistsQ[
   directory <> "graph" <> ToString[n] <> "c_md2_part" <>
    ToString[partcount] <> ".mx"], {
   data =
    Import[directory <> "graph" <> ToString[n] <> "c_md2_part" <>
      ToString[partcount] <> ".mx"];
   hlimit = Length[data];
   data2 = {};

   For[h = 1, h <= hlimit, h++, {
     a = data[[h]];

     dl = {}; (* dl, a list/array of degrees corresponding to the row/
```

```
column of matrix a *)
l = -a; (* l, Laplacian Matrix *)

For[i = 1, i <= n, i++, {
  k = Total[a[[i]]];
  dl = Append[dl, k];
  l[[i]][[i]] = k;
  }];

(*** Upper Triangular Resistance Matrix ***)
r = IdentityMatrix[n] - IdentityMatrix[n]; (* r,
resistance matrix *)
For[i = 1, i <= n, i++, {
  For[j = i + 1, j <= n, j++, {
     tau = l;
     ll = l;
     For[k = 1, k <= n, k++, {
       tau[[k]] = Delete[tau[[k]], i];
       ll[[k]] = Delete[ll[[k]], j];
       ll[[k]] = Delete[ll[[k]], i];
       }];
     tau = Delete[tau, i];
     ll = Delete[ll, j];
     ll = Delete[ll, i];
     r[[i]][[j]] = Det[ll]/Det[tau];
     }];
  }];
```

```
    ki = 0;
    For[i = 1, i <= n, i++, {
      For[j = i + 1, j <= n, j++, {
         ki += r[[i]][[j]];
         }];
      }];
    data2 = Append[data2, ki];
    }];
  Export[
   "graph" <> ToString[n] <> "c_md2_ki_part" <> ToString[partcount] <>
     ".mx", data2];
  PrintTemporary["Part " <> ToString[partcount] <> " processed."];
  partcount += 1;
  total += hlimit;
  }];
Print["Kirchhoff Indices of degree ", n, " for a total of ", total,
  " graphs processed."];
```

```
(*** Duplicate Kirchhoff indices finder ***)
(* MODIFY DEGREE n HERE *)
n = 10;
partsize = 1000;

(* MODIFY IMPORT DIRECTORY HERE *)
directory = "D:\\Google Drive\\Maff Programs\\data\\";
(* MAKE SURE TO INCLUDE \\ AT THE END *)

(***** NO MORE MODIFIERS BELOW *****)
partcount = 0;
```

```
count = 0;
kilist = {};

While[FileExistsQ[
   directory <> "graph" <> ToString[n] <> "c_md2_ki\\graph" <>
    ToString[n] <> "c_md2_ki_part" <> ToString[partcount] <> ".mx"], {
   data =
    Import[directory <> "graph" <> ToString[n] <> "c_md2_ki\\graph" <>
       ToString[n] <> "c_md2_ki_part" <> ToString[partcount] <> ".mx"];
   kilist = Join[kilist, data];
   partcount++;
   count += Length[data];
   }];
Print["Imported ", count, " kirchhoff indices of degree ", n];
PrintTemporary["Now generating the list of duplicates."];

(* This is the finding duplicate function *)
positionDuplicates[list_] := GatherBy[Range@Length[list], list[[#]] &];

positionlist = positionDuplicates[kilist];
positionlist = Select[positionlist, Length@# != 1 &];
paircount = 0;
PrintTemporary[
  "Duplicate list generated. Now generating pairs of matrices with \
same kirchhoff (this will take a while ...)"];

While[positionlist != {},
  templist = positionlist[[-1]];
```

```
  positionlist = Delete[positionlist, -1];

  ilimit = Length[templist];
  If[ilimit > 1, {
    alist = {};
    paircount++;
    For[i = 1, i <= ilimit, i++, {
      alist =
        Append[alist,
         Import[directory <> "graph" <> ToString[n] <>
            "c_md2\\graph" <> ToString[n] <> "c_md2_part" <>
            ToString[Floor[(templist[[i]] - 1)/partsize]] <> ".mx"][[
          Mod[templist[[i]] - 1, partsize] + 1]]];
      }];
    Export[
     "graph" <> ToString[n] <> "c_md2_matching_ki_pair" <>
      ToString[paircount] <> ".mx", alist];
    }];
  ];
Print["There are ", paircount,
  " distinct kirchhoff indices that appears on more than 1 graph."];

(*** Uniqueness Kirchhoff Index calculator ***)
(* MODIFY DEGREE n HERE *)
n = 10;
partsize = 1000;

(* MODIFY IMPORT DIRECTORY HERE *)
directory = "D:\\Google Drive\\Maff Programs\\data\\";
```

```
(* MAKE SURE TO INCLUDE \\ AT THE END *)

(***** NO MORE MODIFIERS BELOW *****)
partcount = 0;
count = 0;
kilist = {};

While[FileExistsQ[
   directory <> "graph" <> ToString[n] <> "c_md2_ki\\graph" <>
    ToString[n] <> "c_md2_ki_part" <> ToString[partcount] <> ".mx"], {
   data =
    Import[directory <> "graph" <> ToString[n] <> "c_md2_ki\\graph" <>
       ToString[n] <> "c_md2_ki_part" <> ToString[partcount] <> ".mx"];
   kilist = Join[kilist, data];
   partcount++;
   count += Length[data];
   }];
Print["Imported ", count, " kirchhoff indices of degree ", n];
PrintTemporary["Now generating the list of duplicates."];

(* This is the finding duplicate function *)
positionDuplicates[list_] := GatherBy[Range@Length[list], list[[#]] &];

positionlist = positionDuplicates[kilist];
Print["Total # of unique kirchhoff indices of degree " <>
   ToString[n] <> ": ", Length[Select[positionlist, Length@# == 1 &]]];
Print["Total # of nonunique kirchhoff indices (distinct indices that \
appears on more than 1 graph) of degree " <> ToString[n] <> ": ",
```

```
  Length[Select[positionlist, Length@# > 1 &]]];

(*** Ihara Zeta Function Generator ***)
(* This program imports md2 graphs in part of %partsize, expected in \
the format .mx as a list, calculate its ihara zeta function and \
outputs a list (.mx) of the same partsize, where the indices' index \
and part number is the same as the original md2 graphs file *)

(* MODIFY DEGREE n HERE *)
n = 10;
partsize = 1000;

(* MODIFY IMPORT DIRECTORY HERE *)
directory =
  "D:\\Google Drive\\Maff Programs\\data\\graph" <> ToString[n] <>
   "c_md2\\";

(***** NO MORE MODIFIERS BELOW *****)
(*ParallelNeeds["ComputerArithmetic`"];*) (*To be added if necessary *)

partcount = 0;
total = 0;

While[FileExistsQ[
   directory <> "graph" <> ToString[n] <> "c_md2_part" <>
    ToString[partcount] <> ".mx"], {
   data =
    Import[directory <> "graph" <> ToString[n] <> "c_md2_part" <>
      ToString[partcount] <> ".mx"];
```

```
hlimit = Length[data];
data2 = {};

For[h = 1, h <= hlimit, h++, {
  a = data[[h]];

  dl = {}; (* dl, a list/array of degrees corresponding to the row/
  column of matrix a *)
  l = -a; (* l, Laplacian Matrix *)

  For[i = 1, i <= n, i++, {
    k = Total[a[[i]]];
    dl = Append[dl, k];
    l[[i]][[i]] = k;
    }];

  edges = Total[dl]/2;
  d = IdentityMatrix[n];
  For[i = 1, i <= n, i++, {
    d[[i]][[i]] = dl[[i]];
    }];
  zf = ((1 - u^2)^(n - edges))/(Det[
      IdentityMatrix[n] - a*u + (d - IdentityMatrix[n])*u^2]);

  data2 = Append[data2, zf];
  }];
Export[
 "graph" <> ToString[n] <> "c_md2_zf_part" <> ToString[partcount] <>
```

```
      ".mx", data2];
   PrintTemporary["Part " <> ToString[partcount] <> " processed."];
   partcount += 1;
   total += hlimit;
   }];
Print["Ihara Zeta Functions of degree ", n, " for a total of ", total,
   " graphs processed."];
(*** Perfect Matching Sets Finder ***)
(* MODIFY DEGREE n HERE *)
n = 9;

(* MODIFY IMPORT DIRECTORY HERE *)
directory =
  "D:\\Google Drive\\Maff Programs\\data\\graph" <> ToString[n] <>
   "c_md2_matching_sets\\";

positionDuplicates[list_] := GatherBy[Range@Length[list], list[[#]] &];

k = 0;
PrintTemporary["Counting number of pairs ..."];
While[FileExistsQ[
   FileNameJoin[{directory,
     "graph" <> ToString[n] <> "c_md2_matching_set" <>
      ToString[k + 1] <> ".mx"}]],
  k++];
iilimit = k;
PrintTemporary["A total of ", k, " pairs/triplets/etc found"];
```

```
spcount = 0;
pcount = 0;

For[ii = 1, ii <= iilimit, ii++, {
   If[Mod[ii, 100] == 0, Print["Completed ", ii, " of ", iilimit]];
   templist =
    Import[
     directory <> "graph" <> ToString[n] <> "c_md2_matching_set" <>
      ToString[ii] <> ".mx"];
   charadjlist = {};
   charlaplist = {};
   charnormlaplist = {};
   For[jj = 1, jj <= Length[templist], jj++, {
     a = templist[[jj]];


     n = Length[a];(* n, degree of the graph i.e. number of vertices *)

     dl = {}; (* dl, a list/array of degrees corresponding to the row/
     column of matrix a *)
     l = -a; (* l, Laplacian Matrix *)

     (* OPTION: Normalized Laplacian ONLY >>>>> *)
      dd = IdentityMatrix[n]; (* dd, degree matrix raised to -1/
     2 power *)
     (* <<<<< Normalized Laplacian ONLY *)

     For[i = 1, i <= n, i++, {
```

```
    k = Total[a[[i]]];
    dl = Append[dl, k];
    l[[i]][[i]] = k;

    (* OPTION: Normalized Laplacian ONLY >>>>> *)
    dd[[i]][[i]] *= k;
    dd[[i]][[i]] = 1/dd[[i]][[i]]^(1/2);
    (* <<<<< Normalized Laplacian ONLY *)
    }];

  charadjlist = Append[charadjlist, CharacteristicPolynomial[a, u]];
  charlaplist = Append[charlaplist, CharacteristicPolynomial[l, u]];
  charnormlaplist =
   Append[charnormlaplist,
    CharacteristicPolynomial[IdentityMatrix[n] - dd.a.dd, u]];
  }];

charadjpositions = positionDuplicates[charadjlist];
charlappositions = positionDuplicates[charlaplist];
charnormlappositions = positionDuplicates[charnormlaplist];

charadjpositions = Select[charadjpositions, Length@# != 1 &];
charlappositions = Select[charlappositions, Length@# != 1 &];
charnormlappositions =
 Select[charnormlappositions, Length@# != 1 &];

If[charadjpositions != {} && charlappositions != {} &&
  charnormlappositions != {} &&
```

```
    charadjpositions == charlappositions == charnormlappositions, {
    pcount++;
    Export[
     "graph" <> ToString[n] <> "c_md2_perfect_matching_set" <>
      ToString[pcount] <> ".mx", templist];
    }, {
    spcount++;
    Export[
     "graph" <> ToString[n] <> "c_md2_semiperfect_matching_set" <>
      ToString[spcount] <> ".mx", templist];
    }];
   }];

Print["Complete! Found ", pcount, " perfectly matching set(s) and ",
  spcount, " semi-perfect matching set(s)."];
```

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