1994

A partial "squeezing theorem" for a particular class of many-valued logics

Stephen Michael Walk
University of Northern Iowa

A PARTIAL "SQUEEZING THEOREM" FOR A PARTICULAR CLASS OF MANY-VALUED LOGICS

An Abstract of a Thesis

Submitted

In Partial Fulfillment

of the Requirements for the Degree

Master of Arts

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May 1994

ABSTRACT

The problem to be studied for this thesis was that of whether the usual statement calculus is a suitable formal system for every many-valued logic in a particular collection of logics. The logics in question are those that fall between the usual two-valued logic and a modified form of the Łukasiewicz-Tarski three-valued logic.

Since this betweenness relationship was an original concept and appeared nowhere in the literature, the first goal in the research plan was to define this relationship precisely. Preliminary concepts included truth value mapping and forgivingness of logics, concepts that, like betweenness, are original to this paper and that facilitate the comparison of many-valued logics. After betweenness was defined, the next stage of the research would be to investigate the logics between the classical logic and the modified L_3 and to see for which of these logics the statement calculus is suitable. This would involve direct calculations with truth tables as well as the use of any published results on the axiomatization and also the comparison of many-valued logics.

Because of the scarcity of work or literature on the problem of comparing many-valued logics, direct calculation turned out to be the most effective method of research. The introduction of a device called a truth class table proved to be invaluable. Such a table allows the logician to work with sets of truth values instead of with individual truth values themselves. Truth class tables were calculated that are characteristic of the logics under consideration.

It was discovered that the usual statement calculus is not suitable for every logic between the two-valued logic and the modified L_3. The main

result of the thesis is a theorem relating sufficient conditions under which a many-valued logic will have the usual statement calculus for a suitable formal system. It is not yet known whether these conditions are necessary as well as sufficient. Concluding remarks demonstrate, as a corollary to the main result, that for any integer n there exist n-valued logics for which the usual statement calculus is suitable.

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This Study by: Stephen M. Walk

Entitled: "A Partial 'Squeezing Theorem' for a Particular Class of Many-Valued Logics"

has been approved as meeting the thesis requirement for the Degree of Master of Arts in Mathematics.

4/21/94
Date — Dr. Michael H. Millar, Chair, Thesis Committee

4/21/94
Date — Dr. Joel K. Haack, Thesis Committee Member

4-21-94
Date — Dr. David R. Duncan, Thesis Committee Member

4/21/94
Date — Mr. David L. Morgan, Thesis Committee Member

5-13-94
Date — Dr. John W. Somervill, Dean, Graduate College

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CHAPTER I

INTRODUCTION

The claim made quite casually by Rosser and Turquette, that "ever since there was first a clear enunciation of the principle 'Every proposition is either true or false,' there have been those who questioned it,"[1] is not supported by those authors in their text, Many-Valued Logics. It is, nonetheless, probably true; or perhaps we should say, in deference to their work, that the claim probably has a truth value close to, but not necessarily equal to, absolute truth. In any event, doubts about the bivalent or true-false nature of logic seem to have existed for some time. Even Aristotle, renowned for his embrace of two-valued logic, developed some arguments that, when interpreted in a certain manner, seemed to indicate that not all propositions were either true or false.[2] It is these doubts that ultimately led to the field we know today as many-valued logic.

Postponing for a moment our examination of the development of many-valued logic, we note that, for some time, ordinary two-valued logic existed primarily in the province of the philosophers. The first distinct systems of logic were that of Aristotle, the "syllogistic," and that of the Stoics, the "dialectic."[3] Łukasiewicz regards the Stoic dialectic as the true ancestor of today's propositional logic, and he traces its further existence and development through the Middle Ages.[4] Despite this early blossoming and long maintenance of logic in the hands of the philosophers, however, it was mathematicians who, hundreds of years later, truly expanded the

[1] Rosser and Turquette [21], p. 10.

[2] Łukasiewicz [7], pp. 125-126.

[3] Łukasiewicz [8], pp. 197-202.

[4] Łukasiewicz [8], p. 198.

discipline. In the words of Łukasiewicz, "modern logic is reborn out of the spirit of mathematics."[5]

The great mathematician and philosopher Gottfried Wilhelm von Leibniz was, according to Lewis and Langford, the "first serious student of symbolic logic,"[6] because of his quest for both a language in which all of science could be expressed and a system for reasoning within that language. However, it was not until the nineteenth century that successful attempts were made to create a symbolic logic. George Boole in 1847 presented a system that would eventually be known as Boolean algebra; Lewis and Langford consider this system to be the foundation for later developments in the field.[7] Gottlob Frege in 1879, Charles Peirce in 1895, and Bertrand Russell and Alfred North Whitehead in 1910 introduced successively better systems of propositional logic,[8] and results involving these systems followed.

Now that our review has reached the twentieth century, we can begin a discussion of many-valued logic, since it was during the first half of this century that many-valued logic began to be considered as a worthwhile area of study. Until then, the two-valued logic had been the system of choice. According to Kosko [4], however, two developments early in this century fostered a sense of doubt in, and dissatisfaction with, the black-and-white, bivalent view of science and mathematics. These developments were the discoveries of Russell's Paradox around 1900 and Heisenberg's Uncertainty Principle in the 1920s. In Kosko's words, "it took the new mathematics of

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[5] Łukasiewicz [8], p. 214.

[6] Lewis and Langford [5], p. 5.

[7] Lewis and Langford [5], p. 9.

[8] Łukasiewicz [7], p.116.

Russell and the new quantum mechanics of Heisenberg to make us first doubt, really doubt, the logic we inherited from Aristotle."[9]

Russell's Paradox involves membership in sets and is constructed, briefly, as follows. Objects can be elements of sets. In fact, sets themselves can be members of sets. Sets can even be members of themselves. For example, the set of all sets is a member of itself, as it is a set; the set of all infinite sets (sets with an infinite number of members) is also a member of itself, since it has an infinite number of members. Of course, not all sets are members of themselves. The set of all numbers, for example, is not a member of itself as it is not a number. The paradox becomes clear when we consider the set (call it F) of all sets that are not members of themselves. Is F a member of itself? If so, then it is not a member of itself, since that is the condition for membership in F. Thus, F is not a member of itself. But if F is not a member of itself, then it satisfies the condition for membership in F, so F is a member of F after all. Thus, F is neither a member of itself nor a nonmember. The statement "F is a member of itself" is neither true nor false--a paradox.

One way to avoid this paradox is to introduce, as Russell did, a system of types and then require that a set of one type can be a member only of sets with the next highest type. Under this restriction, no sets can be members of themselves. However, another solution is to introduce another truth value besides truth and falsity. Then the statement "F is a member of itself" can take this value. Russell himself experimented with this sort of solution in an article called "Vagueness."[10]

[9] Kosko [4], pp. 93-94.
[10] Kosko [4], p. 92n.

The other discovery, Heisenberg's Uncertainty Principle, states that it is impossible to measure both the velocity and the position of a particle. The more precisely the velocity is measured, the less precisely the position can be measured, and vice versa. Instead of stating that a particle has a certain velocity and a certain position, we can only state that the particle has a certain probability distribution for its velocity and another for its position.[11] Thus, statements such as "particle A has this velocity" seem to be neither true nor false--especially as we measure particle A's position with greater and greater precision.

These developments, writes Kosko, pointed away from the usual two-valued logic where every proposition is either 0% true or 100% true, and toward a new viewpoint that led ultimately to the development of fuzzy logic, which we will mention later. However, before we leave the subject of quantum mechanics, we bring up two other concepts from this area that are related to the study of many-valued logics.

First, there is the idea in quantum mechanics that certain quantum events exist not as actual events but as probabilities of events. For example, a particular radioactive atom may have a 50% chance of decaying within a certain amount of time. Until this decay affects something in the macrocosm--the resulting radioactive particle hits a detector or a human chromosome, for example--in other words, until the event is measured, it cannot be considered to have taken place or to have failed to take place. The atom has neither decayed nor failed to decay. Thus, the sentence "the atom has decayed" is neither true nor false until the measurement takes place.[12] While we have found nothing in the literature to indicate that this concept

[11] Kosko [4], pp. 103-107.

[12] Penrose [15], pp. 375-378.

led directly to any consideration of many-valued logics, we feel that it certainly could have led to such a consideration.

Another quantum mechanical concept that directly relates to many-valued logics involves the principle that no causal signal or particle travels faster than light. Putnam [17] calls this the principle of "No action at a distance." Thus, if a star goes supernova four light-years from Earth, we on this planet cannot be affected by it in any way until at least four years have elapsed. However, there have been experimental results in quantum mechanics in which, though no particle was detected actually moving faster than light, there was no way the result could have occurred unless some particle actually had moved faster than light.[13] It was these contradictory results that prompted Hans Reichenbach to recommend a new approach to logic using a three-valued system in which it would be possible for both the "No action at a distance" principle and the experimental results to coexist.[14]

Thus, some of the impetus for studying many-valued logics seems to have come from both the realm of mathematics, in the form of Russell's and other paradoxes, and the realm of physics, as a result of work in quantum mechanics. However, a great portion of the early work on many-valued logics came from a logician named Jan Łukasiewicz of the University of Warsaw. The reason for his pioneering work in the 1920s and 1930s seems to derive from neither mathematical paradoxes nor quantum oddities but instead from a philosophical issue, the question of free will versus determinism, as described below.

[13] Putnam [17], p. 77.

[14] See Putnam [17] for a fuller account.

It was stated earlier that Aristotle developed some arguments that seemed to discredit the principle of bivalence, that is, that every proposition is either true or false. As Łukasiewicz relates [7], Aristotle started with the self-evident Law of the Excluded Middle. This law states that "P or not-P" is true regardless of whether P is true or false. He then argued that from this principle it followed that determinism was correct. However, as Łukasiewicz writes, "Aristotle formulated his argument in support of determinism solely for the purpose of its subsequent rejection as invalid."[15]

Łukasiewicz, like Aristotle, did not wish to accept determinism. However, unlike Aristotle, Łukasiewicz accepted the argument leading from the Law of the Excluded Middle to determinism. To get around this difficulty, then, he developed many-valued logics in which the Law of the Excluded Middle does not hold. The best-known contribution of Łukasiewicz in this area is his work with Alfred Tarski on the systems L_n of n-valued logic. These have since been studied by other authors[16] and will be described in a later chapter.

We make one final note on Łukasiewicz. A careful reading of his argument leading to determinism from the Law of the Excluded Middle shows a mistake in the original step. The idea is to show that a statement "A or not-A" leads to a statement "if P is true, then P has been true at <u>every</u> time t." His argument begins with the choice of an arbitrary time t and consideration of the statement "P is true at time t, or not-P is true at time t." However, this does <u>not</u> have the form "A or not-A."[17] Łukasiewicz (and, if this is Aristotle's argument, then Aristotle also) has begged the question in

[15] Łukasiewicz [7], p. 125.
[16] Ackermann [1], pp. 37-43.
[17] Łukasiewicz [7], pp. 114-117.

his very first step. Nevertheless, noting all of the advances Łukasiewicz made in the field of many-valued logic, we wonder where that field would be today if Łukasiewicz had not made this error.

Another name in the early history of many-valued logic is that of Emil Post. His contribution was primarily in one article [16] in 1921, in which he generalized some two-valued logical concepts to the case of many values. His work is noted by Rosser [20] and also by Rasiowa [18], who remarks that Post's work led eventually to the study of Post algebras. These are algebraic systems that reflect the nature of the L_n logics of Łukasiewicz and Tarski.

These, then, were some of the developments that led initially to the study of many-valued logics. A later landmark was the 1952 monograph <u>Many-Valued Logics</u> by Rosser and Turquette. In this work, drawing on their own and others' previous contributions, Rosser and Turquette laid the basis for the theories of many-valued <u>statement calculi</u> and many-valued <u>predicate calculi</u>.[18] Their contribution in the area of many-valued statement calculi is impressive: they presented an algorithm for constructing, for any finitely many-valued logic satisfying certain standard conditions, an axiom system that reflects the structure of that logic.[19] The importance of that result cannot be overstated, since one of the main objectives in investigating a many-valued logic is to find an axiom system for it.

[18] The difference between a statement calculus and a predicate calculus is roughly as follows. A statement calculus involves methods for inferring, for example, the statement "Q" from the statements "If P, then Q" and the statement "P." A predicate calculus involves methods for inferring, for example, the statement "Socrates is mortal" from the statement "For any man x, x is mortal" and the fact that Socrates is a man.

[19] Rosser and Turquette [21], pp. 10-48.

Later developments in the study of many-valued logic involve Post algebras and fuzzy logic. Post algebras are algebraic structures based on Post's work. According to Rasiowa, they were first investigated by Rosenbloom and, since then, have been considered by Chang and Horn, Dwinger, Rousseau, and others. Post algebras consist of elements corresponding to truth values of a many-valued logic and operators that correspond to the usual statement connectors not, and, or, and if...then.[20] They provide an algebraic framework for representing many-valued logics in much the same way Boolean algebras provide an algebraic framework for ordinary two-valued logic. As mentioned earlier, they correspond to the systems L_n of Łukasiewicz and Tarski.

Fuzzy logic is a related field that is currently playing a role in the development of smart circuitry and machines. The topic originated with Lotfi Zadeh of the University of California at Berkeley in a 1965 paper titled "Fuzzy Sets." Kosko [4] notes that in that paper, Zadeh considered set membership in terms of Łukasiewicz's many-valued logic. Crisp, all-or-nothing membership in sets does not exist in fuzzy logic. Take, for example, the set of tall men. We do not find a cutoff height, say six feet, such that all men taller than six feet are in the set and all men shorter than six feet are outside the set. Instead, we consider all men to be in the set to some degree.[21] One who is six-foot-one may have an eighty per cent membership in the set, while one who is five feet tall may have just a twenty per cent membership. Machinery, such as washing machines, fans, and

[20] Rasiowa [18], pp. 132-133.

[21] In fact, we could consider all human beings to be in the set to some degree, with women and children having lesser degrees of membership. For that matter, we could consider all animals or all living things or all material objects to be in the set to some degree.

trains, that employs fuzzy logic in its circuitry tends to work better than machinery that uses binary circuitry. For an explanation of these technical matters, the reader is referred to Kosko [4]. What is significant for our purpose is that, first, fuzzy logic rejects the old principle of bivalence, and second, fuzzy logic was inspired in part by the work of Łukasiewicz.

Other work in recent years has moved away from mathematics and toward that area where logic was housed for hundreds of years: philosophy. Recent efforts seem to be less interested in axiomatization, which is not surprising considering the extensive work of Rosser and Turquette in this area. Instead, philosophers are trying to interpret many-valued logic. For example, should a truth value reflect a probability? Should it reflect a degree of precision?[22]

One topic that seems to be lacking in the current study of many-valued logics--in fact, it seems never to have been energetically addressed--is the comparison of many-valued logics, and it is to this topic that we turn in the present paper. As we explain later, there have been few attempts to compare arbitrary many-valued logics. The aims of the present paper are to present a general method for comparing many-valued logics and to investigate the behavior of certain logics that, under this comparison method, fall into a certain class.

We begin by considering some relevant concepts in two-valued logic, and we continue by defining many-valued logics in general. A more thorough explanation of our objectives follows, as do the definitions of the necessary new concepts. Finally, our main result is stated and proved, and implications and directions for future research are discussed.

[22] See Marquis [12] and Weston [25].

CHAPTER II

REVIEW OF TWO-VALUED LOGIC AND THE USUAL STATEMENT CALCULUS

Two-Valued Logic and Truth Tables

Before presenting the material on many-valued logics, we will review certain aspects of the usual two-valued logic, with which most people are relatively familiar. The most obvious aspect of this logic is that there are only two truth values, called true and false and abbreviated as "T" and "F" respectively. Using these truth values, we define four truth functions. A truth function is simply a rule that, given a finite list (or n-tuple) of truth values, assigns a single truth value to that list. For example, we might define a truth function h that assigns the value T to the list (T,F) while assigning the value F to the list (F,T), which is different from (T,F) because of the order of its components. In such a case, our notation will be "h(T,F) = T" and "h(F,T) = F," or, more commonly, "T h F = T" and "F h T = F." This is analogous to writing "2 + 3 = 5" instead of "+(2,3) = 5." Any particular truth function will assign truth values only to lists with a certain number n of arguments. If n is 1, then the truth function is unary; if n is 2, the truth function is binary.

The four truth functions associated with the two-valued logic are negation, conjunction, disjunction, and implication. The first is symbolized by "~" and should be thought of as representing the English word not. The function ~ is unary; ~(T) = F and ~(F) = T. Thus, if a sentence P is true, i.e., if P has the value T, then the sentence "not-P" is false, i.e., it has the value F. If P is false, then "not-P" is true.

The second truth function is a binary one, symbolized by "$\wedge$" and representing the English word and. The sentence "P and Q" is true only when P and Q are both true and is false otherwise; accordingly, $T \wedge T = T$, $T \wedge F = F$, $F \wedge T = F$, and $F \wedge F = F$.

The third truth function is binary, symbolized by "$\vee$" and representing the English word or. The sentence "P or Q" is false only when P and Q are both false. Thus, $T \vee T = T$, $T \vee F = T$, $F \vee T = T$, $F \vee F = F$.

The final truth function is another binary one, symbolized by "$\rightarrow$" and representing the English phrase if...then. The sentence "if P, then Q" is false only when P is true and Q is false. For example, if P is the sentence "You buy two" and Q is the sentence "You get one free," then "if P, then Q" is the sentence "If you buy two, then you get one free," which is false only in the instance where you buy two and do not get one free. Thus, $T \rightarrow T = T$, $T \rightarrow F = F$, $F \rightarrow T = T$, and $F \rightarrow F = F$. All of these functions are illustrated in figure 1, which displays the information in truth tables.

It is important to note that the functions $\wedge$ and $\vee$ can be expressed in terms of the functions ~ and $\rightarrow$. Instead of writing "$P \vee Q$," we may write "$\sim P \rightarrow Q$," since, as the reader can verify, these two expressions will have the same truth value no matter what the truth values of P and Q are. For example, suppose P has the value F and Q has the value T. Then $P \vee Q$ has the value T, as we see from the truth table for $\vee$. But ~P has the value ~(F), which is T; so $\sim P \rightarrow Q$ has the value $\sim(F) \rightarrow T$, which is $T \rightarrow T$, and this in turn is T by the truth table for $\rightarrow$.

Similarly, instead of writing "$P \wedge Q$," we may write "$\sim(P \rightarrow \sim Q)$." The significance of these observations is that, though the two-valued logic has negation

$\sim$	P	
	T	F
	F	T

conjunction

$\wedge$	$P \setminus Q$	T	F
	T	T	F
	F	F	F

disjunction

$\vee$	$P \setminus Q$	T	F
	T	T	T
	F	T	F

implication

$\rightarrow$	$P \setminus Q$	T	F
	T	T	F
	F	T	T

Figure 1. Truth tables for the two-valued logic.

four truth functions, we really need only two of them. This fact will be a great help for the work to come.

The Syntax of the Usual Statement Calculus

Our discussion of the two-valued logic has so far been an informal one. We can be forgiven our lack of precision. As long as mathematicians and logicians have studied the two-valued logic--since Aristotle's time, or earlier--it was not until the nineteenth century that formal and precise methods began to be developed for expressing and investigating the logic.[23]

The tool that is now used almost universally to study the two-valued logic is called the statement calculus,[24] and it is the best example of a formal

[23] For an outline of these developments, see Copi [3], p. 7.

[24] The phrase statement calculus is used by some authors in a more general sense. In all chapters to come, the statement calculus defined here will be referred to as the usual statement calculus to avoid confusion.

system. To construct a formal system, we enumerate, first, certain axioms or beginning assertions, and second, certain rules of inference. All later assertions will be deemed acceptable or unacceptable not on the basis of their meaning (no meaning is imparted to the assertions), but on the basis of whether they follow from the axioms by successive applications of the rules of inference.

Thus, if a system is to be truly formal, we should be able to work within it without any particular assumptions as to what it means. The reader should bear in mind, however, that while the statement calculus was formulated for the purpose of investigating the two-valued logic, no explicit mention of that logic will be made in the formulation of the statement calculus.

The formulation of the statement calculus involves a series of definitions, which are more or less standard ones in mathematical logic. We use the definitions given by Margaris [11],[25] as these definitions are very explicit, leaving nothing to the interpretation or the imagination of the reader.

Definition: The following twelve symbols are the formal symbols of the statement calculus:

$$\sim \quad \rightarrow \quad \forall \quad (\quad) \quad , \quad x \quad \alpha \quad \beta \quad \gamma \quad \# \quad |$$

Definitions: A string is any finite concatenation (ordering) of formal symbols. A variable is a string composed of one occurrence of "x" followed by zero or more occurrences of "|." A constant is a string composed of one occurrence of "γ" followed by zero or more occurrences of "|."

[25] Some slight adaptations of the Margaris definitions have been made and are due to Millar [14].

Though we do not formally attach any particular interpretation to the statement calculus, an informal explanation of motivations for these definitions is in order. Variables can be considered in the same sense here as in algebra: a variable is used to represent something that we want to talk about but cannot identify; its value is not fixed. A constant, on the other hand, represents something in particular, and its value is fixed.

The next definitions need to be explained before they are given. A predicate in mathematics, like a predicate in English, is essentially a verb: it expresses something about an object or objects. For example, in the sentence "13 is prime," the predicate is "is prime"; the argument is the object talked about, which in this case is the number thirteen. Hence, the predicate "is prime" is a one-place predicate because it takes only one argument. Two-place predicates include "is equal to," "is greater than," and "is less than." Three-place predicates include "is between," as in, "point A is between point B and point C."

Operations, on the other hand, are functions, like the truth functions discussed earlier. When given an n-tuple of the appropriate number of arguments, an operation assigns a value: for instance, addition ("+") is a two-place operation that assigns to the 2-tuple (2,3) the value 5. Operations, like predicates, can be one-place, two-place, and so on. Unlike predicates, they do not assert anything about objects, but merely name objects. For instance, "2+3" does not say anything about 2 and 3; it just names their sum. However, "2+3 = 5" asserts something about that sum and the number 5, namely, that they are equal.

Thus, with the above discussion in mind of what predicates and operations signify, we define symbols to represent them within the statement calculus.

Definition: An n-place predicate symbol is a string composed of one occurrence of the symbol "α" followed by n occurrences of the symbol "#" followed by zero or more occurrences of the symbol "$|$."

Definition: An n-place operation symbol is a string composed of one occurrence of the symbol "β" followed by n occurrences of the symbol "#" followed by zero or more occurrences of the symbol "$|$."

In the definitions above, the roles of the symbols "#" and "$|$" are clear. The number of occurrences of "#" is the number of arguments of the predicate or the operation. The occurrences of "$|$" are simply a tally; they distinguish between different n-place predicates or operations.

Definition: Term is defined recursively as follows:

1. All variables are terms.
2. All constants are terms.
3. If f is any n-place operation symbol, and $t_1, t_2, \ldots, t_n$ are terms, then $f(t_1, t_2, \ldots, t_n)$ is a term.
4. Only strings are terms, and a string is a term only if its being so follows from (1) to (3) above.

Definition: Formula is defined recursively as follows:

1. If G is any n-place predicate symbol, and $t_1, t_2, \ldots, t_n$ are terms, then $G(t_1, t_2, \ldots, t_n)$ is a formula, called an atomic formula.
2. If P is a formula, then $\sim P$ is a formula.
3. If P and Q are formulas, then $P \rightarrow Q$ is a formula.

4. If P is a formula and v is a variable, then $\forall vP$ is a formula.

5. Only strings are formulas, and a string is a formula only if its being so follows from (1) to (4) above.

The reader may have noticed that we have finally used all of the given formal symbols. Now that we can express sentences in it, the language of our formal system is complete. It remains only to set up a method for deciding which sentences are acceptable and which are not, much like separating the true sentences from the false ones.

One more comment is in order before we continue. If we continue our discussion from above, interpreting predicates as verbs, terms as nouns, and so on, we can interpret the symbols "~" and "→" to mean "not" and "if...then," respectively. No interpretation has yet been given for the symbol "$\forall$." It should be interpreted as "for all." Suppose, for example, that we have decided that, whenever the predicate symbol "$\alpha\#$" occurs within the system, it represents the predicate "is prime." Then the formula $\alpha\#(x)$ asserts that "x is prime." Finally, then, the formula $\forall x\alpha\#(x)$ asserts that "<u>for all x</u>, x is prime," or, "everything is prime." In this case, the formula $\alpha\#(x)$ is said to lie within the <u>scope</u> of the symbol $\forall x$, which is called a <u>quantifier</u>. If a quantifier occurs immediately before a formula in parentheses, as in $\forall x||\ (\alpha\#(x||) \rightarrow \sim\alpha\#\#(x||,x||))$, then everything within the parentheses is said to lie within the scope of the quantifier.

We must now set up the machinery for deciding which statements are acceptable. Such statements will be called <u>theorems</u>. We proceed in the time-honored mathematical tradition of laying down a few beginning formulas, called axioms, which will be regarded as acceptable immediately, and a very few rules of inference for deriving logically satisfactory formulas from other satisfactory formulas.

We begin with a finite number of axiom schemes:

A1. $P \rightarrow (Q \rightarrow P)$.

A2. $(P \rightarrow (Q \rightarrow R)) \rightarrow ((P \rightarrow Q) \rightarrow (P \rightarrow R))$.

A3. $(\sim Q \rightarrow \sim P) \rightarrow (P \rightarrow Q)$.

When P and Q, or P, Q, and R, in the above axiom schemes are replaced with actual formulas, the resulting string is an instance of that particular axiom scheme, or, more simply, an axiom.

Thus, we have an infinite number of axioms, but the key is that we only need a finite number of axiom schemes to express them. Our set of rules of inference is even simpler: there is only one rule of inference, called modus ponens, the well-known rule that from "if P, then Q" and "P," we can infer "Q." The acceptable statements (theorems) are found through proofs.

Definition: A proof is a finite sequence of formulas such that every formula in the sequence is

(i) an axiom, or

(ii) a formula inferred from two earlier formulas in the sequence by modus ponens (that is, a formula Q such that both P and $P \rightarrow Q$ have already occurred, in either order, in the sequence).

Definition: A theorem is a formula that is the last step in some proof.

This, then, is the formal basis of the statement calculus.

Suitability and the Completeness Theorem

The reader has no doubt noticed that the above formulation of the statement calculus made no mention of truth values, and also that the earlier discussion of truth values made no mention of proofs.

There is a good reason for this. In any study of logic, the objective is, in essence, to decide which sentences are acceptable and which are not. However, to be able to decide which sentences are acceptable, we must first have a set of sentences to consider. Our sentences are the formulas just defined. Once this set of sentences exists, there are different contexts in which to consider those sentences. One context is that of <u>syntax</u> or form. Syntactically acceptable sentences are those that are theorems.

The other context is that of <u>semantics</u> or meaning. Semantically acceptable sentences are those that are <u>tautologies</u>. A tautology is a sentence that is always true no matter whether its component parts are true or false. The sentence "I have never seen this person before, and I resent the accusation" is not always true: in fact, it is false if the speaker has seen the person before or if the speaker doesn't really resent the accusation. However, the sentence "if the sky is cloudy, then the sky is cloudy" is always true, no matter whether the sky really is cloudy. Such a sentence is an example of a tautology; its component parts are two instances of the sentence "the sky is cloudy". We can symbolize such a sentence in the statement calculus with a formula of the form $P \rightarrow P$. These two instances of the formula P are then the component parts, or <u>prime constituents</u>, of the formula $P \rightarrow P$. We make the notions of prime constituent and tautology precise with the definitions that follow.

<u>Definitions:</u> A formula is a <u>prime formula</u> if

(i) it has the form $\forall vP$, where v is any variable and P is any formula, or

(ii) it is an atomic formula that does not lie within the scope of a quantifier.

A *prime constituent* of a formula P is any consecutive part of P that is a prime formula. For example, if P is ~α##||| (β#(x),x) →∀x||α##(x||,x||), then the prime constituents of P are α##|||(β#(x),x) and ∀x||α##(x||,x||).

Once we have distinguished the prime constituents of a formula, we can *assign* a truth value to each prime constituent. We agree that identical prime constituents are assigned the same truth value. Then, if we interpret the symbols ~ and → in the formula to be the truth functions ~ and →, we see that the entire formula *receives* a truth value on the basis of the values assigned to its prime constituents.

Definition: A *tautology* is a formula that receives the value T under *every* assignment of truth values to its prime constituents.

Example: As noted above, the sentence "if the sky is cloudy, then the sky is cloudy" is a tautology, and this can be shown as follows. Suppose we are representing the predicate "is cloudy" within our system by the symbol "α#|||." Then, letting the constant γ| represent the sky, the above sentence becomes α#|||(γ|) → α#|||(γ|). The only prime constituent is α#|||(γ|); when it is assigned the value T, the sentence receives the value T→T = T. When the prime constituent is assigned the value F, the sentence receives the value F→F = T.

In fact, it is not always necessary to distinguish every prime constituent of a formula in order to pronounce the formula a tautology. For example, any formula with the form P→P will be a tautology, whether P itself is prime as in the above example, or whether P is made up of hundreds of millions of prime constituents. This is so because no matter what values the prime constituents of P are assigned, P itself will always receive either the value T or the value F. We then need only consider the cases T→T and F→F, as above, to conclude that we have a tautology.

The tautologies, then, can be thought of as the formulas that are acceptable under the logic. Since we earlier thought of the theorems of a formal system as the acceptable formulas within the formal system, the link between truth values and formal systems becomes clear: a formal system will be said to be <u>suitable</u> for a logic if the formulas that are theorems in the formal system are exactly the formulas that are tautologies of the logic. We do not define this concept precisely at this point, since we have not yet defined <u>tautology</u> for any logic other than the classical two-valued one; indeed, we have not even defined <u>logic</u> yet! However, the reader has probably guessed by now that the statement calculus is a suitable formal system for the classical two-valued logic.

This result, called the Completeness Theorem,[26] shows that the statement calculus does exactly what it was designed to do: it reflects which statements are tautologies in the classical two-valued logic. Once we generalize our discussion to include logics with more than two values, we will be concerned again with such a question: given a particular many-valued logic, what kind of formal system has to be set up to reflect the tautologies of that logic?

Certainly, we must introduce some more definitions to tell exactly what is meant by "many-valued logic" and "tautology," and these definitions will appear in the next chapter. However, a closing note for this chapter is in order, and it concerns the character of the given formulation--Margaris's--of the statement calculus. Many authors, when discussing many-valued

[26] For a proof of the Completeness Theorem, see Margaris [11].

logics, and logics in general, postulate at the beginning a set of "sentential variables" p, q, r,....[27] However, this avoids the question of just what a "sentence" is, i.e., what objects can be substituted for a sentential variable. It also fails to state what the sentential variables themselves are, once the letters p through z have been used in the above progression. The Margaris framework is superior to any such vague formulation, in that it defines every part of the structure explicitly and thus leaves nothing up to the discretion of the reader. This is a desirable feature for any formal system.

The reader should also be aware that while this formulation of the statement calculus does include an infinite number of variables, constants, predicates, and operations, it does not require an infinite number of symbols to name them. To add a new name to our system, we just add another "#" or "|" mark. The entire system is expressible in terms of the given twelve symbols. This finitely expressible character is another feature that makes this formulation attractive. We shall return to the topic of finite expressibility.

[27] See Łukasiewicz [8], Rosser and Turquette [21], Mendelson [13], Chang [2].

CHAPTER III

GENERALIZATION TO MANY VALUES

Requirements of a Many-Valued System

Our aim in this chapter is to define precisely the concept of a many-valued logic. To this end, we consider certain notions that other authors have included, or have at least mentioned, in their definitions of the concept. Our definition will be essentially that of Mendelson [13], with slight alterations in deference to Łukasiewicz [6] and Rosser and Turquette [21].

The reader will recall, from the last chapter, the meaning of atomic formulas of a formal system. Such formulas are those that, quite simply, cannot be decomposed into smaller formulas; they are the basic sentences of the language of the formal system. Similarly, in the study of many-valued logics, it is assumed that there exists some set of formulas, each of which will take one of the truth values of the logic, or of formula variables, each of which will range over the set of truth values of the logic when the set of truth values is defined. Usually these formulas (or formula variables) are referred to as statements (or statement variables). Margaris [12] defines a statement as a formula in which no variable v occurs outside the scope of the quantifier $\forall v$. However, the term is usually used in a more general sense to mean any sentence, and we adopt such usage here.

After the manner of Rosser and Turquette [21], we make these statements (or formulas) our first requirement for a many-valued logic. Thus, a many-valued logic will be said to be defined for a set of formulas. In all cases we will consider, this set of formulas will be exactly those constructed in the manner of Margaris [12], described in the last chapter.

Second, we must choose a set of truth values to represent varying degrees of acceptibility for the statements. There are many ways to choose these values (for example, Chang's [2] choice of all values in the interval [-1,1]), but the two most common choices are as follows. First, for an M-valued logic, Rosser and Turquette [21] and Mendelson [13] choose the integers from 1 to M as their truth values, with 1 representing absolute truth and M absolute falsity. Such a truth-value set has two disadvantages from the standpoint of the present paper. First, it is counterintuitive, in that greater numbers correspond to lesser degrees of truth. Second, our purpose is to compare logics that may differ in their number of truth values, and comparisons cannot readily be made between, for example, a logic whose truth values are 1, 2, and 3 and one whose truth values range from 1 to 1000. The other manner of choosing truth values, used by Łukasiewicz in the majority of his papers,[28] is the one we will use for the present paper. This system places falsity at zero and truth at one, so that it not only agrees with our intuition that a greater truth value should correspond to a greater degree of truth but also facilitates comparisons between logics. It further fits our intuition when compared to Boole's Laws of Thought or to probability theory. Finally, this placement of truth and falsity also admits logics with an infinite number of truth values.

Third, we need a set of truth functions, which are functions on the set of truth values. For example, the ~ of the classical two-valued logic is a truth function from {0,1} to {0,1} such that ~(0) = 1 and ~(1) = 0. We will suppose that these truth functions can be symbolized by the statement connectors among the formal symbols that constitute our set of formulas. For

[28] See Łukasiewicz [7], p. 141, and Łukasiewicz and Tarski [10], p. 87n.

example, the truth functions ~ and → are symbolized by "~" and "→" within the formulas we constructed in the preceding chapter.

The distinction between a statement connector and its corresponding truth value function is a conceptually important but fairly obvious one; consequently, while some authors (notably Rosser and Turquette [21] and Mendelson [13]) are always careful to draw the distinction, others (Łukasiewicz [8], Chang [2]) seem to regard it as sufficiently clear without further elaboration or different notation. In the present paper, we note the distinction, but we use the same symbol to denote both the statement connector and its corresponding truth function. The usage will always be obvious from the context.

Finally, Rosser and Turquette [21] and Mendelson [13] require the choice of an integer S such that $1 \leq S < M$ (recall that here the integers 1 to M are the truth values of the logic). Once this choice has been made, the integers 1, ..., S are called <u>designated values</u>, and should be thought of as the values that are "true enough" for a specific purpose. The integers S+1, ..., M are then the <u>undesignated values</u>, those that are not "true enough." It is worth noting that Łukasiewicz's work does not include the general concept of designated values; when designated values are considered at all, the only designated value is absolute truth.[29] However, we will follow the procedure of Rosser and Turquette and Mendelson, with a definition for designated values on our zero-to-one scale that is analogous to the definition on their scale. We will also define designated values for logics with infinite numbers of values.

[29] See, for example, Łukasiewicz [6], pp. 283ff.

Many-Valued Logics and Suitability

With the above considerations in mind, then, we make the necessary definitions as follows.

Definitions: An n-valued logic is a quadruple $\langle P, V, \mathfrak{F}, S \rangle$, where

(i) P is a set of formulas;

(ii) V is the set $\{k/(n-1) : k = 0, 1, \ldots, n-1\}$;

(iii) $\mathfrak{F}$ is a finite collection of functions on the set V, such that each element of $\mathfrak{F}$ is represented by a statement connector among the formal symbols of P; and

(iv) S is a nonzero element of V.

Then the set V is called the set of truth values of the n-valued logic. The elements of $\mathfrak{F}$ are called the truth functions of the n-valued logic. The elements of the subset $\{x \in V : x \geq S\}$ are called the designated values of the n-valued logic, and elements of the complement $\{x \in V : x < S\}$ are called the undesignated values of the n-valued logic.

Definitions: An infinite-valued logic is a quintuple $\langle P, V, \mathfrak{F}, V_1, V_0 \rangle$, where

(i) P is a set of formulas;

(ii) V is the interval $[0,1]$ or V is the set $\{x \in [0,1] : x \text{ is rational}\}$;

(iii) $\mathfrak{F}$ is a finite collection of functions on the set V, such that each element of $\mathfrak{F}$ is represented by a statement connector among the formal symbols of P; and

(iv) V_1 and V_0 are disjoint subsets of V such that $V_1 \cup V_0 = V$ and, for every x_1 in V_1 and every x_0 in V_0, $x_1 > x_0$.

Truth values and truth functions are defined as for an n-valued logic. The elements of the set V_1 are called the designated values of the infinite-valued logic, and the elements of the set V_0 are called the undesignated values of the infinite-valued logic.

Definition: A many-valued logic (or, a logic) is an n-valued logic (n a positive integer) or an infinite-valued logic.

The reader can verify that these definitions comply with the requirements listed in the previous section. Notice that we could have defined designated and undesignated values in the n-valued case the same way we defined them in the infinite-valued case, that is, by delineating sets V_1 and V_0 rather than by choosing a number S to serve as the lowest designated value. However, because of the requirements and practice of Rosser and Turquette [21] and Mendelson [13], choosing a lowest designated value seems more appropriate historically in the n-valued case. Thus, we use this definition for the n-valued case. Such a definition would be needlessly restrictive for an infinite-valued logic, which may or may not have a lowest designated (or a highest undesignated) value.

Our purpose, given a many-valued logic, is to find a formal system that "matches" the logic, i.e., a system that is suitable for the logic; we define this concept below. The reader will notice that the definitions for prime constituent, tautology, and suitability are analogous to the definitions made for the same terms within the context of the classical two-valued logic.

Definitions: Let L be a many-valued logic defined on a set of formulas whose statement connectors $F_1, F_2, \ldots, F_m$ correspond to the truth functions $f_1, f_2, \ldots, f_m$ of the logic L. Let P be a formula in this set. Then prime constituent of P is defined recursively as follows:

(i) if P is an atomic formula, that is, P contains no occurrences of the statement connectors $F_1, F_2, \ldots, F_m$ except those that are inside the scope of quantifiers, then the prime constituent of P is P itself;

(ii) if P is $F_i(Q_1, Q_2, \ldots, Q_{i_j})$, where i_j is the number of arguments of the connector F_i, and where $Q_1, Q_2, \ldots, Q_{i_j}$ are formulas, then the prime constituents of P are all of the prime constituents of $Q_1, Q_2, \ldots, Q_{i_j}$;

(iii) only formulas of the posited set are prime constituents, and a formula in this set is a prime constituent only if its being so follows from conditions (i) and (ii) above.

A formula P is a <u>tautology</u> of the logic L if, under any assignment of truth values of L to the prime constituents of the formula P, the corresponding truth function takes a designated value.

A formal system is <u>suitable</u> for a many-valued logic if the theorems of the formal system are precisely those formulas that are tautologies of the many-valued logic.

As an example, consider the many-valued logic L_3 introduced by Jan Łukasiewicz [9] and later studied, with other, similar logics, by Łukasiewicz and Tarski [10]. The set of truth values of L_3 is $\{0, 1/2, 1\}$. The truth functions of L_3 correspond, as might be expected, to <u>not</u>, <u>and</u>, <u>or</u>, and <u>if...then</u> and are illustrated in figure 2. These truth functions are symbolized, respectively, by **N**, **K**, **A**, and **C**, and they can be expressed numerically as follows. If the statement P_1 has truth value p_1, then $\mathbf{N}P_1$ has truth value $1 - p_1$. If statements P_1 and P_2 have truth values p_1 and p_2, respectively, then $P_1 \mathbf{K} P_2$ has the truth value $\min\{p_1, p_2\}$; $P_1 \mathbf{A} P_2$ has the

N

P_1	
1	0
1/2	1/2
0	1

C

$P_1 \setminus P_2$	1	1/2	0
1	1	1/2	0
1/2	1	1	1/2
0	1	1	1

K

$P_1 \setminus P_2$	1	1/2	0
1	1	1/2	0
1/2	1/2	1/2	0
0	0	0	0

A

$P_1 \setminus P_2$	1	1/2	0
1	1	1	1
1/2	1	1/2	1/2
0	1	1/2	0

Figure 2. Truth functions of the logic L_3.

truth value $\max\{p_1, p_2\}$; and P_1 **C** P_2 has the truth value $\min\{1, p_2 - p_1+1\}$.[30] For example, suppose that the statement P_1 has the truth value 1/2 and P_2 has the truth value 0. Then, by the above formula, P_1 **C** P_2 has the value $\min\{1, 0 - 1/2 + 1\} = \min\{1, 1/2\} = 1/2$, as we verify by consulting the table. As in the two-valued logic, it is possible to define some of these functions in terms of the others. Given statements P and Q, regardless of their individual truth values, P **A** Q always takes the same truth value as (P **C** Q) **C** Q; thus, whenever a formula of the form P **A** Q appears, we can replace it with the appropriate expression of the form (P **C** Q) **C** Q. Similarly, P **K** Q always takes the same truth value as **N**((**N**P) **A** (**N**Q)), or,

[30] The functions given for **N** and **C** above are as given by Ackerman [1], p. 40, and by Rosser [19], p. 137. The others are adapted from the functions given by Rosser and Turquette [21], p. 15, where the truth values are instead the set {1, 2, 3}.

in light of the above observation, by **N** (((NP) **C** (NQ)) **C** (NQ)).[31] Thus, every truth function of L_3 can be expressed in terms of **N** and **C**.

Other systems L_n, where n is the number of truth values for the logic, were studied by Łukasiewicz and Tarski[32]; in these systems, the functions **N** and **C** are defined numerically as above, and the functions **K** and **A** are defined in terms of them. The logic L_2 is then the classical two-valued logic. Similar functions are defined for the infinite-valued Łukasiewicz-Tarski logic, whose truth values range over the interval [0,1].[33] In each case, there is only one designated value which, of course, is 1.

For each logic L_n, with $n > 2$, there exist formulas that are tautologies in L_2 but are not tautologies in L_n. For instance, P **A** (NP), the Law of the Excluded Middle, is not a tautology in L_3, since if P has truth value 1/2, then NP has truth value 1/2, and so P **A** (NP) has truth value max {1/2, 1/2} = 1/2. In fact, the Law of the Excluded Middle is not a tautology in any L_n with $n > 2$. We assume that this fact was noticed by Łukasiewicz, who, by his own account,[34] wished to avoid that particular law. However, suppose we loosen our requirements a bit for L_3, and designate instead the values 1 and 1/2. Then the Law of the Excluded Middle and many other formulas that are not tautologies of L_3 become tautologies of the new logic, which we will henceforth denote as L_3'.[35]

[31] Łukasiewicz dispensed with parentheses through a notation known as Polish notation; as Polish notation is not necessary, we omit discussion of it here and use parentheses instead.

[32] See Ackerman [1].

[33] See Rosser [19].

[34] See Łukasiewicz [7].

[35] For example, **N** (P **K** (NP)), which says that a statement and its negation are not both acceptable, is a tautology in L_3' but not in L_3, as the reader can easily verify.

Our original indication was that such a logic would have as tautologies exactly those formulas that are tautologies in the classical two-valued logic. However, some of the steps leading to this conclusion were faulty, and we are unsure whether it is in fact so. It is known that the set of tautologies of L_3' is a subset of the set of tautologies of the classical two-valued logic, which we will henceforth denote as C. Whether these sets are in fact the same remains to be proved.

The importance of the logic L_3' for the present work will be made clear in the following chapter. In the remaining discussions, we shall denote the truth functions N and C of L_3' as "~" and "→", respectively.

A final word is in order before we continue to the work of the present paper. The purpose in studying a many-valued logic is, at least for us, to find a suitable formal system for it. However, as Mendelson notes,[36] this can always be done in at least a trivial manner: we simply take, as axioms for a formal system, the set of all tautologies of the logic; and we dispense with rules of inference altogether. Such a procedure, however, would be uninteresting and, from a mathematical point of view, wasteful, since some of the tautologies could perhaps be derived from others, given appropriate rules of inference. It is our aim instead to find a finite axiomatization--that is, one with a finite number of axiom schemes and a finite number of rules of inference. Such a structure is more interesting from a mathematical point of view and fulfills our desire, mentioned in the last chapter, for finite expressibility.

That such a finite axiomatization exists for a large class of finitely many-valued logics was demonstrated by Rosser and Turquette [21]. It is

[36] See Mendelson [13], p. 37.

our aim in the present paper to specify, for a certain collection of many-valued logics, that they can in fact be finitely axiomatized by the usual statement calculus.

CHAPTER IV
PAST RESEARCH ON COMPARING MANY-VALUED LOGICS

A survey of the literature indicates that little has been attempted in the area of comparing many-valued logics. In fact, there seem to be just two contexts in which such comparisons have been considered. The first involves the systems L_n described before. As mentioned, some formulas that are tautologies of C (= L_2) are not tautologies for any L_n with $n > 2$. In fact, the set of tautologies of any system L_n with $n > 2$ is a proper subset of the set of tautologies of C. In this sense, C is being compared to the other systems L_n.

The second context is discussed in the article by Schock [23]. Neither of these cases involves the sort of generality that is of interest for the present study. The systems L_n all involve the particular functions **N** and **C** as described earlier, and Schock assumes certain "standard conditions" on the logics in question.

The assumptions we make in the present paper will be much less restrictive than those made for the L_n or those made by Schock. On the other hand, our assumptions will be sufficiently powerful to guarantee a relatively uniform structure for the logics in which we are interested. The main concept is that of betweenness, which was conceived during the writing of the 1992 paper [24]. In this paper, a three-valued logic M was investigated and found to have the usual statement calculus as a formal system. This logic M is nearly identical to the logic L_3'; the only difference occurs in the (1/2) → (1/2) entry of the truth table for implication. In the logic L_3', (1/2) → (1/2) has the value 1; in M, it has the value 1/2. The usual statement calculus is suitable for the logic C and appeared to be suitable for L_3' as well. It is also suitable for the logic M, which is between C and L_3' in the sense mentioned above. The question that presents itself is as follows: Is the usual statement calculus suitable for <u>all</u> logics between C and L_3'?

To answer this question, we must make precise our notion of betweenness and all of the concepts involved in comparing many-valued logics. We do so in the next chapter.

CHAPTER V

PRELIMINARY CONCEPTS

The aim of the present paper is to investigate a certain class of logics: those that are between the classical two-valued logic C and the modified Lukasiewicz three-valued logic L_3'. In this chapter, we make precise this notion of betweenness by introducing some machinery for comparing logics. We also discuss some characteristics of the logics between C and L_3'.

Standardization of Truth Values

The notion of betweenness involves a comparison of many-valued logics. Such a comparison, in turn, requires a certain uniformity between the logics to be compared. We have taken a step toward such uniformity by standardizing the sets of truth values, setting truth to one and false to zero for any many-valued logic. However, further standardization is necessary. If we are to compare logics, then it seems appropriate to do so by comparing their respective truth functions. However, difficulties soon surface if the truth values of the logics have not been standardized beyond the zero-to-one scaling.

The first difficulty is that truth function tables for logics of differing numbers of truth values cannot properly be compared. It seems reasonable that we should call one truth function f more forgiving than another truth function g if, given any ordered n-tuple of truth values as arguments, the value of f applied to those arguments is either the same as, or closer to truth than, the value of g applied to those arguments. For example, we would consider the function $\rightarrow$ of the classical two-valued logic to be more forgiving than the $\wedge$ of the same logic, since $F \rightarrow T = T$ while $F \wedge T = F$, $F \rightarrow F =$

T while F$\wedge$F = F, and the functions agree on the other possible argument values. Apparently, then, given the truth tables in figure 3, in which f and g agree on all common ordered pairs except (1,0), we should consider the function f (from a four-valued logic) more forgiving than the function g (from a three-valued logic), since f(1,0) = 2/3 and g(1,0) = 1/2, and 2/3 is closer to one than 1/2 is.

f	1	2/3	1/3	0
1	1	1	1	2/3
2/3	1	1	1	1
1/3	1	1	1	1
0	1	1	1	1

g	1	1/2	0
1	1	1	1/2
1/2	1	1	1
0	1	1	1

Figure 3. Functions on different sets of truth values.

However, of 1/2 and 2/3, each is only one truth value away from absolute truth in its respective logic. Further, while we have standardized each logic to make 1 absolute truth and 0 absolute falsity, our assignment of the fractions k/(n-1) to the middle truth values was arbitrary. These considerations suggest that it is improper to see the 2/3 of one logic as "truer than" the 1/2 of another logic, so it is a dubious move for us to pronounce f more forgiving than g in this manner. In addition, f, because it ranges over a greater number of truth values, has lines in its truth table that do not even exist in the truth table for g. We cannot directly compare the truth tables for f and g without either leaving these lines out or constructing analogous lines for the g truth table, and neither course of action seems proper.

The second difficulty that presents itself is that even if two logics have the <u>same</u> number of truth values, they may differ in their sets of designated values. Certainly any comparison between logics should take designated values into account. This is especially true when the study is tied intimately to the determination of which tautologies are theorems, and thus to which formulas are tautologies. Thus, if the sets of designated values differ, it is again improper to compare truth tables directly. Consider, for example, the functions φ_1 and φ_2 whose truth tables are displayed in figure 4. The tables themselves are identical. However, if φ_1 is a truth function in a logic whose designated values are 1, 2/3, and 1/3, and φ_2 is a truth function in a logic whose only designated value is 1, then φ_1 should be considered more forgiving than φ_2. In other words, the tables by themselves cannot discriminate between the logics.

As another example of identical truth functions in logics with differing sets of designated values, compare the truth functions from the three-valued logic L_3 with the corresponding functions in our adaptation L_3' of L_3.

To eliminate the difficulties outlined above, we introduce the concept of a <u>truth-value mapping</u>.

<u>Definition:</u> Let J and K be j- and k-valued logics, respectively, where $j \geq k$. (We will understand this to include the cases where J is an infinite-valued logic and K is a k-valued logic, and where J and K are infinite-valued logics but K has countably many truth values.) A <u>truth-value mapping</u> from J to

φ_1	1	2/3	1/3	0
1	1	2/3	1/3	0
2/3	1	1	2/3	0
1/3	1	1	1	0
0	1	1	1	2/3

φ_2	1	2/3	1/3	0
1	1	2/3	1/3	0
2/3	1	1	2/3	0
1/3	1	1	1	0
0	1	1	1	2/3

Figure 4. Functions on logics with different sets of designated values.

K is a function f from the set of truth values of J onto the set of truth values of K such that

(i) $f(1_J) = 1_K$ and $f(0_J) = 0_K$, that is, f maps the truth and falsity of J to the truth and falsity, respectively, of K;

(ii) if J has a least designated value not equal to 1, then this value is mapped to the least designated value of K (if it exists); similarly, if J has a greatest undesignated value not equal to 0, this value is mapped to the greatest undesignated value of K if it exists; and

(iii) for any distinct truth values x and y of J, if $x > y$, then $f(x) \geq f(y)$.

Example: Let J and K be five- and four-valued logics, respectively. Let 1 and 3/4 be the designated values for J, and let 1 and 2/3 be the designated values for K. Then figure 5 illustrates a truth-value mapping from J to K. There is only one other truth-value mapping from J to K; it is as in figure 5, except that it maps $1/4_J$ to $1/3_K$. Note the form of the diagram. Diagramatically, a truth-value mapping is a mapping for which none of the arrows cross.

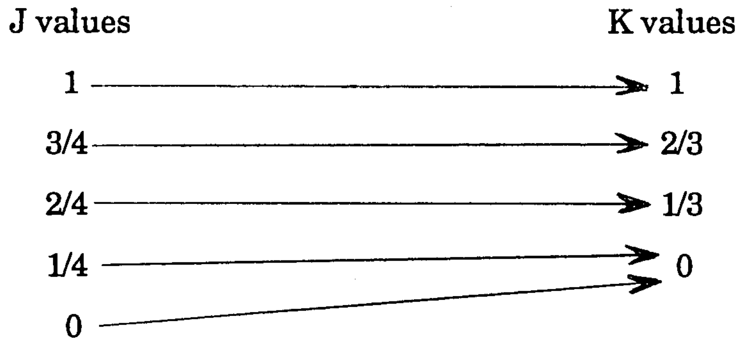

Figure 5. A truth-value mapping.

Example: Consider the three-valued logic M investigated in [24]. Only one truth-value mapping g from M to L_3' exists. The designated values for each of these logics are 1 and 1/2, so g must map 1 to 1, 1/2 to 1/2, and 0 to 0. Also, there is only one truth-value mapping h from M to the classical two-valued logic: it maps 1 to 1, 1/2 to 1, and 0 to 0.

With the concept of a truth-value mapping in place to standardize our logics further, we can make the necessary comparisons between truth functions. We now make precise the concepts of more forgiving and less forgiving.

Definitions: Let J and K be j- and k-valued logics, respectively, where $j \geq k$. Let f be a truth-value mapping from J to K. Let φ_1 be an m-place truth function of J, and let φ_2 be an m-place truth function of K.

Suppose that, for any truth values $x_1, x_2, \ldots, x_m$ of J, the truth value $f(\varphi_1(x_1, x_2, \ldots, x_m))$ is greater than or equal to the truth value

$\varphi_2(f(x_1), f(x_2), \ldots, f(x_m))$. Then φ_1 is said to be as forgiving as φ_2 with respect to f. If φ_1 is as forgiving as φ_2 and strict inequality holds for some m-tuple of truth values--that is, if $f(\varphi_1(x_1, x_2, \ldots, x_m)) > \varphi_2(f(x_1), f(x_2), \ldots, f(x_m))$ for some $x_1, x_2, \ldots, x_m$--then φ_1 is more forgiving than φ_2 with respect to f.

Similarly, suppose that, for any truth values $x_1, x_2, \ldots, x_m$ of J, the truth value $f(\varphi_1(x_1, x_2, \ldots, x_m))$ is less than or equal to the truth value $\varphi_2(f(x_1), f(x_2), \ldots, f(x_m))$. Then φ_1 is no more forgiving than φ_2 with respect to f. If strict inequality holds for some m-tuple of truth values, then φ_1 is less forgiving than φ_2 with respect to f.

If φ_1 is neither no more forgiving than nor as forgiving as φ_2 with respect to f, then φ_1 and φ_2 are not comparable with respect to f; otherwise, they are comparable with respect to f.

It is not known whether a truth function that is less or more forgiving than another truth function under one truth-value mapping will have the same relation to the second function under another truth-value mapping. However, for the results obtained in the present paper, we only need one truth-value mapping to exist for any relationship of forgivingness. Thus, if the particular truth-value mapping is obvious from the context, we drop the phrase "with respect to f."

Examples: With respect to the truth-value mapping h described earlier, the truth function ~ from the logic M is more forgiving than the truth function ~ from the classical two-valued logic C. Equality holds in the definition of more forgiving, except in the ~(1/2) case: $h(\sim_M(1/2)) = h(1/2) = 1 > 0 = \sim_C 1 = \sim_C(h(1/2))$. Also, the function → from the logic M is more forgiving than the function → from C. Equality holds in the definition of more forgiving, except in the (1/2) → 0 case: $h((1/2) \rightarrow_M 0) = h(1/2) = 1 > 0 = 1 \rightarrow_C 0 = h(1/2) \rightarrow_C h(0)$.

As the reader can verify, with respect to the truth-value mapping g described earlier, the function ~ from M is no more forgiving than the function ~ from L_3', and the function → from M is less forgiving than the function → from L_3'.

Now we can compare truth functions. When we take the next step--comparing entire logics--it seems reasonable that if all of the truth functions of a logic A_1 are as forgiving as the truth functions of a logic A_2, then we should consider A_1 to be "as forgiving as" A_2. We must be cautious, however, about which truth functions we compare. Obviously we cannot compare a unary function of A_1 to a binary function of A_2. More generally, if m is not equal to n, we cannot compare any m-place function of A_1 to any n-place function of A_2. In addition, two different logics may have differing numbers of truth functions, as figure 6 illustrates. Even with a truth-value mapping in place, we are unable to compare the logics A_1 and A_2, because we have no way to decide whether to compare the function f to g_1 or to g_2.

We eliminate such a difficulty by assuming that for any logic under consideration, every truth function of the logic can be expressed in terms of two truth functions f and g that may or may not be among the truth functions of the logic. Because of our special interest in the logics C and L_3', and because for each of these there exist a unary function ~ and a

Logic A_1

f	1	2/3	1/3	0
1	1	1/3	1/3	1/3
2/3	1	2/3	1/3	1/3
1/3	1	1	2/3	1/3
0	1	1	2/3	2/3

Logic A_2

g_1	1	1/2	0
1	1	1/2	0
1/2	1	1/2	1/2
0	1	1	1/2

g_2	1	1/2	0
1	1	1	1
1/2	1	1/2	0
0	1/2	1/2	0

Figure 6. Logics with different numbers of truth functions.

binary function → in terms of which all truth functions of the logic can be expressed, we will further assume that one of {f,g} is unary and the other binary.[37] This is not such a harsh demand as it may seem. Salomaa [22] notes that in any finitely many-valued logic, it is possible to find a unary and a binary truth function such that every truth function of the logic can be expressed in terms of those two functions.[38] Also, certain infinite-valued logics have such functions, such as the infinite-valued Lukasiewicz logic.[39]

[37] This second assumption is not necessary. We can define more forgiving and less forgiving for logics in which f and g are not unary or binary, and in fact we can make these definitions for logics where it takes more than just two functions (though a finite number) to express all the truth functions. However, the notation involved is cumbersome, and since such a level of generality is not necessary for the present paper, we omit it.

[38] Actually, Salomaa notes that in any finitely many-valued logic, it is possible to find a Sheffer function, i.e., a binary function in terms of which we can express ev ery possible function on the truth values of the logic. Certainly, then, given a Sheffer function, all the truth functions of the logic (a subset of the set of functions on the truth values) can be expressed in terms of it.

[39] See Rosser [19].

Once the needed unary and binary truth functions are formulated for each logic under consideration, we can compare these functions in the obvious manner. It is from this discussion that the following definitions arise.

Definitions: Let J and K be j- and k-valued logics, respectively, where $j \geq k$. Let f be a truth-value mapping from J to K. Let d_1 and d_2 be a unary and a binary function, respectively, on the truth values of J such that every truth function of J can be expressed in terms of d_1 and d_2. Let e_1 and e_2 be a unary and a binary function, respectively, on the truth values of K such that every truth function of K can be expressed in terms of e_1 and e_2.

If d_1 is as forgiving as e_1 and d_2 is as forgiving as e_2, then J is said to be as forgiving as K,[40] and K is no more forgiving than J. If J is as forgiving as K and, in addition, one of the d_i is more forgiving than the corresponding e_i, then J is more forgiving than K, and K is less forgiving than J.

If d_1 is no more forgiving than e_1 and d_2 is no more forgiving than e_2, then J is said to be no more forgiving than K, and K is as forgiving as J. If J is no more forgiving than K and, in addition, one of the d_i is less forgiving than the corresponding e_i, then J is less forgiving than K, and K is more forgiving than J.

If one of the d_i is more forgiving than the corresponding e_i and the other d_j is less forgiving than the corresponding e_j, or if either pair is not

[40] We could be more precise and say that J is more forgiving than K with respect to f, d_1, d_2, e_1, and e_2. Since, as in the definition of forgivingness for functions, our purposes only require that one truth-value mapping exist, as well as one choice of d_1, d_2, e_1, and e_2, and since context will always prevent confusion, we dispense with this terminology for the obvious reason.

comparable, then the logics are not comparable. Otherwise, the logics are comparable.

Example: The logic M is more forgiving than the classical two-valued logic (with respect to the truth-value mapping h described earlier and the sets $\{\sim_M, \rightarrow_M\}$ and $\{\sim_C, \rightarrow_C\}$). Also, M is less forgiving than L_3'.

Example: Any logic is both as forgiving as, and no more forgiving than, itself. This can be seen by considering, as a truth-value mapping, the identity mapping on its set of truth values.

Definition: Let J, K, and B be many-valued logics such that J is more forgiving than K. Then B is said to be between J and K (or B is said to be between K and J) if

(i) B is as forgiving as K and B is no more forgiving than J, and

(ii) B is more forgiving than K or B is less forgiving than J.

Example: Any logic L is between itself and any other comparable logic. However, L is not between L and L, since L cannot be either more forgiving or less forgiving than itself.

Example: The reader can verify that the logic L_3' is more forgiving than the logic C. Then, as demonstrated earlier, the logic M is between the logics C and L_3'.

Now that the notion of betweenness has been defined, we can finally investigate the logics that, like M, are between C and L_3'. As stated before, our objective is to discover whether all of these logics have the usual statement calculus as a suitable formal system.

Truth Class Tables

Our investigation will be aided by the introduction of truth class tables, which are similar to truth tables. The difference is that, whereas a truth table gives the value of $x \rightarrow y$ for individual truth values x and y, a truth class table tells what <u>subset</u> of truth values $x \rightarrow y$ is in when x and y are in particular <u>subsets</u> of truth values. By working with truth class tables, we are freed from considering fully detailed truth tables. This enables us to make general statements about many different truth functions (on different sets of truth values) simultaneously, and also prevents us from writing out truth tables that have a very large number of entries.

The development of truth class tables for the present work is as follows. Suppose we are considering a logic I (for "intermediate") that is between C and L_3'. Then we know by the definition of betweenness that, with respect to some truth-value mappings, I is as forgiving as C and no more forgiving than L_3'. We will show first that the mappings in question are mappings from I to C and from I to L_3', and not from these logics to I. In other words, the mappings in question map "outward" from the intermediate logic.

Since I is as forgiving as C, then there must be a truth-value mapping from I to C or from C to I. However, C has only two truth values, and for a truth-value mapping to exist from C to I, I must itself have only two values. In this case, I is the logic C, as we show now. Since the truth of C must map to the truth of I, and the falsity of C maps to the falsity of I, we can compare directly the truth tables for $\rightarrow$ and $\sim$. Since I is as forgiving as C, both truth tables of I must contain the truth value 1 everywhere but in the ~ 1 and the $1 \rightarrow 0$ case. Considering the only possible truth-value mapping from L_3' to I, and considering that under this mapping L_3' is as forgiving as I, we see that in the logic I, ~ 1 must be 0 and $1 \rightarrow 0$ must be 0 also, as the reader can easily verify from the definition of <u>as forgiving as</u> and the truth tables for L_3'. Then the truth functions $\sim$ and $\rightarrow$ are the same as those of C, so the two logics are the same. Thus, the only two-valued logic between C and L_3' is C itself.

Now, if I is a logic between C and L_3', we can assume I is not C, for otherwise, there is nothing to investigate. Then I has more than two truth values, so the truth-value mapping in question, i.e., the mapping with respect to which I is as forgiving as C, is a mapping from I to C.

We also know that I is no more forgiving than L_3' with respect to a truth-value mapping from I to L_3' or from L_3' to I. Suppose the truth-value mapping is from L_3' to I; then I must have two values (in which case I is C, as noted above) or three values. If I has three values, then the truth-value mapping in question, since it is onto, must in fact be the identity mapping on the set {1, 1/2, 0}. Since this is the case, when we apply the definition of as forgiving as, we are simply comparing directly the truth tables of I to the truth tables of L_3'. If we consider the identity mapping as a mapping not from L_3' to I but from I to L_3', we will again compare the truth tables directly, and the same forgivingness relationship will hold.

Thus, either I has two values, a case we do not need to consider, or I has three values, in which we can use the identity mapping from I to L_3' as the truth-value mapping for our investigations, or I has more than three values, in which case any truth-value mapping must map from I to L_3'. Therefore, in all cases we can assume that the truth-value mappings with respect to which I is between L_3' and C are in fact truth-value mappings from I "outward."

Thus, we know there exist a truth-value mapping f from I to L3' and a truth-value mapping g from I to C. Now, consider the truth values of I. Specifically, consider them as three disjoint subsets: {x: f(x) = 1}, {x: f(x) = 1/2}, and {x: f(x) = 0}. These are the "truth classes" to be used for the truth class tables and will be denoted as τ, υ, and ϕ, respectively.[41] By the definition of truth-value mapping, every element of τ is greater than every element of υ, and every element of υ is greater than every element of ϕ. It follows that we can consider ϕ to be a subset of [0,a], υ a subset of [a,b], and τ a subset of [b,1] for appropriately chosen a and b.[42] However, what is most important is simply the fact that we can divide the truth values in this way. Notice that, since τ and υ contain all the values that f maps to 1 or to 1/2, it must be that τ and υ contain all the designated values of I. Therefore, g maps any element of τ or of υ to 1. The subset ϕ, on the other hand, contains all the undesignated values of I.

We know by the definitions of <u>less forgiving</u> and <u>more forgiving</u> for logics that there exist a unary function ~ and a binary function → on the truth values of I such that all truth functions of I can be expressed in terms of ~ and →. Now we can construct <u>truth class tables</u> for these functions ~ and →. These will enable us to study the logic I by means of the subsets τ, υ, and ϕ, instead of by means of the truth values directly. The construction is as follows: To find an entry for the truth class table--the entry for ~τ, for example--we choose a truth value x that is in τ. Then, using the fact that I

[41] In [24], the truth values 1, 1/2, and 0 of the logic M were denoted T (for "truth"), U (for "uncertainty"), and F (for "falsity"). The set names τ, υ, and ϕ are intended to reflect these denominations.

[42] The word <u>subset</u> is important here. It is impossible for ϕ to be the whole interval [0,a] <u>and</u> υ the whole interval [a,b], or υ the whole interval [a,b] <u>and</u> τ the whole interval [b,1], since a cannot be in both ϕ and υ, and b cannot be in both υ and τ.

is as forgiving as C and no more forgiving than L_3', we calculate which subsets, τ, υ, or ϕ, can contain ~x. For this example, it turns out that if x is in τ, then ~x must be in ϕ, so the "~τ" entry in the truth class table is ϕ. The completed truth class tables for ~ and → are in figure 7. A comma in an entry means that the truth value in question could lie in either given subset. For example, if x and y are both in τ, then x→y may be in τ or in υ.

→	τ	υ	ϕ
τ	τ,υ	υ	ϕ
υ	τ,υ	τ,υ	υ,ϕ
ϕ	τ,υ	τ,υ	τ,υ

~	
τ	ϕ
υ	υ,ϕ
ϕ	τ,υ

Figure 7. Truth class tables for logics I between C and L_3'.

The full derivation of these truth class tables is given in Appendix B, where it is also shown that these tables characterize the logics between C and L_3'. That is, any logic I that is represented by the tables in figure 7 is in fact between C and L_3'.

The object of our investigation is to seek a "squeezing theorem" for logics between C and L_3'. That is, we want to know whether every logic between C and L_3' has the usual statement calculus as a suitable formal system. In light of the preceding discussion, this becomes the question of whether every logic that is represented by the truth class tables in figure 7 has the usual statement calculus as a suitable formal system.

The results of this search are presented in the next chapter.

CHAPTER VI

A PARTIAL SQUEEZING THEOREM FOR LOGICS I BETWEEN C AND L_3'

Nonexistence of a Full Squeezing Theorem

The objective described in the last chapter is impossible. Logics I exist between C and L_3' that are represented by the given truth class tables, and yet do not have the usual statement calculus as a formal system.

Such a logic is illustrated in figure 8. If we let 1, 2/3, and 1/3 be the designated values of this logic, then there is a truth-value mapping g such that g(1) = 1, g(2/3) = g(1/3) = 1/2, and g(0) = 0. Then $\tau = \{1\}$, $\upsilon = \{2/3, 1/3\}$, and $\phi = \{0\}$. The truth class tables for this logic, then, are given in table 9, and they can be easily calculated by the reader. For example, consider the $\upsilon \rightarrow \phi$ case. The truth value 2/3 is in υ, and 0 is in ϕ, and $2/3 \rightarrow 0 = 0$ is in ϕ. However, 1/3 is also in υ, and $1/3 \rightarrow 0 = 2/3$ is in υ. Thus, the entry for the $\upsilon \rightarrow \phi$ case is "υ,ϕ." The other entries for the tables are filled in similarly.

Clearly this logic is between C and L_3'. For example, if x and y are any elements of τ, then $x \rightarrow y$ is an element of υ, so $x \rightarrow y$ is certainly an element of τ <u>or</u> υ, so this part of the characteristic truth class table for $\rightarrow$ is fulfilled. Similarly, the other entries in the truth class tables for this logic fit the characteristic truth class tables, so this logic is between C and L_3'. However, it does not have the usual statement calculus as a formal system. In fact, even the Axiom Schemes A1-A3 are not all tautologies in this logic. A1 is, but A2 receives the undesignated value (0) when P is assigned 1 or

→	1	2/3	1/3	0
1	2/3	2/3	2/3	0
2/3	2/3	2/3	2/3	0
1/3	2/3	2/3	2/3	2/3
0	2/3	2/3	2/3	2/3

~	
1	0
2/3	0
1/3	0
0	1/3

Figure 8. A counterexample for the putative "squeezing theorem."

→	τ	υ	φ
τ	υ	υ	φ
υ	υ	υ	υ,φ
φ	υ	υ	υ

~	
τ	φ
υ	φ
φ	υ

Figure 9. The truth class tables for the counterexample.

2/3, Q is assigned 1/3, and R is assigned 0. Further, A3 fails when P is assigned 1 or 2/3 and Q is assigned 0. This is certainly not the only logic for which these axioms fail. Many of the truth table entries can be changed without making A2 or A3 tautologies, and yet preserving the betweenness relationship of the separate logics.

Though we are unable to find the desired "Squeezing Theorem," we can state a corresponding partial result.

The Partial Squeezing Theorem for Logics I between C and L_3'

The partial result gives conditions that are sufficient for a logic between C and L_3' to have the usual statement calculus as a formal system. Its proof will require some preliminary lemmas, but we state the theorem here.

Metatheorem: Let I be a many-valued logic between C and L_3'. Suppose that either of the following conditions holds:

(i) the set $\tau \cup \phi$ is closed under the operations ~ and →, and, for any truth values x_1, x_2 in υ and any truth value y in ϕ, the truth value $x_1 \rightarrow x_2$ is in υ, and the truth values $x_1 \rightarrow y$ and $\sim x_1$ are in υ; or

(ii) for any truth value x in υ and any truth value y in ϕ, the truth values $x \rightarrow y$ and $\sim x$ are in ϕ.

Then the usual statement calculus is a suitable formal system for the logic I.

Any logic I between C and L_3' will have truth class tables as in figure 7. The conditions in the metatheorem simply narrow the possibilities for the table entries. For example, condition (ii) changes the $\upsilon \rightarrow \phi$ and $\sim\upsilon$ entries from "υ,ϕ" to "ϕ." The altered truth class tables for logics meeting either of these conditions separately are shown in figures 10 and 11.

In our proofs of the metatheorem and of the lemmas leading to it, we assume, since all truth functions of I can be expressed in terms of the functions ~ and →, that these are in fact the only functions used.

$\rightarrow$	τ	υ	ϕ
τ	τ	υ	ϕ
υ	τ,υ	υ	υ
ϕ	τ	τ,υ	τ

$\sim$	
τ	ϕ
υ	υ
ϕ	τ

Figure 10. Truth class tables for a logic I satisfying condition (i).

$\rightarrow$	τ	υ	ϕ
τ	τ,υ	υ	ϕ
υ	τ,υ	τ,υ	ϕ
ϕ	τ,υ	τ,υ	τ,υ

$\sim$	
τ	ϕ
υ	ϕ
ϕ	τ,υ

Figure 11. Truth class tables for a logic I satisfying condition (ii).

In the results that follow, let I be a logic between C and L_3' such that condition (i) is satisfied, that is, the truth values of I can be divided into sets τ, υ, and ϕ that fulfill the truth class tables in figure 10.

Lemma 6.1: Let P be a formula. If no prime constituent of P is assigned a truth value in υ, then P does not receive a truth value in υ.

Proof: By strong induction on the number of occurrences of $\rightarrow$ and $\sim$ in P.

Basis: P has no occurrences of $\rightarrow$ or $\sim$. Then P is an atomic formula or has the form $\forall$vQ. Then P is a prime formula and is its own prime constituent, so if P is not assigned a value in υ, it will not receive a value in υ.

Inductive step: Suppose P has n occurrences of $\rightarrow$ and $\sim$, and assume that the lemma holds for formulas with fewer than n such occurrences. Suppose no prime constituent of P is assigned a truth value in υ.

Case 1: P is ~Q for some formula Q that has n - 1 occurrences of $\rightarrow$ and $\sim$. The prime constituents of Q are identically those of P, so none of them has been assigned any value in υ. By the induction hypothesis, Q does not receive a value in υ. Then Q receives a value in τ or ϕ, so that, by the truth class table for $\sim$, P receives a value in ϕ or τ, and hence not in υ.

Case 2: P is Q$\rightarrow$R for some formulas Q and R, each with fewer than n occurrences of $\rightarrow$ and $\sim$. None of the prime constituents of Q or of R has been assigned a value in υ, since they are all prime constituents of P. Then, by the induction hypothesis, Q receives a value in τ or a value in ϕ, and R receives a value in τ or a value in ϕ, so there are four possibilities in all. The truth class table for $\rightarrow$ shows that P receives a value in τ or ϕ for each of these possibilities. In each case, therefore, P does not receive a value in υ.

By the strong form of the principle of mathematical induction, therefore, the theorem holds for any formula P with any number of occurrences of $\rightarrow$ and $\sim$, that is, it holds for any formula P.

Lemma 6.2: Let P be a formula. Let A be an assignment of truth values from τ ϕ to the prime constituents of P. Let B be the assignment of the truth values 1 and 0 of C to the prime constituents of P such that B assigns 1 to the prime constituents that were assigned values in τ, and B assigns 0 to the prime constituents that were assigned values in ϕ. If P receives a value in τ under the assignment A (where $\sim$ and $\rightarrow$ represent the truth functions associated with I), then it receives the value 1 under the assignment B (where $\sim$ and $\rightarrow$ represent the truth functions associated with C), and if P receives a value in ϕ under the assignment A, then it receives 0 under the assignment B.

Proof: Let A and B be as in the statement of the lemma. The proof of the lemma is by strong induction on the number of occurrences of $\rightarrow$ and $\sim$ in P.

Basis: P has no occurrences of $\rightarrow$ or $\sim$. Then P is its own prime constituent and receives whichever truth value it is assigned. The result is vacuously true by the choice of the assignment B.

Inductive step: Suppose P has n occurrences of $\rightarrow$ and $\sim$, and assume the lemma holds for formulas with fewer than n such occurrences.

Case 1: P is $\sim$Q for some formula Q that has n - 1 occurrences of $\rightarrow$ and $\sim$. By Lemma 6.1, since no prime constituent of P was assigned a value in υ, P receives a value in τ or in ϕ under the assignment A. If P receives a value in τ under the assignment A, then Q receives a value in ϕ

by the truth class table for ~. By the induction hypothesis, then, Q receives 0 under the assignment B. Then, by the truth table for ~ in C, P receives 1 under that assignment. If, on the other hand, P receives a value in ϕ, then Q receives a value in τ or υ by the truth class table for ~. But the prime constituents of Q are exactly the prime constituents of P, so none of them was assigned a value in υ under the assignment A. Thus, by Lemma 6.1, Q cannot receive a value in υ, and hence it receives a value in τ. Then, by the induction hypothesis, Q receives 1 under the assignment B. Therefore, by the truth table for ~ in C, P receives 0 under that assignment.

Case 2: P is Q→R for some formulas Q and R, each with fewer than n occurrences of → and ~. By Lemma 6.1, P receives a value in τ or a value in ϕ under the assignment A.

Subcase 2.1: P receives a value in τ. Suppose, for purposes of contradiction, that P receives the value 0 under the assignment B. Then, by the truth table for → in C, Q receives the value 1 and R receives the value 0. Since the prime constituents of Q and of R are all prime constituents of P, and since none of these has been assigned a value in υ, under the assignment A each of Q and R receives a value in τ or in ϕ by Lemma 6.1. Then, by the induction hypothesis, since Q receives the value 1 under the assignment B, and since it cannot receive a value in both τ and ϕ, it receives a value in τ under the assignment A. Similarly, R receives a value in ϕ under the assignment A. Then, by the truth class table for →, P receives a value in ϕ, which contradicts the original assumption. Therefore, P cannot receive 0 under the assignment B, so that it must receive 1 under that assignment.

Subcase 2.2: P receives a value in ϕ. Then, by the truth class table for $\rightarrow$, Q receives a value in τ, and R receives a value in ϕ. Again, by Lemma 6.1, Q cannot receive a value in υ, so Q receives a value in τ. Then, by the induction hypothesis, under the assignment B, Q receives 1 and R receives 0, so that P receives 0 by the truth table for $\rightarrow$ in C.

By the strong form of the principle of mathematical induction, the theorem holds for any formula P with any number of occurrences of $\rightarrow$ and $\sim$, that is, it holds for any formula P.

Lemma 6.3: If a formula P is a tautology in I, then it is a tautology in C.

Proof: Let P be any formula that is not a tautology in C. Then, for some assignment of 0's and 1's to the prime constituents of P, P receives the value 0. Create a new assignment A that assigns the absolute truth of I to every prime constituent that was assigned 1 and assigns the absolute falsity of I to every prime constituent that was assigned 0. By Lemma 6.1, since no prime constituent of P is now assigned a value in υ, P receives a value in τ or in ϕ. Suppose P receives a value in τ. Then, by Lemma 6.2, P receives the value 1 under the original assignment of 0's and 1's, contradicting the assumption about P. Therefore, P must receive a value in ϕ under the assignment A, so that P is not a tautology of I.

Metatheorem 6.4: Every tautology of I is a theorem of the usual statement calculus.

Proof: By Lemma 6.3, every tautology of I is a tautology of C. Margaris [12] demonstrates that every tautology of C is a theorem of the usual statement calculus. The metatheorem follows.

In the discussion that follows, the term tautology refers exclusively to tautologies of I. Recall that a tautology of I is a formula such that for any assignment of truth values to the prime constituents of P, P receives a value in τ or in υ.

Lemma 6.5: Let P be a formula. Let an assignment of truth values of I be made to the prime constituents of P.

(i) Suppose P receives a value in ϕ. If the truth-value assignment is altered so that the prime constituents previously assigned values in υ are assigned 1 instead, and all other assignments are retained, then P receives a value in ϕ under the new assignment.

(ii) Suppose P receives a value in τ. If the truth-value assignment is altered so that the prime constituents previously assigned values in υ are assigned 1 instead, and all other assignments are retained, then P receives a value in τ under the new assignment.

(Note that in this theorem, the "1" referred to is absolute truth in I.)

Proof: By induction on the number of occurrences of $\rightarrow$ and $\sim$ in P.

Basis: P has no occurrences of $\rightarrow$ or $\sim$. Then P is its own prime constituent and, if it receives a value in ϕ or τ, must have been assigned the same value. The lemma is vacuously true in this case.

Inductive step: Suppose P has n occurrences of $\rightarrow$ and ~, and assume that the lemma holds for all formulas with fewer than n such occurrences.

Suppose that, under the given assignment of truth values, P receives a value in ϕ.

Case 1: P is ~Q for some formula Q with n - 1 occurrences of $\rightarrow$ and ~. Then, by the truth class table for ~, Q receives a value in τ under the original assignment, so that by the induction hypothesis. Q receives a value in τ under the new assignment. Then, by the truth table for ~, P receives a value in ϕ under the new assignment.

Case 2: P is Q$\rightarrow$R for some formulas Q and R, each with fewer than n occurrences of $\rightarrow$ and ~. By the truth class table for $\rightarrow$, Q receives a value in τ and R receives a value in ϕ under the original assignment of truth values, so under the new assignment Q receives a value in τ and R receives a value in ϕ. Then P receives a value in ϕ under the new assignment by the truth table for $\rightarrow$.

Now suppose that, under the given assignment of truth values, P receives a value in τ.

Case 3: P is ~Q for some formula Q. Then by the truth class table for ~, Q receives a value in ϕ under the original assignment, so by the induction hypothesis, Q receives a value in ϕ under the new assignment as well. Then, by the truth class table for ~, P receives a value in τ under the new assignment.

<u>Case 4:</u> P is Q→R for some formulas Q and R, each with fewer than n occurrences of → and ~. Then, by the truth class table for →, Q receives a value in ϕ or R receives a value in τ.

<u>Subcase 4.1:</u> Q receives a value in ϕ under the original assignment. Then, by the induction hypothesis, Q receives a value in ϕ under the new assignment as well. By the truth class table for →, P then receives a value in τ under the new assignment.

<u>Subcase 4.2:</u> R receives a value in τ under the original assignment. Then, by the induction hypothesis, R receives a value in τ under the new assignment as well. Once again, by the truth table for →, P receives a value in τ under the new assignment.

Thus, by the strong form of the principle of mathematical induction, the lemma holds for any formula with any number of occurrences of → and ~, so that it holds for any formula.

<u>Lemma 6.6:</u> Let P and Q be any formulas. If P and P→Q are both tautologies, then Q is a tautology as well. In other words, modus ponens preserves tautology.

<u>Proof:</u> Suppose that P and P→Q are tautologies and that Q is not a tautology, that is, Q receives a value in ϕ under some assignment of truth values to its prime constituents. By Lemma 6.5, there is an assignment of truth values from τ and from ϕ to the prime constituents of Q such that Q receives a value in ϕ under that assignment also. Create a new assignment of truth values to the prime constituents of P and Q in the following way. To every

prime formula that is a prime constituent of Q, assign the same truth value as in the assignment just described using Lemma 6.5. To every prime formula of P that is not a prime constituent of Q, assign the (absolute truth) value 1. By Lemma 6.1, since no prime constituent of P was assigned a value in υ, P must receive a value in τ or in ϕ, and since it is a tautology, P must receive a value in τ. But then, since P receives a value in τ and Q receives a value in ϕ, $P \rightarrow Q$ receives a value in ϕ by the truth class table for $\rightarrow$, contradicting the assumption that $P \rightarrow Q$ was a tautology. Thus, Q cannot receive a value in ϕ under any assignment of truth values to its prime constituents, so that Q is a tautology.

Lemma 6.7: Any instance of Axiom Scheme A1, A2, or A3 is a tautology.

Proof: The relevant truth class tables are displayed in figures 12 to 14. These tables can be easily calculated from figure 10. Note that none of the truth class tables for A1, A2, or A3 includes the set ϕ as a final entry.

Lemma 6.8: Any step in a proof is a tautology.

Proof: Let P be a formula that is the nth step in some proof. By the definition of proof, P is an instance of an Axiom Scheme or is inferred from two earlier steps by modus ponens. The proof is by induction on n.

Basis: P is step 1 or step 2. Then P is an instance of an Axiom Scheme and thus is a tautology by Lemma 6.7.

Inductive step: P is step n, where steps 1 to n - 1 are tautologies. Then P is an instance of an Axiom Scheme or is inferred from two earlier steps, steps j and k ($1 \leq j < k$), by modus ponens. In the first case, P is a tautology by Lemma 6.7. In the second, steps j and k are tautologies by the induction

P	Q	Q→P	P→(Q→P)
τ	τ	τ	τ
τ	υ	τ,υ	τ,υ
τ	φ	τ	τ
υ	τ	υ	υ
υ	υ	υ	υ
υ	φ	τ,υ	τ,υ
φ	τ	φ	τ
φ	υ	υ	τ,υ
φ	φ	τ	τ

Figure 12. The truth class table for Axiom Scheme A1.

P	Q	~Q	~P	~Q→~P	P→Q	(~Q→~P) → (P→Q)
τ	τ	φ	φ	τ	τ	τ
τ	υ	υ	φ	υ	υ	υ
τ	φ	τ	φ	φ	φ	τ
υ	τ	φ	υ	τ,υ	τ,υ	τ,υ
υ	υ	υ	υ	υ	υ	υ
υ	φ	τ	υ	υ	υ	υ
φ	τ	φ	τ	τ	τ	τ
φ	υ	υ	τ	τ,υ	τ,υ	τ,υ
φ	φ	τ	τ	τ	τ	τ

Figure 13. The truth class table for Axiom Scheme A3.

P	Q	R	P→(Q→R)	(P→Q)→(P→R)	(P→(Q→R)) → ((P→Q)→(P→R))
τ	τ	τ	τ	τ	τ
τ	τ	υ	υ	υ	υ
τ	τ	ϕ	ϕ	ϕ	τ
τ	υ	τ	τ,υ	τ,υ	τ,υ
τ	υ	υ	υ	υ	υ
τ	υ	ϕ	υ	υ	υ
τ	ϕ	τ	τ	τ	τ
τ	ϕ	υ	τ,υ	τ,υ	τ,υ
τ	ϕ	ϕ	τ	τ	τ
υ	τ	τ	τ,υ	τ,υ	τ,υ
υ	τ	υ	υ	υ	υ
υ	τ	ϕ	υ	υ	υ
υ	υ	τ	τ,υ	τ,υ	τ,υ
υ	υ	υ	υ	υ	υ
υ	υ	ϕ	υ	υ	υ
υ	ϕ	τ	τ,υ	τ,υ	τ,υ
υ	ϕ	υ	τ,υ	υ	υ
υ	ϕ	ϕ	τ,υ	υ	υ
ϕ	τ	τ	τ	τ	τ
ϕ	τ	υ	τ,υ	τ,υ	τ,υ
ϕ	τ	ϕ	τ	τ	τ
ϕ	υ	τ	τ,υ	τ,υ	τ,υ
ϕ	υ	υ	τ,υ	τ,υ	τ,υ
ϕ	υ	ϕ	τ,υ	τ,υ	τ,υ
ϕ	ϕ	τ	τ	τ	τ
ϕ	ϕ	υ	τ,υ	τ,υ	τ,υ
ϕ	ϕ	ϕ	τ	τ	τ

Figure 14. The truth class table for Axiom Scheme A2.

hypothesis, so P is a tautology by Lemma 6.6. By the strong form of the principle of mathematical induction, every step in a proof is a tautology.

Metatheorem 6.9: Every theorem of the usual statement calculus is a tautology in I.

Proof: Every theorem is, by definition, the last step of a proof. By Lemma 6.8, then, every theorem is a tautology of I.

Metatheorem 6.10: The usual statement calculus is a suitable formal system for the given logic I.

Proof: Metatheorems 6.4 and 6.9.

The first part of the partial squeezing theorem has been proved.

In the results that follow, let J be a logic between C and L_3' such that condition (ii) is satisfied, that is, the truth values of J can be divided into sets τ, υ, and ϕ that fulfill the truth class tables in figure 11.

Lemma 6.11: Let P be a formula. Let A be an assignment of truth values of J to the prime constituents of P, and let B be an assignment of truth values of the two-valued logic C to the prime constituents of P, such that

(i) B assigns 1 to those prime constituents that were assigned designated values of J under the assignment A, and

(ii) B assigns 0 to those prime constituents of P that were assigned undesignated values of J under the assignment A.

Then the formula P receives a designated value of J under the assignment A if and only if P receives the value 1 under the assignment B, and P receives an undesignated value of J under the assignment A if and only if P receives the value 0 under the assignment B.

Proof: By strong induction on the number of occurrences of ~ and → in P.

<u>Basis:</u> P has no occurrences of ~ or →. Then P is its own prime constituent, and the result holds by the construction of B.

<u>Inductive step:</u> Suppose P has n occurrences of ~ and →, and assume the result holds for all formulas with fewer than n such occurrences. Suppose P receives a designated value of J under the assignment A.

<u>Case 1:</u> P is ~Q for some formula Q with n - 1 occurrences of ~ and →. Then, by the truth class table for $\sim_J$, since P received a designated value (that is, a value from τ or υ), Q received an undesignated value (that is, a value from ϕ). Then, by the induction hypothesis, Q receives 0 under the assignment B, and hence P receives 1 by the truth table for $\sim_C$.

<u>Case 2:</u> P is Q→R for some formulas Q and R, each with fewer than n occurrences of ~ and →. Then, by the truth class table for $\rightarrow_J$, either Q received an undesignated value or R received a designated value under the assignment A.

<u>Subcase 2.1:</u> Q received an undesignated value under A. Then, by the induction hypothesis, Q receives 0 under B. Then P, which is Q→R, receives 1 under B by the truth table for $\rightarrow_C$.

<u>Subcase 2.2:</u> R received a designated value under A. Then, by the induction hypothesis, R receives 1 under B. Then P receives 1 by the truth table for $\rightarrow_C$.

Now suppose P receives an undesignated value of J under the assignment A.

<u>Case 3:</u> P is ~Q for some formula Q with n - 1 occurrences of ~ and →. Then, by the truth class table for $\sim_J$, Q received a designated value under A;

so, by the induction hypothesis, Q receives 1 under B. Then P, which is ~Q, receives 0 under B by the truth table for $\sim_C$.

Case 4: P is Q→R for some formulas Q and R, each with fewer than n occurrences of ~ and →. Then, by the truth class table for $\rightarrow_J$, Q received a designated value and R received an undesignated value under the assignment A. Thus, by the induction hypothesis, Q receives 1 under B and R receives 0 under B, so P that receives 0 under B by the truth table for $\rightarrow_C$.

Thus, by the strong form of the principle of mathematical induction, the stated result holds for a formula with any number of occurrences of ~ and →, and hence it holds for any formula. Note that while this induction proof has established only the "if" part of Lemma 6.11, the "only if" part follows trivially from the fact that any truth value of J is either a designated value or an undesignated value.

Lemma 6.12: A formula P is a tautology in the logic J if and only if it is a tautology in the two-valued logic C.

Proof: Let P be a formula that is not a tautology of J. Then, for some assignment A of truth values of J to the prime constituents of P, P receives an undesignated value. Then, if B is an assignment of truth values of C to the prime constituents of P, constructed as in Lemma 6.11, then P receives the value 0 under the assignment B, and P is not a tautology in C.

Now let P be a formula that is not a tautology in C. Then there is some assignment B_0 of truth values of C to the prime constituents of P such that P receives the value 0 under the assignment B_0. Let A be the assignment of truth values of J to the prime constituents of P such that A assigns 1_J to every prime constituent to which B_0 assigns 1_C, and also assigns 0_J to every prime constituent to which B_0 assigns 0_C.

If we now construct an assignment B of truth values of C to the prime constituents of P, as in Lemma 6.11, then this assignment B is the same as our assignment B_0. If P received a designated value under the assignment A, then, by Lemma 6.11, P would receive 1_C under the assignment B (which is B_0), contradicting the choice of B_0. Therefore, P receives an undesignated value under the assignment A, so P is not a tautology of J.

Metatheorem 6.13: A formula P is a tautology of J if and only if P is a theorem of the usual statement calculus; that is, the usual statement calculus is a suitable formal system for the logic J.

Proof: Lemma 6.12 and the Completeness Theorem.

Metatheorem 6.14: Let I be a many-valued logic between C and L_3'. Suppose that either of the following conditions holds:

(i) the set $\tau \cup \phi$ is closed under the operations ~ and →, and, for any truth values x_1, x_2 in υ and any truth value y in ϕ, the truth value $x_1 \rightarrow x_2$ is in υ, and the truth values $x_1 \rightarrow y$ and $\sim x_1$ are in υ; or

(ii) for any truth value x in υ and any truth value y in ϕ, the truth values $x \rightarrow y$ and $\sim x$ are in ϕ.

Then the usual statement calculus is a suitable formal system for the logic I.

Proof: Metatheorems 6.10 and 6.13.

CHAPTER VII
CONCLUDING REMARKS

The significance of Metatheorem 6.14 is, in part, that it is a generalization of an earlier result, and generalizations are always welcome in mathematics. If Metatheorem 6.14 had been available when the 1992 paper [24] was written, it would have saved the author a sizeable amount of work. In fact, it would have made the 1992 paper altogether unnecessary. The logic M studied in that paper is a special case of Metatheorem 6.14. Recall that the truth functions of M are as in figure 15. Since M has three truth values, these values are mapped to the truth values of L_3' by the identity mapping, so that $\tau = \{1\}$, $\upsilon = \{1/2\}$, $\phi = \{0\}$. Then the truth class tables of M are as in figure 16, as the reader can verify, and these tables are certainly a special case of condition (i) of Metatheorem 6.14.

More significant is our discovery that there are an infinite number of many-valued logics that have the usual statement calculus as a suitable formal system. In fact, we have essentially discovered a method for creating, for any n, an n-valued logic for which the usual statement calculus is suitable. To see this, let n be a positive integer greater than or equal to two. We define truth functions on the set of truth values $\{k/(n-1): 0 \leq k \leq n-1\}$ in the following manner. First, we separate the truth values into three sets τ, υ, and ϕ such that $x > y$ for all x in τ and all y in υ, and such that $y > z$ for all y in υ and all z in ϕ. Beyond this basic requirement, the separation of the truth values is a matter of personal choice. With the separation accomplished, we simply construct truth tables for $\sim$ and $\rightarrow$ in

$\rightarrow$	1	1/2	0
1	1	1/2	0
1/2	1	1/2	1/2
0	1	1	1

$\sim$	
1	0
1/2	1/2
0	1

Figure 15. Truth functions of the logic M.

$\rightarrow$	τ	υ	ϕ
τ	τ	υ	ϕ
υ	τ	υ	υ
ϕ	τ	τ	τ

$\sim$	
τ	ϕ
υ	υ
ϕ	τ

Figure 16. Truth class tables of the logic M.

such a way that Metatheorem 6.14 applies. In other words, we construct functions $\sim$ and $\rightarrow$ whose truth class tables are as in figure 10 or figure 11.

It seems that since there are such a variety of many-valued logics for which the usual statement calculus is suitable, the study of many-valued logics has a certain credibility. Critics cannot say that many-valued logics produce methods of reasoning that are alien to our natural way of thinking, because we have just seen that a large class of logics obey exactly the rules which govern our natural way of thinking. The same logical arguments that have been used for centuries with the <u>black-and-white</u> two-valued system are still valid for a shades-of-gray interpretation. All that change are the degrees of truth that can be assigned to the atomic statements, and these were never addressed by the system anyway.

However, while this partial squeezing theorem for these particular logics has led to these discoveries, questions remain as to even more general results, specifically the following:

(i.) Are the sufficient conditions given in Metatheorem 6.14 necessary conditions as well?

(ii.) Can betweenness be redefined to make possible a full squeezing theorem for logics between C and L_3'? If so, can that result be generalized still further?

It is not yet clear how question (i) will be answered, although, of the logics so far investigated that fail to satisfy the conditions for Metatheorem 6.14, the usual statement calculus is suitable for none of them.[43] However, as stated earlier, it is still an open question whether the usual statement calculus is suitable for L_3'. If so, then question (i) must be answered in the negative, since L_3' does not meet either condition of Metatheorem 6.14. Condition (i) is violated because $1/2 \rightarrow 1/2 = 1$, so that $\upsilon \rightarrow \upsilon$ is τ, not υ.

Condition (ii) is violated because $\sim(1/2) = 1/2$, so that $\sim\upsilon$ is υ, not ϕ, and because, similarly, $\upsilon \rightarrow \phi$ is not ϕ.

If the usual statement calculus is in fact suitable for L_3', we would have to add some conditions to Metatheorem 6.14, and then ask question (i) again. This seems a small price to pay for the nice symmetry we would have in such a case.

[43] See Appendix C.

Question (ii), on the other hand, can be considered as follows. Since the truth class tables, which governed the behavior of the logics we studied, came directly from the definitions for forgivingness and betweenness, and these in turn depended on the definition of truth-value mapping, it is reasonable to wonder whether an alteration in those definitions might produce a stronger result. If more restrictions are placed on the betweenness relation, then a smaller set of logics will be between C and L_3'. Can we, in fact, restrict the definition in such a way that this smaller set is made up entirely of logics for which the usual statement calculus is suitable?

Early attempts at new definitions are not promising. An examination of the betweenness and forgivingness definitions suggests that they are already as restrictive as we can expect them to be[44], so apparently the target of the alterations should be the definition of truth-value mapping.

Suppose, then, that we wish to define truth-value mapping so that all logics that are between C and L_3' under the new definition meet the conditions of Metatheorem 6.14, part (i), in other words, so that the truth class tables of such a logic have the form illustrated in figure 17. That is, certain table entries that originally could be narrowed down only to $\tau \quad \upsilon$ are now required to be τ specifically.

[44] In fact, the only reasonable change appears to be in the definition of betweenness: We could require that, in order for a logic I to be between C and L_3', I must be both more forgiving than C <u>and</u> less forgiving than L_3'. However, since more- and less-forgivingness require only <u>one</u> strict inequality within comparison of <u>one</u> pair of truth tables (and there seems to be no reasonable way to increase those demands), the effect of such a change on our truth class tables would be minimal.

$\rightarrow$	τ	υ	ϕ
τ	τ	υ	ϕ
υ	τ,υ	υ	υ
ϕ	τ	τ,υ	τ

$\sim$	
τ	ϕ
υ	υ,ϕ
ϕ	τ

Figure 17. Truth class tables fulfiliing condition 6.14.i.

The reader will remember the reason that, in the construction of the original truth class tables[45], these entries were left as "τ,υ." For example, if x and y are values in τ, we discovered that $x \rightarrow y$ had to be a value that mapped to two-valued truth. In other words, $x \rightarrow y$ was designated, so it could be from either τ or υ. We were unable to pin down its value any further.

To guarantee the construction of truth class tables like the ones in figure 17, then, we would have to require that only elements of τ can map to two-valued truth. Then if $x \rightarrow y$ maps to two-valued truth, $x \rightarrow y$ must be in τ, and the table entry is as desired.

However, a new problem presents itself. If we require that only elements of τ can map to two-valued truth, then we are defining one truth-value mapping (that from I to C) on the basis of another truth-value

[45] See Appendix B.

mapping (that from I to L_3', without which we would have no set τ to discuss). This is clearly an improper state of affairs; the only way around it is to restrict which values may map to truth under any given mapping. For instance, if we stipulate that the absolute truth of one logic is the only value of that logic that can map to the absolute truth of another logic, then it will work out, in the case of mappings from I to C and to L_3', that the only element of τ is 1_I, which is also the only truth value of I that maps to the absolute truth of C. Then if, while constructing a truth class table, we discover that one of the entries must map to 1_C, then we know this entry must be τ, the set whose only element is the element that maps to 1_C.

However, this creates another problem. If the absolute truth of I is the only truth value of I that maps to two-valued truth, then all other designated values of I, as well as all the undesignated values, will map to two-valued falsity. Suppose that we now construct a new truth class table with τ, υ, and ϕ retaining their original meanings (note, however, that in this case $\tau = \{1\}$, since only absolute truth can map to the 1 of L_3'). Suppose x is in υ and y is in ϕ. Let g be the truth-value mapping from I to C. Then $g(x) = g(y) = 0$, so $g(x) \rightarrow g(y) = 1$. Since I is as forgiving as C, $g(x \rightarrow y) \geq g(x) \rightarrow g(y)$, so $g(x \rightarrow y) = 1$. Then $x \rightarrow y = 1$. Let f be the truth-value mapping from I to L_3'. Then $f(x) = 1/2$ and $f(y) = 0$ by the definitions of υ and ϕ. Since I is no more forgiving than L_3', $f(x \rightarrow y) \leq f(x) \rightarrow f(y) = 1/2$. Thus, $x \rightarrow y$ does not map to 1 and therefore cannot be 1, so we have contradicted our earlier finding. The same happens for the $\sim\upsilon$ case.

The only solution to this problem is to make υ empty, so that these problematic cases ($\upsilon \rightarrow \phi$, $\sim\upsilon$) never arise. The resulting general definition of truth-value mapping becomes a rather uninteresting one: the truth of the logic J maps to the truth of the logic K, and all other values of J map to the falsity of K. If we retain the name "τ" for the set of truth values that map to the 1 of L_3', and we denote the rest by "ρ" (for "rest"), we can construct a new sort of truth class table for such a logic. As illustrated in figure 18, such truth class tables are remarkably similar to the original truth tables of C. Thus, it should be easy to prove that any logic satisfying these truth class tables has the usual statement calculus as a suitable formal system.

$\rightarrow$	τ	ρ
τ	τ	ρ
ρ	τ	τ

$\sim$	
τ	ρ
ρ	τ

Figure 18. Truth class tables for a new definition of truth-value mapping.

However, the above definition of truth-value mapping seems unnaturally restrictive, and it also violates our expectation that designated values map to designated values and undesignated values map to undesignated values. Moreover, the resulting concept of betweenness seems remarkably artificial.

Whether such a concept of betweenness is nonetheless useful--and whether it leads to a general squeezing theorem for these and other many-valued logics--is a question for the future to decide.

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APPENDIX A

A DEMONSTRATION THAT THE USUAL STATEMENT CALCULUS IS SUITABLE FOR THE THREE-VALUED LOGIC M

To show that the usual statement calculus is a suitable formal system for the logic M, we must demonstrate, first, that every formula that is a tautology under M is a theorem of the usual statement calculus, and second, that every theorem of the statement calculus is a tautology of M.

Since all connectors of the logic M can be written in terms of $\rightarrow$ and ~, we assume, in the following discussion, that $\rightarrow$ and ~ are the only connectors used. The proof is from Walk [24], with slight modifications.

Lemma A.1: Let P be a formula. If no prime constituent of P is assigned the truth value 1/2, then P does not receive the truth value 1/2.

Proof: By strong induction on the number of occurrences of $\rightarrow$ and ~ in P.

Basis: P has no occurrences of $\rightarrow$ or ~. Then P is an atomic formula or has the form $\forall v Q$. Then P is a prime formula and is its own prime constituent, so if P is not assigned 1/2, it will not receive 1/2.

Inductive step: Suppose P has n occurrences of $\rightarrow$ and ~, and assume that the lemma holds for formulas with fewer than n such occurrences. Suppose no prime constituent of P is assigned the truth value 1/2.

Case 1: P is ~Q for some formula Q that has n - 1 occurrences of $\rightarrow$ and ~. The prime constituents of Q are identically those of P, so none of them has been assigned 1/2. By the induction hypothesis, Q does not receive 1/2. Then Q receives 0 or 1, so, by the truth table for ~, P receives 1 or 0.

Case 2: P is $Q \rightarrow R$ for some formulas Q and R, each with fewer than n occurrences of $\rightarrow$ and ~. None of the prime constituents of Q or of R has been assigned 1/2, since they are all prime constituents of P. Then, by the induction hypothesis, Q receives 0 or 1, and R receives 0 or 1, so there are four possibilities in all. The truth table for → shows that P receives 0 or 1 for each of these possibilities.

Then, by the strong form of the principle of mathematical induction, the theorem holds for any formula P with any number of occurrences of → and ~, that is, for any formula P.

Lemma A.2: Let P be a formula. If an assignment of 0's and 1's is made to the prime constituents of P, then the truth value P receives will be the same whether the connectors → and ~ in P are regarded as connectors in the classical two-valued logic C or as connectors in M.

Proof: Let an assignment of 0's and 1's be made to the prime constituents of P. The proof of the lemma is by strong induction on the number of occurrences of → and ~ in P.

Basis: P has no occurrences of → or ~. Then P is its own prime constituent and receives whichever truth value, 0 or 1, it is assigned. The result is vacuously true in this case.

Inductive step: Suppose P has n occurrences of → and ~, and assume the lemma holds for formulas with fewer than n such occurrences.

Case 1: P is ~Q for some formula Q that has n - 1 occurrences of → and ~. By Lemma A.1, P receives 0 or 1 if the connectors → and ~ are regarded as truth functions of M. If P receives 0, then Q receives 1 by the truth table for ~ in M. Then, by the induction hypothesis, Q receives 1 when → and ~ are regarded as truth functions of C. Therefore, by the truth table for ~ in C, P receives 0. If, on the other hand, P receives 1, then Q receives 0 by the truth table for ~ in M, and by the induction hypothesis, Q receives 0 when $\rightarrow$ and ~ are regarded as truth functions of C. Thus, by the truth table for ~ in C, P receives 1.

Case 2: P is Q$\rightarrow$R for some formulas Q and R, each with fewer than n occurrences of $\rightarrow$ and ~. By Lemma A.1, P receives 0 or 1 if the connectors $\rightarrow$ and ~ are regarded as truth functions of M.

Subcase 2.1: P receives 0. Then, by the truth table for $\rightarrow$ in M, Q receives 1 and R receives 0. Then, by the induction hypothesis, when $\rightarrow$ and ~ are regarded as truth functions in C, Q receives 1 and R receives 0 again, so that P receives 0 by the truth table for $\rightarrow$ in C.

Subcase 2.2: P receives 1. Then, by the truth table for $\rightarrow$ in M, Q receives 0 or R receives 1. Suppose Q receives 0. Then, by the induction hypothesis, Q receives 0 when $\rightarrow$ and ~ are regarded as truth functions of C, so by the truth table for $\rightarrow$ in C, P receives 1. Now suppose R receives 1. Then, by the induction hypothesis, R receives 1 when $\rightarrow$ and ~ are regarded as truth functions of C, so by the truth table for $\rightarrow$ in C, P receives 1.

Then, by the strong form of the principle of mathematical induction, the theorem holds for any formula P with any number of occurrences of $\rightarrow$ and ~; that is, for any formula P.

Lemma A.3: If a formula P is a tautology in M, then it is a tautology in C.

Proof: Let P be any formula that is not a tautology in C. Then, for some assignments of 0's and 1's to the prime constituents of P, P receives the value 0. By Lemma A.2, P receives the value 0 under this assignment when the connectors in P are regarded as truth functions of M. Then P is not a tautology in M.

Metatheorem A.4: Every tautology of M is a theorem of the usual statement calculus.

Proof: By Lemma A.3, every tautology of M is a tautology of C. Margaris [12] demonstrates that every tautology of C is a theorem of the usual statement calculus. The metatheorem follows.

In the discussion that follows, the term tautology refers exclusively to tautologies of M.

Lemma A.5: Let P be a formula. Let an assignment of 0's, 1/2's, and 1's be made to the prime constituents of P.

(i) Suppose P receives the value 0. If the truth-value assignment is altered so that the prime constituents previously assigned 1/2 are assigned 1 instead, and all other assignments are retained, then P receives the value 0 under the new assignment.

(ii) Suppose P receives the value 1. If the truth-value assignment is altered so that the prime constituents previously assigned 1/2 are assigned 1 instead, and all other assignments are retained, then P receives the value 1 under the new assignment.

Proof: By induction on the number of occurrences of $\rightarrow$ and $\sim$ in P.

Basis: P has no occurrences of $\rightarrow$ or $\sim$. Then P is its own prime constituent and, if it receives 0 or 1, must have been assigned 0 or 1. The lemma is vacuously true in this case.

Inductive step: Suppose P has n occurrences of $\rightarrow$ and $\sim$, and assume that the lemma holds for all formulas with fewer than n such occurrences.

Suppose that, under the given assignment of 0's, 1/2's, and 1's, P receives the value 0.

Case 1: P is $\sim$Q for some formula Q with n - 1 occurrences of $\rightarrow$ and $\sim$. Then, by the truth table for $\sim$, Q receives the value 1 under the original assignment, so by the induction hypothesis. Q receives the value 1 under the new assignment. Then, by the truth table for ~, P receives the value 0 under the new assignment.

Case 2: P is Q→R for some formulas Q and R, each with fewer than n occurrences of → and ~. By the truth table for →, Q receives 1 and R receives 0 under the original assignment of truth values, so Q receives 1 and R receives 0 under the new assignment as well. Then P receives 0 by the truth table for →.

Now suppose that, under the given assignment of 0's, 1/2's, and 1's, P receives 1.

Case 3: P is ~Q for some formula Q. Then by the truth table for ~, Q receives 0 under the original assignment, so by the induction hypothesis, Q receives 0 under the new assignment as well. Then, by the truth table for ~, P receives 1 under the new assignment.

Case 4: P is Q→R for some formulas Q and R, each with fewer than n occurrences of → and ~. Then, by the truth table for →, Q receives 0 or R receives 1.

Subcase 4.1: Q receives 0 under the original assignment. Then, by the induction hypothesis, Q receives 0 under the new assignment as well. Then, by the truth table for →, P receives 1 under the new assignment.

Subcase 4.2: R receives 1 under the original assignment. Then, by the induction hypothesis, R receives 1 under the new assignment as well. Then, by the truth table for →, P receives 1 under the new assignment. Thus, by the strong form of the principle of mathematical induction, the lemma holds for any formula with any number of occurrences of → and ~, so it holds for any formula.

Lemma A.6: Let P and Q be any formulas. If P and P→Q are both tautologies, then Q is a tautology as well. In other words, modus ponens preserves tautology.

Proof: Suppose that P and P→Q are tautologies and that Q is not a tautology; that is, Q receives 0 under some assignment of truth values to its prime constituents. By Lemma A.5, there is an assignment of 0's and 1's to the prime constituents of Q such that Q receives the value 0 under that assignment also. Create a new assignment of truth values to the prime constituents of P and Q in the following way. To every prime formula that is a prime constituent of Q, assign the same truth value as in the zeroes-and-ones assignment just described. To every prime formula of P that is not a prime constituent of Q, assign the value 1. By Lemma A.1, since no prime constituent of P was assigned the value 1/2, P must receive 0 or 1, and since it is a tautology, P must receive 1. But then, since P receives 1 and Q receives 0, P→Q receives 0, contradicting the choice of P→Q as a tautology. Thus, Q cannot receive 0 under any assignment of truth values to its prime constituents, so Q is a tautology.

Lemma A.7: Any instance of Axiom Scheme A1, A2, or A3 is a tautology.

Proof: The relevant truth tables are displayed in figures 19 to 21.

Lemma A.8: Any step in a proof is a tautology.

Proof: Let P be a formula that is a step in some proof. By the definition of proof, P is an instance of an Axiom Scheme or is inferred from two earlier steps by modus ponens.

Case 1. P is an instance of an Axiom Scheme. Then P is a tautology by Lemma A.7.

P	Q	$Q \rightarrow P$	$P \rightarrow (Q \rightarrow P)$
0	0	1	1
0	1/2	1/2	1
0	1	0	1
1/2	0	1	1
1/2	1/2	1/2	1/2
1/2	1	1/2	1/2
1	0	1	1
1	1/2	1	1
1	1	1	1

Figure 19. The truth table for Axiom Scheme A1.

P	Q	$\sim Q$	$\sim P$	$\sim Q \rightarrow \sim P$	$P \rightarrow Q$	$(\sim Q \rightarrow \sim P) \rightarrow (P \rightarrow Q)$
0	0	1	1	1	1	1
0	1/2	1/2	1	1	1	1
0	1	0	1	1	1	1
1/2	0	1	1/2	1/2	1/2	1/2
1/2	1/2	1/2	1/2	1/2	1/2	1/2
1/2	0	1	1/2	1/2	1/2	1/2
1	0	1	0	0	0	1
1	1/2	1/2	0	1/2	1/2	1/2
1	1	0	0	1	1	1

Figure 20. The truth table for Axiom Scheme A3.

P	Q	R	P→(Q→R)	(P→Q)→(P→R)	(P→(Q→R)) → ((P→Q)→(P→R))
0	0	0	1	1	1
0	0	1/2	1	1	1
0	0	1	1	1	1
0	1/2	0	1	1	1
0	1/2	1/2	1	1	1
0	1/2	1	1	1	1
0	1	0	1	1	1
0	1	1/2	1	1	1
0	1	1	1	1	1
1/2	0	0	1	1/2	1/2
1/2	0	1/2	1	1/2	1/2
1/2	0	1	1	1	1
1/2	1/2	0	1/2	1/2	1/2
1/2	1/2	1/2	1/2	1/2	1/2
1/2	1/2	1	1	1	1
1/2	1	0	1/2	1/2	1/2
1/2	1	1/2	1/2	1/2	1/2
1/2	1	1	1	1	1
1	0	0	1	1	1
1	0	1/2	1	1	1
1	0	1	1	1	1
1	1/2	0	1/2	1/2	1/2
1	1/2	1/2	1/2	1/2	1/2
1	1/2	1	1	1	1
1	1	0	0	0	1
1	1	1/2	1/2	1/2	1/2
1	1	1	1	1	1

Figure 21. The truth table for Axiom Scheme A2.

Inductive step: P is step n, and steps 1 to n - 1 are tautologies. Then P either is an instance of an Axiom Scheme or is inferred from two earlier steps, Steps j and k ($1 \leq j < k$), by modus ponens. By the induction hypothesis, Step j and Step k are tautologies. By Lemma A.6, since Step j and Step k are tautologies, P is a tautology as well.

<u>Metatheorem A.9:</u> Every theorem of the usual statement calculus is a tautology in M.

<u>Proof:</u> Every theorem is, by definition, the last step of a proof. By Lemma A.8, then, every theorem is a tautology of M.

<u>Metatheorem A.10:</u> The usual statement calculus is a suitable formal system for M; that is, the formulas that are theorems of the usual statement calculus are precisely those that are tautologies of M.

<u>Proof:</u> Metatheorems A.4 and A.9.

APPENDIX B

THE CHARACTERISTIC TRUTH CLASS TABLES FOR LOGICS I BETWEEN C AND L_3'

In this appendix, we supply the full derivation of the truth class tables introduced in table 6, and we demonstrate that these tables are, as claimed, characteristic of the logics between C and L_3'.

The derivation of the truth class tables is as follows. Let I be a logic between C and L_3', and let $\rightarrow$ and $\sim$ be a binary and a unary truth function, respectively, on the truth values of I, such that all truth functions of I can be expressed in terms of $\rightarrow$ and $\sim$.

Recall from chapter 5 that, because I is between C and L_3', there exist a truth value mapping g from I to C such that I is as forgiving as C with respect to the mapping g and the sets $\{\rightarrow,\sim\}$ and $\{\rightarrow_C,\sim_C\}$, and a truth value mapping f from I to L_3' such that I is no more forgiving than L_3' with respect to the mapping f and the sets $\{\rightarrow,\sim\}$ and $\{\rightarrow_L,\sim_L\}$.

For the remainder of this appendix, we dispense with subscripts and use the symbol $\rightarrow$ for the specified binary truth functions and $\sim$ for the specified unary truth functions. Further, we use the symbols 0 and 1 for truth and falsity in each logic. Context will clarify whether the object is to be considered a function or a truth value of I, C, or L_3'.

Denote as τ the set of truth values of I whose image under f is 1. Denote as υ the set of truth values of I whose image under f is 1/2. Note that by the definition of truth-value mapping, τ and υ contain only designated values of I, so that for any truth value x in τ or in υ, $g(x) = 1$. Denote as ϕ the set of truth values of I whose image under f is 0. Note that ϕ contains only undesignated values of I, so that for any truth value x in ϕ, $g(x) = 0$.

We construct the truth class table for $\rightarrow$. Let x and y be any elements of τ. Then $f(x) = f(y) = 1$, so $f(x)\rightarrow f(y) = (1\rightarrow 1) = 1$. Since I is no more forgiving than L_3', $f(x\rightarrow y) \leq f(x)\rightarrow f(y) = 1$, so $f(x\rightarrow y) = 0$, 1/2, or 1; thus, $x\rightarrow y$ can be in any of the sets τ, υ, or ϕ. Further, $g(x) = g(y) = 1$, so that $g(x)\rightarrow g(y) = (1\rightarrow 1) = 1$. Now, I is as forgiving as C, so $g(x\rightarrow y) \geq g(x)\rightarrow g(y) = 1$. Hence $g(x\rightarrow y)$ must be 1, from which it follows that $x\rightarrow y$ must be in τ or in υ. We record this fact with the notation "τ,υ" at the appropriate place in the table.

Let x be any element of τ, and let y be any element of υ. Then $g(x) = g(y) = 1$, so that $g(x)\rightarrow g(y) = (1\rightarrow 1) = 1$. Since I is as forgiving as C, $g(x\rightarrow y) \geq g(x)\rightarrow g(y) = 1$, or $g(x\rightarrow y) = 1$. Then $x\rightarrow y$ must be in τ or in υ. Also, $f(x) = 1$ and $f(y) = 1/2$, and hence $f(x)\rightarrow f(y) = (1\rightarrow 1/2) = 1/2$. Since I is no more forgiving than L_3', $f(x\rightarrow y) \leq f(x)\rightarrow f(y) = 1/2$, so $f(x\rightarrow y)$ is 0 or 1/2. Then $x\rightarrow y$ is in ϕ or in υ. However, we just showed that $x\rightarrow y$ cannot be in ϕ. Thus, $x\rightarrow y$ is in υ.

Let x be any element of τ, and let y be any element of ϕ. Then $g(x) = 1$ and $g(y) = 0$, so that $g(x)\rightarrow g(y) = (1\rightarrow 0) = 0$. Since I is as forgiving as C, $g(x\rightarrow y) \geq g(x)\rightarrow g(y) = 0$, so $g(x\rightarrow y)$ is 0, 1/2, or 1; thus, $x\rightarrow y$ can be in any of the sets τ, υ, and ϕ. However, $f(x) = 1$ and $f(y) = 0$, so that $f(x)\rightarrow f(y) = (1\rightarrow 0) = 0$. Since I is no more forgiving than L_3', $f(x\rightarrow y) \leq 0$, so $f(x\rightarrow y) = 0$. Then $x\rightarrow y$ is in ϕ by definition.

Moving to the next row in the table for $\rightarrow$, let x be any element of υ, and let y be any element of τ. Then $f(x) = 1/2$ and $f(y) = 1$, so that $f(x)\rightarrow f(y) = (1/2\rightarrow 1) = 1$. Since I is no more forgiving than L_3', $f(x\rightarrow y) \leq f(x)\rightarrow f(y) = 1$, so $f(x\rightarrow y)$ is 0, 1/2, or 1, and $x\rightarrow y$ can be in any of the sets τ, υ, and ϕ. Also, $g(x) = g(y) = 1$, so that $g(x)\rightarrow g(y) = (1\rightarrow 1) = 1$. Since I is as forgiving as C, $g(x\rightarrow y) \geq g(x)\rightarrow g(y) = 1$, so $g(x\rightarrow y) = 1$. Then $x\rightarrow y$ is in τ or in υ.

Let x and y be elements of υ. Then $f(x) = f(y) = 1/2$. Hence $f(x)\rightarrow f(y) = (1/2\rightarrow 1/2) = 1$. Since I is no more forgiving than L_3', $f(x\rightarrow y) \leq f(x)\rightarrow f(y) = 1$, so $f(x\rightarrow y)$ is 0, 1/2, or 1, and $x\rightarrow y$ can be in any of the sets τ, υ, and ϕ. Also, $g(x) = g(y) = 1$, and thus $g(x)\rightarrow g(y) = (1\rightarrow 1) = 1$. Since I is as forgiving as C, $g(x\rightarrow y) \geq g(x)\rightarrow g(y) = 1$, so $g(x\rightarrow y) = 1$. Then $x\rightarrow y$ must be in τ or in υ.

Let x be any element of υ, and let y be any element of ϕ. Then $g(x) = 1$ and $g(y) = 0$, so that $g(x)\rightarrow g(y) = (1\rightarrow 0) = 0$. Since I is as forgiving as C, $g(x\rightarrow y) \geq g(x)\rightarrow g(y) = 0$. Consequently, $g(x\rightarrow y)$ is 0 or 1, from which it folllows that $x\rightarrow y$ can be in any of the sets τ, υ, and ϕ. Further, $f(x) = 1/2$ and $f(y) = 0$, so that $f(x)\rightarrow f(y) = (1/2\rightarrow 0) = 1/2$. Since I is no more forgiving than L_3', $f(x\rightarrow y) \leq f(x)\rightarrow f(y) = 1/2$, so that $f(x\rightarrow y)$ is 0 or 1/2. Thus, $x\rightarrow y$ is in υ or in ϕ.

Let x be any element of ϕ, and let y be any truth value of I. Then $f(x) = 0$, so that $f(x)\rightarrow f(y) = (0\rightarrow f(y)) = 1$ for any y. Since I is no more forgiving than L_3', $f(x\rightarrow y) \leq f(x)\rightarrow f(y) = 1$, so that $f(x\rightarrow y)$ is 0, 1/2, or 1. Thus, $x\rightarrow y$ can be in any of the sets τ, υ, and ϕ. However, $g(x) = 0$, so that $g(x)\rightarrow g(y) = (0\rightarrow g(y)) = 1$

for any y. Since I is as forgiving as C, $g(x \rightarrow y) \geq g(x) \rightarrow g(y) = 1$, and thus $g(x \rightarrow y) = 1$. It follows that $x \rightarrow y$ is in τ or in υ.

The truth class table for $\rightarrow$ is complete. Next, we construct the truth class table for ~. Let x be any element of τ. Then $g(x) = 1$, so that $\sim g(x) = \sim 1 = 0$. Since I is as forgiving as C, $g(\sim x) \geq \sim g(x) = 0$, so that $g(\sim x)$ is 0 or 1, and $\sim x$ can be in any of the sets τ, υ, and ϕ. Also, $f(x) = 1$, so that $\sim f(x) = \sim 1 = 0$. Since I is no more forgiving than L_3', $f(\sim x) \leq \sim f(x) = 0$, and hence $f(\sim x) = 0$. Then $\sim x$ is in ϕ.

Let x be any element of υ. Then $g(x) = 1$, and $\sim g(x) = \sim 1 = 0$. Since I is as forgiving as C, $g(\sim x) \geq \sim g(x) = 0$, so $g(\sim x)$ is 0 or 1, and $\sim x$ can be in any of the sets τ, υ, and ϕ. However, $f(x) = 1/2$, so that $\sim f(x) = \sim 1/2 = 1/2$. Since I is no more forgiving than L_3', $f(\sim x) \leq \sim f(x) = 1/2$, so that $f(\sim x) = 0$ or $1/2$. Then $\sim x$ is in υ or in ϕ.

Finally, let x be any element of ϕ. Then $f(x) = 0$, so that $\sim f(x) = \sim 0 = 1$. Since I is no more forgiving than L_3', $f(\sim x) \leq \sim f(x) = 1$, so $f(\sim x)$ is 0, 1/2, or 1, and $\sim x$ can be in any of the sets τ, υ, or ϕ. Further, $g(x) = 0$, so that $\sim g(x) = \sim 0 = 1$. Since I is as forgiving as C, $g(\sim x) \geq \sim g(x) = 1$, and consequently $g(\sim x) = 1$. Then $\sim x$ is in τ or in υ. We have completed the truth class tables shown in figure 22.

We now demonstrate that any logic represented by the truth class tables in figure 22 is between the logics C and L_3'.

Theorem: Suppose I is a logic all of whose truth functions can be expressed in terms of the binary truth function $\rightarrow$ and the unary truth function ~.

$\rightarrow$	τ	υ	ϕ
τ	τ,υ	υ	ϕ
υ	τ,υ	τ,υ	υ,ϕ
ϕ	τ,υ	τ,υ	τ,υ

$\sim$	
τ	ϕ
υ	υ,ϕ
ϕ	τ,υ

Figure 22. Truth class tables for logics I between C and L_3'.

Suppose further that the truth values of I can be separated into three sets--τ, υ, and ϕ--such that these sets satisfy the truth class tables for $\rightarrow$ and $\sim$ shown in figure 22. Then the logic I is between the logics C and L_3'.

Proof: We show first that I is more forgiving than C.

Let g be the truth value mapping from I to C such that

$$g(x) = \begin{cases} 1 \text{ if x is in } \tau \text{ or } \upsilon; \\ 0 \text{ if x is in } \phi. \end{cases}$$

The reader is reminded that the 1 and 0 above are the truth values of the two-valued logic C and that when, below, we refer to 1 and 0 as values of the functions $\sim$ and $\rightarrow$, we are considering $\sim$ and $\rightarrow$ as negation and implication for the logic C. We show that the function $\sim$ of I is as forgiving as the function $\sim$ of C. Let x be a truth value of I.

Case 1: x is in τ. Then $\sim x$ is in ϕ by the truth class table for $\sim$. Then $g(\sim x) = 0 = \sim 1 = \sim g(x)$ by the definition of g.

Case 2: x is in υ. Then $\sim x$ is in υ or $\sim x$ is in ϕ.

Subcase 2.1: $\sim x$ is in υ. Then $g(\sim x) = 1 > 0 = \sim 1 = \sim g(x)$ by the definition of g.

Subcase 2.2: $\sim x$ is in ϕ. Then $g(\sim x) = 0 = \sim 1 = \sim g(x)$ by definition of g.

Case 3: x is in ϕ. Then $\sim x$ is in τ or $\sim x$ is in υ. In either case, $g(\sim x) = 1$ by the definition of g, so $g(\sim x) = 1 = \sim 0 = \sim g(x)$.

We show that the function $\rightarrow$ of I is as forgiving as the function $\rightarrow$ of C. Let x and y be truth values of I.

Case 1: x is in τ.

Subcase 1.1: y is in τ. Then $x\rightarrow y$ is in τ or $x\rightarrow y$ is in υ, by the truth class table for $\rightarrow$. In either case, $g(x\rightarrow y) = 1 = (1\rightarrow 1) = g(x)\rightarrow g(y)$.

Subcase 1.2: y is in υ. Then $x\rightarrow y$ is in υ by the truth class table for $\rightarrow$, and $g(x\rightarrow y) = 1 = (1\rightarrow 1) = g(x)\rightarrow g(y)$ by the definition of g.

Subcase 1.3: y is in ϕ. Then $x\rightarrow y$ is in ϕ by the truth class table for $\rightarrow$, and $g(x\rightarrow y) = 0 = (1\rightarrow 0) = g(x)\rightarrow g(y)$ by the definition of g.

Case 2: x is in υ.

Subcase 2.1: y is in τ. Then $x\rightarrow y$ is in τ or $x\rightarrow y$ is in υ. In either case, $g(x\rightarrow y) = 1 = (1\rightarrow 1) = g(x)\rightarrow g(y)$.

Subcase 2.2: y is in υ. Then $x\rightarrow y$ is in τ or $x\rightarrow y$ is in υ. In either case, $g(x\rightarrow y) = 1 = (1\rightarrow 1) = g(x)\rightarrow g(y)$.

Subcase 2.3: y is in ϕ. Then $x\rightarrow y$ is in υ or $x\rightarrow y$ is in ϕ.

(a) If $x\rightarrow y$ is in υ, then $g(x\rightarrow y) = 1 > 0 = (1\rightarrow 0) = g(x)\rightarrow g(y)$.

(b) If $x \to y$ is in ϕ, then $g(x \to y) = 0 = (1 \to 0) = g(x) \to g(y)$.

Case 3: x is in ϕ.

Subcase 3.1: y is in τ. Then $x \to y$ is in τ or $x \to y$ is in υ. In either case, $g(x \to y) = 1 = (0 \to 1) = g(x) \to g(y)$.

Subcase 3.2: y is in υ. Then $x \to y$ is in τ or $x \to y$ is in υ. In either case, $g(x \to y) = 1 = (0 \to 1) = g(x) \to g(y)$.

Subcase 3.3: y is in ϕ. Then $x \to y$ is in τ or $x \to y$ is in υ. In either case, $g(x \to y) = 1 = (0 \to 0) = g(x) \to g(y)$.

For any x and y that are truth values of I, we have seen above that $g(\sim x) \geq \sim g(x)$ and $g(x \to y) \geq g(x) \to g(y)$. Thus, I is as forgiving as C. Now, by the strict inequalities in subcase 2.1 for ~ and subcase 2.3 for →, we see that I is more forgiving than C.

We now show that I is less forgiving than L_3'. Let f be the truth value mapping from I to L_3' such that

$$f(x) = \begin{cases} 1 \text{ if x is in } \tau; \\ 1/2 \text{ if x is in } \upsilon; \\ 0 \text{ if x is in } \phi. \end{cases}$$

The reader is reminded that, when 1, 1/2, and 0 are referred to as values of the functions ~ and →, we are considering ~ and → as truth functions in L_3'. We show that the function ~ of I is no more forgiving than the function ~ of L_3'. Let x be a truth value of I.

Case 1: x is in τ. Then ~x is in ϕ by the truth class table for →. Then $f(\sim x) = 0 = \sim 1 = \sim f(x)$.

Case 2: x is in υ. Then ~x is in υ or ~x is in ϕ.

Subcase 2.1: ~x is in υ. Then f(~x) = 1/2 = ~1/2 = ~f(x).

Subcase 2.2: ~x is in ϕ. Then f(~x) = 0 < 1/2 = ~1/2 = ~f(x).

Case 3: x is in ϕ. Then ~x is in τ or ~x is in υ.

Subcase 3.1: ~x is in τ. Then f(~x) = 1 = ~0 = ~f(x).

Subcase 3.2: ~x is in υ. Then f(~x) = 1/2 < 1 = ~0 = ~f(x).

We show that the function $\rightarrow$ I is no more forgiving than the function $\rightarrow$ of L_3'. Let x and y be truth values of I.

Case 1: x is in τ.

Subcase 1.1: y is in τ. Then x$\rightarrow$y is in τ or x$\rightarrow$y is in υ.

(a) If x$\rightarrow$y is in τ, then f(x$\rightarrow$y) = 1 = (1$\rightarrow$ 1) = f(x)$\rightarrow$f(y).

(b) If x$\rightarrow$y is in υ, then f(x$\rightarrow$y) = 1/2 < 1 = (1$\rightarrow$ 1) = f(x)$\rightarrow$f(y).

Subcase 1.2: y is in υ. Then x$\rightarrow$y is in υ. Then f(x$\rightarrow$y) = 1/2 = (1$\rightarrow$ 1/2) = f(x)$\rightarrow$f(y).

Subcase 1.3: y is in ϕ. Then x$\rightarrow$y is in ϕ. Then f(x$\rightarrow$y) = 0 = (1$\rightarrow$0) = f(x)$\rightarrow$f(y).

Case 2: x is in υ.

Subcase 2.1: y is in τ. Then x$\rightarrow$y is in τ or x$\rightarrow$y is in υ.

(a) If x$\rightarrow$y is in τ, then f(x$\rightarrow$y) = 1 = (1/2$\rightarrow$ 1) = f(x)$\rightarrow$f(y).

(b) If x$\rightarrow$y is in υ, then f(x$\rightarrow$y) = 1/2 < 1 = (1/2$\rightarrow$ 1) = f(x)$\rightarrow$f(y).

Subcase 2.2: y is in υ. Then x$\rightarrow$y is in τ or x$\rightarrow$y is in υ.

(a) If $x \rightarrow y$ is in τ, then $f(x \rightarrow y) = 1 = (1/2 \rightarrow 1/2) = f(x) \rightarrow f(y)$.

(b) If $x \rightarrow y$ is in υ, then $f(x \rightarrow y) = 1/2 < 1 = (1/2 \rightarrow 1/2) = f(x) \rightarrow f(y)$.

<u>Subcase 2.3:</u> y is in ϕ. Then $x \rightarrow y$ is in υ or $x \rightarrow y$ is in ϕ.

(a) If $x \rightarrow y$ is in υ, then $f(x \rightarrow y) = 1/2 = (1/2 \rightarrow 0) = f(x) \rightarrow f(y)$.

(b) If $x \rightarrow y$ is in ϕ, then $f(x \rightarrow y) = 0 < 1/2 = (1/2 \rightarrow 0) = f(x) \rightarrow f(y)$.

<u>Case 3:</u> x is in ϕ.

<u>Subcase 3.1:</u> y is in τ. Then $x \rightarrow y$ is in τ or $x \rightarrow y$ is in υ.

(a) If $x \rightarrow y$ is in τ, then $f(x \rightarrow y) = 1 = (0 \rightarrow 1) = f(x) \rightarrow f(y)$.

(b) If $x \rightarrow y$ is in υ, then $f(x \rightarrow y) = 1/2 < 1 = (0 \rightarrow 1) = f(x) \rightarrow f(y)$.

<u>Subcase 3.2:</u> y is in υ. Then $x \rightarrow y$ is in τ or $x \rightarrow y$ is in υ.

(a) If $x \rightarrow y$ is in τ, then $f(x \rightarrow y) = 1 = (0 \rightarrow 1/2) = f(x) \rightarrow f(y)$.

(b) If $x \rightarrow y$ is in υ, then $f(x \rightarrow y) = 1/2 < 1 = (0 \rightarrow 1/2) = f(x) \rightarrow f(y)$.

<u>Subcase 3.3:</u> y is in ϕ. Then $x \rightarrow y$ is in τ or $x \rightarrow y$ is in υ.

(a) If $x \rightarrow y$ is in τ, then $f(x \rightarrow y) = 1 = (0 \rightarrow 0) = f(x) \rightarrow f(y)$.

(b) If $x \rightarrow y$ is in υ, then $f(x \rightarrow y) = 1/2 < 1 = (0 \rightarrow 0) = f(x) \rightarrow f(y)$.

Thus, for any truth values x and y of I, we see that $f(\sim x) \leq \sim f(x)$ and $f(x \rightarrow y) \leq f(x) \rightarrow f(y)$. Thus, I is <u>no more forgiving than</u> L_3'. By the <u>strict</u> inequalities in subcases i.2.2 and i.3.2 for the function ~ and subcases 1.1.b, ii.2.1.b, ii.2.2.b, ii.2.3.b, ii.3.1.b, ii.3.2.b, and ii.3.3.b for the function $\rightarrow$, we see that I is in fact <u>less forgiving than</u> L_3'.

Then, by the definition of betweenness, I is between C and L'.

APPENDIX C

LOGICS BETWEEN C AND L_3' FOR WHICH THE USUAL STATEMENT CALCULUS IS NOT SUITABLE

As mentioned in chapter VII, a relevant question to consider now is whether the conditions given in Metatheorem 6.14 are in fact necessary. That is, given a logic I between C and L_3' for which the usual statement calculus is suitable, must condition (i) or condition (ii) hold?

This question is open. As noted in chapter 7, if L_3' is found to have the usual statement calculus as a suitable formal system, then the question will be answered in the negative, since neither condition (i) nor condition (ii) holds for L_3'.

Of the other logics so far studied for which conditions (i) and (ii) fail to hold, none has the statement calculus as a suitable formal system. An example is the logic given as counterexample in the first section of chapter 4. By comparing the truth class tables for this logic to the truth class table for a logic I satisfying condition (ii) of Metatheorem 6.14, we can see how near a miss there is, that is, a difference only in the "$\upsilon \rightarrow \phi$" entry. But--as the saying goes--"near misses count only with horseshoes and grenades."[1]

If we concentrate on that "$\upsilon \rightarrow \phi$" entry, we see that it is "υ,ϕ" for this logic, so that a truth value in such a case would be either in υ or in ϕ. In this logic, as noted, Axiom Scheme 3 is not a tautology. The usual statement calculus is not a suitable formal system because this logic does not have enough tautologies. If we narrow our focus to include, instead, only those logics whose "$\upsilon \rightarrow \phi$" entry is "$\upsilon$," we tend to find that if such a

[1] Millar [14].

logic does not satisfy condition (i) of Metatheorem 6.14, then the logic has too many tautologies to have the usual statement calculus as a formal system.

Whether this is so in general, that is, for all logics between C and L_3' that fail Metatheorem 6.14 and have "υ" as a "$\upsilon \rightarrow \phi$" entry, is not yet known. The following are four particular instances of such a situation. In each case, the presence of a tautology that is not a theorem of the usual statement calculus indicates that that calculus is not suitable for such a logic. Figures 23 to 26 display the truth class tables for the four types of logics under consideration.

Observation: Let I be a logic between C and L_3' such that, for any truth value x in the set υ and any truth value y in the set ϕ, the truth value $x \rightarrow y$ is in the set υ. Then the following results hold, as can be verified by referring to figures 23 to 26.

(i.) If $\sim x$ is in υ for all x in ϕ, then $\sim Q \rightarrow Q$ is a tautology.

(ii.) If $x \rightarrow y$ is in υ for all x and y in ϕ, then $(Q \rightarrow Q) \rightarrow Q$ is a tautology.

(iii.) If $x \rightarrow y$ is in υ for all x in ϕ and all y in τ, and if $x \rightarrow y$ is in τ for all x and y in ϕ, then $(Q \rightarrow (Q \rightarrow Q)) \rightarrow Q$ is a tautology.

(iv.) If $x \rightarrow y$ is in υ for all x and y in τ, and if $x \rightarrow y$ is in τ for all x and y in ϕ, then $((Q \rightarrow Q) \rightarrow (Q \rightarrow Q)) \rightarrow Q$ is a tautology.

→	τ	υ	φ
τ	τ,υ	υ	φ
υ	τ,υ	τ,υ	υ
φ	τ,υ	τ,υ	τ,υ

~	
τ	φ
υ	υ,φ
φ	υ

Figure 23. Truth class tables for situation (i).

→	τ	υ	φ
τ	τ,υ	υ	φ
υ	τ,υ	τ,υ	υ
φ	τ,υ	τ,υ	υ

~	
τ	φ
υ	υ,φ
φ	τ,υ

Figure 24. Truth class tables for situation (ii).

→	τ	υ	φ
τ	τ,υ	υ	φ
υ	τ,υ	τ,υ	υ
φ	υ	τ,υ	τ

~	
τ	φ
υ	υ,φ
φ	τ,υ

Figure 25. Truth class tables for situation (iii).

$\rightarrow$	τ	υ	ϕ
τ	υ	υ	ϕ
υ	τ,υ	τ,υ	υ
ϕ	τ,υ	τ,υ	τ

$\sim$	
τ	ϕ
υ	υ,ϕ
ϕ	τ,υ

Figure 26. Truth class tables for situation (iv).