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# Proportional Reasoning: Understanding Student Thinking with Direct and Inverse Proportions

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# PROPORTIONAL REASONING:

## *UNDERSTANDING STUDENT THINKING WITH DIRECT AND INVERSE PROPORTIONS*

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## I. Title

Proportional Reasoning:

*Understanding Student Thinking with Direct and Inverse Proportions*

## II. Purpose

The topic of proportions in mathematics is often misunderstood by students, yet proportional reasoning is the foundation for higher-level math. The purpose of this study is to understand how students are currently solving direct and inverse proportion problems and how the degree of difficulty increases when solving inverse proportion problems. There is little research done about student thinking with inverse proportions. Primary data has been collected from students and will be used to analyze students' thinking about inverse proportion as well as a comparative look between direct and inverse proportional reasoning. By understanding students' thinking about proportions, educators can better prepare lessons that will strengthen the students' conceptual understanding rather than just procedural knowledge of direct and inverse proportions.

## III. Literature Review

Proportional reasoning is a critical foundation of mathematics for students as they progress through the school grade levels and into the real world. A proportion is a relationship between two ratios that can be set equivalent to each other (Christou & Philippou, 2002). There are two main types of proportional relationships, direct and inverse. A direct proportion is defined as when one quantity increases, the other quantity increases by a steady rate of change. Lobato and Ellis defined direct proportion saying, "When two quantities are related proportionally, the ratio of one quantity to the other is invariant as the numerical values of both quantities change by the same factor" (2010, p. 11). Inverse proportions define the opposite. As one quantity increases, the other decreases at a steady rate of change. Another way to define inverse relationships is "changing one quantity by a factor of k means that the other quantity needs to be changed by a factor of 1/k," (Markworth, 2012, p. 541). Direct and inverse proportions are types of proportional relationships.

**Student Mathematical Development**

Students begin learning about proportionality in kindergarten by looking at the relationship between numbers and number pairs. As they progress, the depth to which students use qualitative relationships increases. Third through fifth graders begin learning multiplicative reasoning and patterns. The concept and teaching of proportional reasoning is beneficial because students are learning about other mathematical ideas like multiplication, constant rate of change, linear functioning, and more. According to the Iowa Department of Education, the actual term "proportions" makes a greater appearance between sixth through eighth grade (Pforts, 2010). Students start with a basic understanding of the qualitative relationship of direct and inverse proportions. Then, they begin to add numbers into those qualitative relationships, turning them into quantitative problems. Most students begin their proportion journey by trying repeated addition strategies and slowly start to make multiplicative connections with math problems that are doubling or halving the quantities given. After that, students are expected to construct a quantitative understanding of the two qualitative relationships. They must see the rate of change between or within-measure spaces and be able to apply it to the missing quantity (Ferrandez-Reinisch, 1985). Connecting mathematical ideas across grade levels is critical. Proportional reasoning allows students to construct high-level explanations of similar traits among ideas to increasingly sophisticated problems. The ability to recognize and deeply understand proportional relationships is needed for future success in algebra and beyond (Seeley & Schielack, 2007). Students' conceptual understanding of proportional relationships will be a guide as they continue to build upon this mathematical knowledge throughout schooling and potentially in their future careers.

**Elements that Influence Student Thinking**

One of the elements that influence students is the number relationships in a proportional situation. Fernández and his team found that when the ratios have a multiplicative relationship that is a non-integer, they were less successful. The data they collected had 46.4% of students correctly answer problems with integer relationships and only 34.3% of students answered correctly with non-integer problems (Fernández, Ceneida, et al., 2010). With Dr. Riehl and Dr. Steinthorsdottir's research (2019), the same goes for "enlarging" over "shrinking" problems. When the ratio can be scaled down, that is classified as a shrink problem, and when the target is larger than the given situation or "scaled up", that is classified as an enlarge problem.

**Valid Direct Proportion Strategies**

When looking at proportional reasoning strategies, research has grouped them into four distinct, correct categories. To explain the strategies, I will use the following missing value proportion problem: Kelly is making cranberry juice for a party. Her original recipe is 3 cups of cranberry juice for every 2 cups of water. If Kelly has enough supplies for 15 cups of cranberry juice, how many cups of water should she add if she wants to keep the same taste?

The first strategy is the "build-up" method. Students take the given quantities and develop a soft relationship between them. For example, as cranberry juice increases by 3 cups, the water will increase by 2 cups. They see this relationship but use repeated addition measures to get to the missing value. This could be represented in a picture, written numbers, or a ratio table. Figure 1 provides an example of how students could set up the problem to find how much water is needed for 15 cups of cranberry juice.

**Figure 1**

*Ratio Table Example*

| | | | | | |
|---|---|---|---|---|---|
| Cranberry (cups) | 3 | 6 | 9 | 12 | 15 |
| Water (cups) | 2 | 4 | 6 | 8 | 10 |

In the Figure above, the student found the solution by adding another round of cranberry juice. Three cups plus another 3 cups of cranberry juice get them to 6 cups of cranberry juice. Then, they must do the same for water. Two cups of water added gets them a total of 4 cups of water. This process is repeated until the student stops at 15 cups of cranberry juice and reveals the answer of 10 cups of water as the missing value. There are many ways to display the "build-up" strategy from repeated addition to multiplicative building. Students are building their understanding using a rudimentary method. Students do not fully understand the relationship within (scalar method) or between (unit rate) measure spaces but the build-up strategy can become a transition step into more sophisticated forms of reasoning (Ercole, Frantz, & Ashline, 2011). Students used different strategies as they progressed through different grades. Student thinking tends to get more robust, losing the initial build-up strategy, their strategies become more sophisticated. Repeated addition strategies shift to proportional strategies as students transition from primary to secondary school (Fernández, Llinares, et al., 2010). In some cases, the build-up method decreases as students encounter more systematic teaching. In this instance, students' problem-solving skills using the "build-up strategy" tend to be rewired to follow a given set of instructions. For inverse proportions, students must recognize that the relationship between quantities is inverse so that as they build “down” on one of the quantities, the other is building “up.” As one quantity increases, the other shrinks.

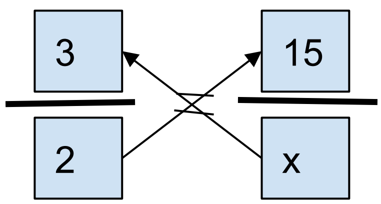
The second strategy is finding the scale factor. Students who use this method find the multiplicative relationship *within* two quantities of the same variable. This is often called the “within-measure space multiplicative relationship” or scalar method. With integer problems, students tended to use the within-measure space multiplicative relationship strategy to solve more often (Riehl & Steinthorsdottir, 2019). In the example, students will recognize that the two quantities covary. The students recognize that 5 groups of 3 cups of cranberry juice are equivalent to fifteen cups of cranberry juice, therefore, they get a scale factor of 5. Students would then multiply the 2 cups of water with a scale factor of 5 to get the answer of 10 cups of water. They are multiplying between the same variable which is the key to a within-measure space relationship.

The third strategy involves finding the unit rate. Students who use this method look at the relationship *between* two quantities of different variables. This multiplicative relationship is often called “between-measure space” or the constant of proportionality. The quantities that make up a ratio covary, but the relationship between them remains invariant (Silvestre & Ponte, 2011; Langrall & Swafford, 2000). It breaks the problem down into finding how much of a single quantity would affect the second quantity. For example, students could take the 3 cups of cranberry juice and divide it by the 2 cups of water to get a unit rate of 1.5. For every 1 cup of water, there are 1.5 cups of cranberry juice. Then, they would divide the 15 cups of cranberry juice by 1.5 to get 10 cups of water. The important part of the “between-measure space” relationship is that students are solving for the unit rate between the two different variables.

The fourth strategy is almost entirely procedural and often fails to identify the proportional relationship between the quantities. This is the cross-multiplication method. This method is not recommended, but if taught, should be introduced once students have a solid conceptual understanding of proportions (Chapin & Anderson, 2003). Figure 2 provides an example of how students could set up the problem using a cross-multiplication method.

**Figure 2**

*Cross-Multiplication Example*




Then, they would multiply across the quantities to get 3x=30 and solve algebraically to get x=10 cups of water. It is an efficient algorithm, especially when dealing with noninteger numbers, but can loop students into the procedure of using the numbers without understanding what they mean (Ercole, Frantz, & Ashline, 2011). Sense-making is important before moving on to procedures like this when needed in proportion problems. Cross-multiplying is not good for students' sense-making (Lobato, et al., 2010). Most importantly, cross multiplication cannot be applied to inverse proportions which furthers the divide in students' conceptual understanding of proportional relationships.

**Invalid Direct Proportion Strategies**

Students can correctly solve proportional reasoning problems after lots of direct instruction, but they have a much stronger ability to problem solve and use proportional reasoning skills when discovering concepts through problem-based teaching. Exploration and investigation lead to correct reasoning and justification. It takes educators carefully planning a sequence of problems for students to explore that will give students the best learning experience. However, no matter the preparation and teaching provided, many students still face difficulties, resulting in erroneous methods and misconceptions (Miller & Fey, 2000). It is important for educators to be aware of these common errors so that instruction can be geared toward combating them.

With direct proportions, the common errors include non-proportional reasoning, miscalculations, or additive errors. (Langrall & Swafford, 2000). When students use additive comparison, they are finding the difference between the two quantities in the given ratio rather than comparing them multiplicatively. Additive reasoning is in direct contrast to proportional reasoning yet it is the most frequent error. This type of reasoning is very common among students and often begins their rudimentary problem-solving of proportions (Bright, Joyner, & Wallis, 2003). With inverse proportions, similar issues can occur, but it is also common to misinterpret the problem by solving it proportionally. In the research done by De Bock et al. (2015), when errors occurred with inverse proportion problems, it was most often due to proportional errors.

Some errors cannot be seen by looking at a single problem. There can be multiple problems of varying difficulty even though the broad category of a "direct" or "inverse" problem exists for all of them. To see the true thinking of the student, teachers need to allow students to apply their knowledge in different situations, with different degrees of difficulty (Bright, Joyner, & Wallis, 2003). This will reveal commonalities of errors that educators can focus on to help the entire class succeed with direct and inverse proportions.

## IV. Research Question to Be Answered

The goal of this research is to better understand students' proportional reasoning skills and strategies for inverse proportions and how they compare to students' reasoning for direct proportions.

1. How are students reasoning about inverse proportion?
2. What common errors are occurring in inverse proportion problems?
3. Does student thinking differ when dealing with direct versus inverse proportions?

## V. Methodology

This study analyzed data about proportional reasoning collected by Dr. Steinthorsdottir and Dr. Riehl in their study "Route to Reason" (Riehl & Steinthorsdottir, 2014; 2019). They gave direct and inverse proportion problems to 409 5th to 8th graders from a Midwest school district in 2017. For this study, I analyzed the written work of 243 students in 7th and 8th grade on problems addressing inverse proportion. At this stage of schooling, students have worked with proportional reasoning for some time now, more so in 8th grade than in 7th grade. These two grades were selected because they have had more time to develop proportional reasoning skills (over 5th and 6th graders) which will make observation and analysis of their strategies more effective.

The research done in the areas of non-integer vs integer relationships in proportions informed my decision on students' questions to analyze. According to the research, enlarging proportion problems with an integer relationship tend to have the most correct answers from students. With this information, I chose an enlarging, non-integer inverse proportion problem. This would give me the clearest view of a strategy. Sometimes with enlarging, integer relationships, students can find a way to reach the correct answer without understanding the strategy behind their work. If a student understands the multiplicative nature of proportion and can apply a multiplicative strategy, it should not matter if it is an integer or non-integer number in relation to the problem. In addition, by using an “enlarging” problem, fewer students will face confusion after multiple experiences with solving direct proportion problems about enlarging. Shrinking problems are a different category of work that I will not be looking into at this time. I chose to select a problem that was enlarging and a noninteger relationship. I wanted a problem that was more familiar to students while still being challenging. The inverse proportion problem I chose to investigate was question #8 on the instrument collected by Dr. Steinthorsdottir and Dr. Riehl. It reads, “The Green Soles Nature Club is going to clean up the stream bank. They figure it will take 12 members all working together 6 hours. They decided to recruit more helpers. If 18 members work together, how long will it take? Explain your thinking.” It is a more familiar enlarging problem with 12 members increasing to 18 members, while the noninteger, within-measure space relationships of 1.5 adds a challenge.

The literature review explained multiple strategies students use when solving proportion problems. To best analyze how students are reasoning about inverse proportions, the data from the inverse problem above was separated into different categories based on the strategy used by the student. First, students’ thinking was separated into correct and erroneous answers. The strategies that resulted in a correct answer were then analyzed, using the framework of strategies research has identified for direct proportion problems. The strategies are (1) Build-Up strategies, ranging from rudimentary unit by unit “build up” to more complex multiplicative build up, (2) Within Measure Space Comparison strategies, (3) Between Measure Space Comparison strategies, (4) Cross-Multiplication strategies, and (5) strategy unclear. This helped me to see what ways students were reasoning about inverse proportions. The erroneous category was broken down into different answers with an invalid reasoning behind each one. I grouped the data based on an answer. I did expect errors based on additive thinking and errors where students interpret the problem as direct proportion. The category for random answers was based on whether there was no explanation given for an invalid answer or if there was only one answer that was not one of the most common errors. I further analyzed the mathematical reasons for the errors as I let it inform my results on what common errors there are in inverse proportion problems. The categories revealed where student misunderstandings are most likely taking place.

Finally, research already completed by Dr. Steinthorsdottir and Dr. Riehl in their study "Route to Reason" (Riehl & Steinthorsdottir, 2014; 2019) was used in comparison with the results on student understanding of inverse proportion problems. They looked at the strategies used to solve direct proportion problems. I compared an enlarging, noninteger direct proportion problem to the enlarging, noninteger inverse proportion problem. The data for the direct proportion problem had been analyzed according to strategies. I used the analysis conducted by Dr. Steinthorsdottir and Dr. Riehl.

Afterward, I synthesized my results to answer my research questions and shared what students were currently thinking about proportions. My analysis and categorization revealed the most common thinking patterns and the most common errors. I took this information to make generalizations of students' conceptual understanding of inverse proportions.

## VI. Results

To begin, the percentage of correct answers was very low for both 7th and 8th grade data. There were 92 total 7th graders and 16% of the 7th graders answered this inverse proportion problem correctly. Similarly, there were 151 total 8th graders and 23% of the 8th graders answered this inverse proportion problem correctly. There was an increase in correct answers from 7th to 8th grade, which can partially be explained by the additional year of mathematics learning for 8th graders. The data revealed a range of strategies used by the students to solve the inverse proportion problem. As explained previously, there was certain student thinking that made clear categories of strategies with direct proportions. In inverse proportional reasoning, however, not all of these same categories could be described in the same way. The analysis of strategies revealed how students were reasoning about inverse proportions. The strategies that emerged from the analysis were (1) Build-Up method, (2) Within Measure Space Relationship method, (3) Cross Multiplication Hybrid Relationship method, (3) Rate per One method, and (4) No clear explanation (see Table 1). The following sections will explain the different strategies used by students.

**Table 1**
*Valid Strategies for Problem 8*

| | Build-Up | Within-Measure Space | Cross Multiplication Hybrid | Rate per One | Unclear | Total |
|---|---|---|---|---|---|---|
| Number of 7th Graders | 4 (26.7%) | 4 (26.7%) | 3 (20%) | 0 (0%) | 4 (26.7%) | 15 (100%) |
| Number of 8th Graders | 8 (23.5%) | 6 (17.6%) | 4 (11.8%) | 3 (8.8%) | 13 (38.2%) | 34 (100%) |

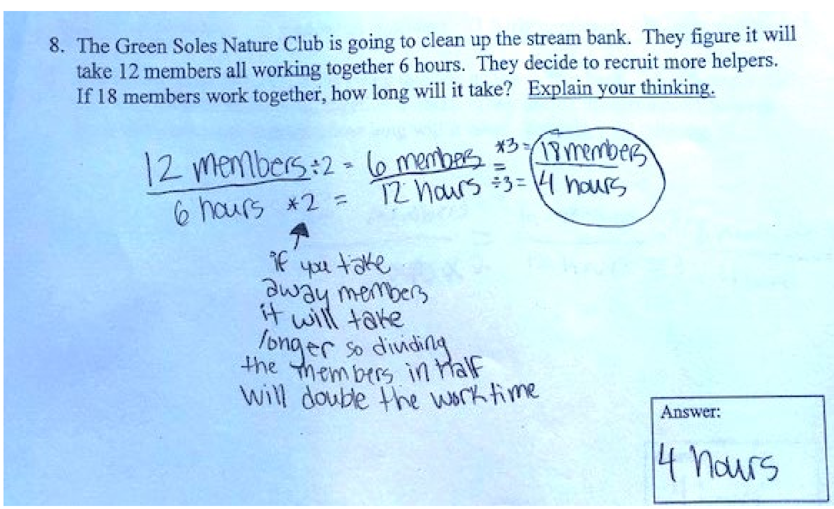
## Valid Inverse Proportion Strategies

### *Build-Up Strategies*

Taking a look at the strategies used, one can see a similar percentage of 7th and 8th grade students using the "build-up" strategy to solve the problem. For 7th graders, 26.7% of the students used this strategy while 23.5% of 8th graders did (see Table 1). Figure 3 is an example of a "build-up" strategy used by an 8th grader. In the strategy, the student sets up the information given to them in the problem as a proportion: for 12 members working together, it takes 6 hours. They divide the members by two and multiply the hours by two to get 6 members working for 12 hours. The student understands that this problem is an inverse proportion because they write, "If you take away members it will take longer so dividing the members in half will double the work time". The student understands that as one quantity increases, the other quantity decreases at a constant rate of change. Finally, the student multiplies 6 members by 3 to reach 18 members, and by understanding the inverse relationship the 12 hours needs to be divided by 3 to find the answer.  This is a "build-up" strategy because the student works with the "within-measure space" relationship but by breaking it down into smaller steps, "builds down" first and then "builds up". They built the 3/2 relationship to reach their desired answer of 18 members. This, then, reveals the answer of 4 hours.

**Figure 3**

*8th Grader's "Build-Up" Strategy*

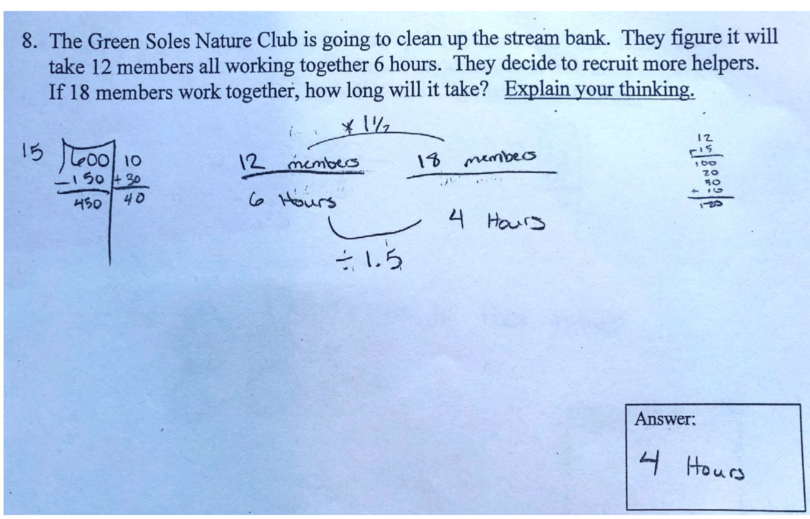



***Within-Measure Space Strategies***

The strategy of the within-measure space relationship is seen when students find a scale factor of 1.5 or 3/2. One efficient way to demonstrate this is in Figure 4. The student sets up a within-measure space relationship between the 12 and 18 members and finds that 18 is 3/2 of 12. Using this scale factor inversely, the student sees that 4 is ⅔ of 6. The student represents this, however, by multiplying 12 by 1.5 and dividing the 6 hours by 1.5 to get the total of 4 hours. With the data broken into strategies, the use of within-measure space in 7th grade is higher at 26.7% while the 8th graders use it 17.6% of the time (see Table 1). One factor that might be affecting the lower use of build-up strategies is the high percentage of unclear explanations in the 8th grade data. More students might have been using this strategy mentally without writing down their thinking. These strategies exhibit students' understanding of within-measure space relationships. Their use of the within-measure space relationship demonstrates that they are understanding the properties of an inverse relationship.

**Figure 4**

*8th Grader's "Within-Measure Space" Strategy*

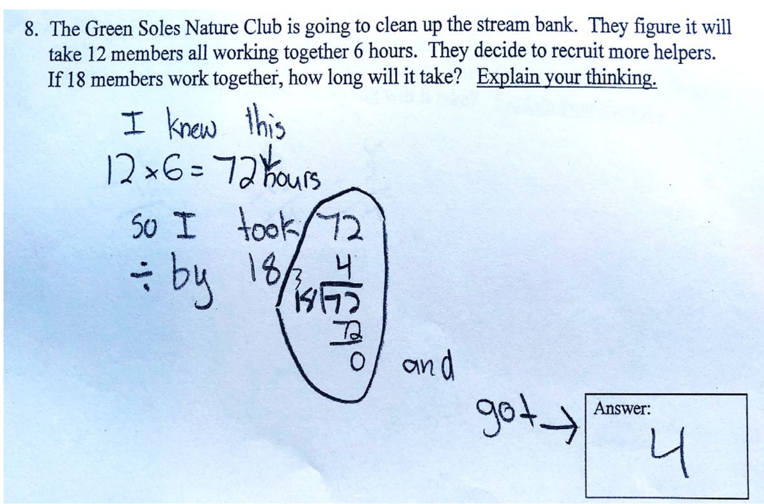



***Between Measure Space - Unit Per One Strategies***

A new strategy that I did not expect was found in three 8th graders, with only 8.8% of correct answers (see Table 1). They found the hourly rate per member. The student in Figure 5 knew that 12 members times 6 hours gets 72 total hours. To understand what that means, the student understands that each of the 12 members would work 6 hours which equals a total of 72 hours needed to complete the work. This gives us the rate per person. Therefore, when there are 18 members, the 72 hours of total work is divided among them to get 4 hours of work per member. This new strategy was thought-provoking as I analyzed it. Three students solved the problem using this strategy, showing that there is student thinking happening. The strategy works well for inverse proportion problems because it can be used for any problem with two quantities. For these students, it shows a deep understanding of the story problem and the mathematical principles behind the numbers.

**Figure 5:**

*8th Grader's "Rate per Person" Strategy*




***Between Measure Space - Cross Multiplication Hybrid Strategies***

As I was analyzing the between-measure space relationship, I noticed that it would not work the same way as a direct proportion would. In a direct proportion, the constant of proportionality or "k" is $\frac{a}{b}$ , however, an inverse proportion has the "k" as $a \times b$. Finding the relationship ⅓ between the given 6 hours and the target of 18 members, which are two opposite quantities, does not have a unit that makes sense within the context of the problem. Mathematically it can be explained with what I have labeled as a hybrid cross multiplication and between-measure space strategy (see Figure 6).

**Figure 6**

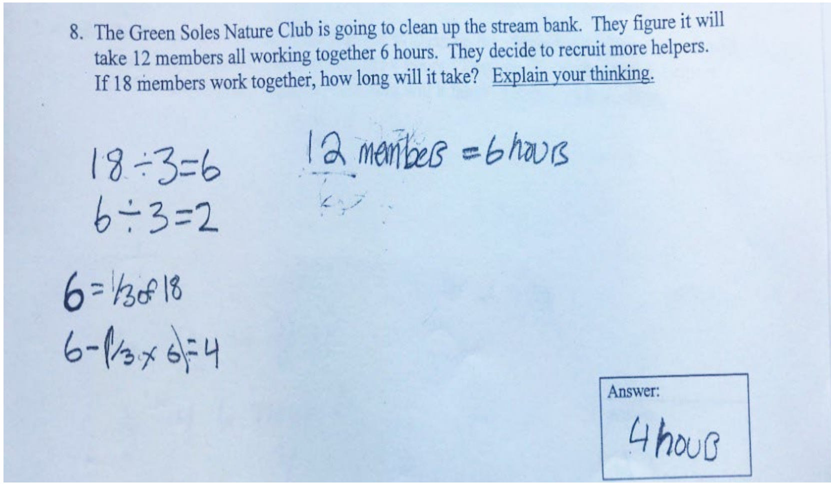
*Explanation of the Mathematics Behind the Hybrid Cross Multiplication Strategy*

$$\frac{12 \times 6}{18} = 12 \times \frac{6}{18} = 12 \times \frac{1}{3} = 2 \times \left(6 \times \frac{1}{3}\right) = 2 \times 2 = 4$$

Looking between Figure 6 and the student's work in Figure 7, the student finds ⅓ of 6 hours which is the answer of 2. Next, they subtract the 2 hours from 6 to get an answer of 4 hours. This aligns with the mathematical explanation in Figure 6 because 12 members times ⅓ gets the answer of 4 hours. 2 times 6 times ⅓ is the same equation. The only difference with Figure 7 is that the student subtracts their answer of 2 by 6 to get the answer of 4 hours while the mathematical equation multiplies that 2 by 2 hours.  With this new hybrid strategy, one can see a large number of students using it; 20% of 7th graders and 11.8% of 8th graders used this strategy (see Table 1).

**Figure 7**

*8th Graders Cross Multiplication and Between-Measure Space Hybrid*




**Invalid Inverse Strategies**

While there was analysis of the correct answers, there is also value in analyzing the strategies of incorrect answers. After separating the answers into their categories, I followed students' thinking through the work they wrote down with their answers. There were 92 total 7th graders and 77 invalid answers given. 83.7% of sampled 7th graders answered this inverse proportion problem using an invalid strategy (see Table 2). There were 151 total 8th graders and 117 invalid answers given. 77.5% of sampled 8th graders answered this inverse proportion problem using an invalid strategy (see Table 2). Of the common answers, I found a pattern of students still using direct proportional reasoning with the inverse proportion problem. There were also additive reasoning errors and invalid build-up strategies.

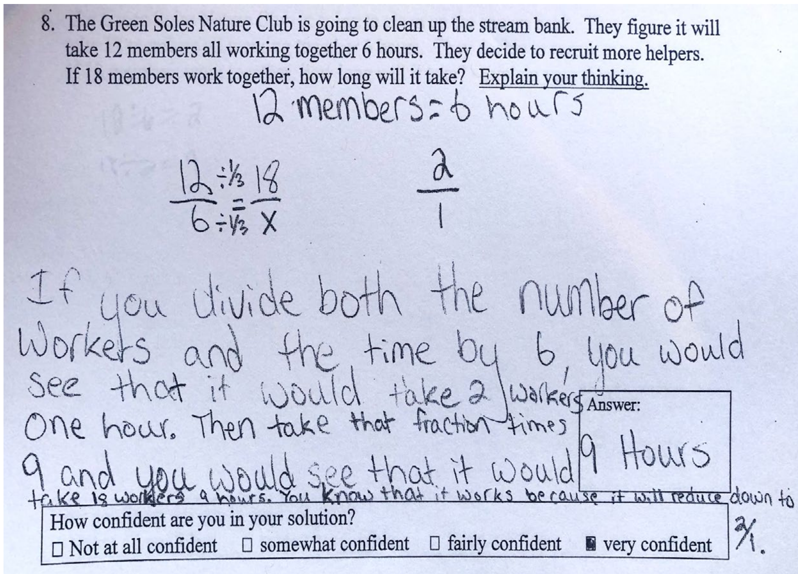
**Table 2**
*Invalid Answers for Problem 8*

| | 2 hours | 3 hours | 4.5 hours | 9 hours | 12 hours | Random | Total |
|---|---|---|---|---|---|---|---|
| Number of 7th graders | 5 (6.5%) | 35 (45.5%) | 5 (6.5%) | 14 (18.2%) | 5 (6.5%) | 13 (16.9%) | 77 (100%) |
| Number of 8th graders | 6 (5.1%) | 44 (37.6%) | 4 (3.4%) | 39 (33.3%) | 8 (6.8%) | 16 (13.7%) | 117 (100%) |

The first category I looked at was the answer of 9 hours. 18.2 % of 7th graders and 33.3% of 8th graders answered with "9 hours" which is a significant amount (see Table 2). Many students found the scale factor of 3/2 or built up to 9 hours using the thinking of direct proportionality. A third of 8th graders were solving this problem as a direct proportion which is a concern. Figure 8 reveals one student's thought process. They found the between-measure space relationship with a unit rate of 2. They divided 18 members by 2 to get their answer of "9 hours." This reveals that the student is thinking about a direct relationship, however, if you think about the problem logically, you can see that more members would mean that the job could be done in less time. Part of the problem is that students are missing the important part of thinking about the math behind the given problem. I believe that if the problem was given with no numbers first, and they were asked to think about how more members would impact the number of hours, many students would conclude that the answer would have to be less than the current number of hours. They would be thinking about the mathematical principles behind the story problem, rather than relying on the pattern of problems that they had solved prior. Then, when students are handed the original problem, the answer of 9 would cause students to pause and reevaluate their thought process. Students finding an answer of 9 hours did not connect that this problem was inverse proportions and not a direct proportion.

**Figure 8**

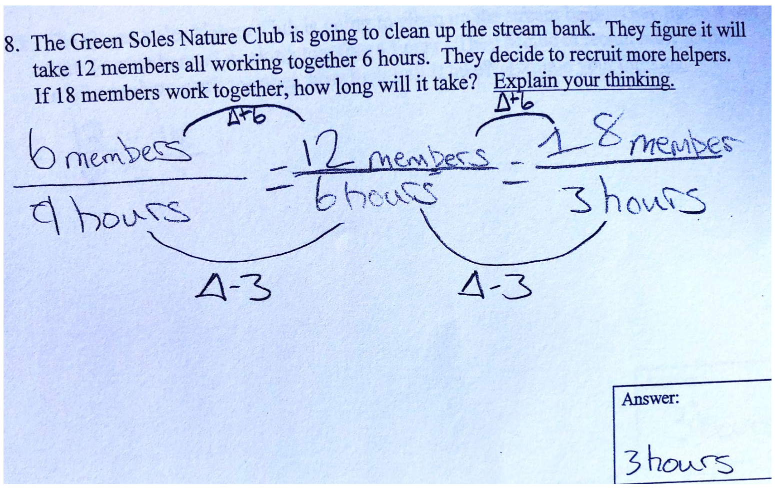
*8th Grader's "9 hour" Strategy*




The next category of answers was 3 hours. Many 7th and 8th grade students also fell into additive reasoning errors. 45.5% of 7th graders and 37.6% of 8th graders had an answer of 3 hours (see Table 2). This was the largest category of error in both grade levels. As I analyzed this common error, I saw a lot of different strategies being used, most of them were additive. It was interesting that there were so many different strategies yet they all got 3 hours. One strategy was a mix of a build-up and multiplicative strategy. The student in Figure 9 found the unit rate of 2 by dividing 12 by 6. Then the student divided 12 by 2 and 6 by 2 to get the answers of 6 and 3. They switched over to the build-up strategy and added 6 to 12 members to get 18 members. Next, they subtracted 3 hours to 6 hours to get their final answer of 3 hours. The student could see the inverse relationship a little because they understood that as one quantity increases, the other decreases which is why they added 6 to the top quantity but subtracted 3 from the bottom quantity. Another student reasoned in their response like this, "If you add half as much people to the first amount, it will take half of the time." They are explaining that if you add 6 more members (which is half of 12) to 12 members to get 18 members, you need to divide the hours by half, which is 3 hours. A commonality among students that I observed was that they were looking at the 6 extra members that were added from 12 to get 18 members. This fueled a lot of the different strategies that I saw.

**Figure 9**

*8th Grader's "3 hour" Strategy*




There were a few answers that were not common, being under 7% of students but had repetition for a few students (see Table 2). The "12 hours" answer tended to be just an additive and multiplicative error as well. Many students saw an increase of 6 members between 12 and 18 members and added 6 to the number of hours as well. The answers of "2 hours" and "4.5 hours" had no consistent strategy but some were solved with an additive error as well. One interesting strategy of a student was that they built up to 24 members by doubling the 12 members. 18 is the median between 12 and 24 members. Then, they halved the 6 hours to 3 hours and found the middle of 6 and 3. This got them the answer of 4.5 hours. This student understood the concept of the problem being an inverse but did not show an understanding of the multiplicative relationship involved in the proportion. These 3 answers were found to be repeated by a few students but there was no consistency in strategies.

**Direct vs. Inverse Comparison**

Using analysis already completed by Dr. Steinthorsdottir and Dr. Riehl in their study "Route to Reason," I compared the breakdown of strategies used with an enlarging, noninteger, direct proportion problem. Question #3 says “Grandma has some special plant food that comes in powder form. She uses 3 scoops to feed 9 plants. There are 11 scoops of plant food left. Using the same strength, how many plants can she feed? Explain your thinking.” 50% of 7th and 8th grade students got this direct proportion problem correct while about 20.2% of these same students got the inverse proportion problem correct. Looking at the strategies displayed in Table 3, 70.8% of students used the build-up method with the direct proportion problem. There is a dramatic decrease in the use of the build-up method with the inverse problem of 25.4% in Table 4. This reveals that the build-up method is more difficult for students to use effectively in inverse proportion problems because it requires the student to build up one quantity and decrease the other. Using additive building will not result in the correct answer, but it has to be a multiplicative building. This seems to have been more difficult for students and it goes along with Dr. Steinthorsdottir and Dr. Riehl’s research. They found that more students used additive build-up strategies (31%) than multiplicative build-up (19%).  It makes sense that fewer students used the multiplicative build-up strategy with the inverse proportion problem because the additive one gave the wrong answer.

**Table 3**
*Valid Direct Proportion Strategies for Problem 3 for 7th and 8th Graders*

| | Build-Up Method | Within-Measure Space Relationship | Between-Measure Space Relationship | Cross-Multiplication | Random but correct | Total |
|---|---|---|---|---|---|---|
| Number of correct | 92 | 8 | 27 | 2 | 1 | 130 |

| answers | | | | | | |
|---|---|---|---|---|---|---|
| Percentage of correct strategies | 70.8% | 6.2% | 20.8% | 1.5% | 0.8% | 100% |

**Table 4**
*Valid Inverse Proportion Strategies in Problem 8 for 7th and 8th Graders*

| | Build-Up Method | Within-Measure Space Relationship | Cross Multiplication Hybrid Relationship | Rate per one relationship | Random but correct | Total |
|---|---|---|---|---|---|---|
| Number of correct answers | 12 | 10 | 7 | 3 | 17 | 49 |
| Percentage of correct strategies | 24.5% | 20.4% | 14.3% | 6.1% | 34.7% | 100% |

The direct proportion problem had 6.2% of students solve using the within-measure space relationship and 20.8% of students solve using the between-measure space relationship (see Table 3). Students chose to find the whole number unit rate of 3 by dividing 9 plants by 3 scoops of feed. The integer was more obvious than finding the within-measure space relationship of 11/3. However, the opposite was true for the inverse proportion problem. 20.4% of students used the within-measure space strategy and 14.3% used the between-measure space strategy (see Table 4). As was found earlier, although it was more obvious to see the 12 members/6 hours relationship, the inverse proportion doesn't work directly between the quantities. Instead, it is more of a cross-multiplication relationship. Therefore, more students chose to look at the within-measure space relationship even though it was not an integer scale factor.

Different strategies were more common with direct vs inverse proportion problems which was an interesting finding. The build-up strategy was the most common for the direct problem by a lot while the build-up strategy and within-measure space were used in similar frequency with inverse problems. There was also 2.5 times the amount of correct answers for the direct proportion problem than the inverse proportion problem. There is less of a conceptual understanding of inverse proportion relationships than direct proportion relationships.

## VII. Application

After analyzing the data on inverse proportion relationships and comparing it to the direct proportion problems, it became evident that there is less conceptual understanding of inverse relationships than direct proportion relationships. The implication of these findings is a need for teachers to include more inverse proportion problems in their teaching. I believe that if it was introduced more to students with an intentionality of teaching the difference between the two, students would more quickly recognize the differences when they encounter different types of proportion problems.

One idea I can envision is to have students work with story problems that do not yet include numbers. For example, I would give students the same problem as problem 8 but take out the 18 members and instead describe that more workers were coming in. Teachers could ask students questions that would cause them to think deeply about the mathematics going on before giving them a chance to solve. Such questions could include,

- *"What is going on in this story problem?"*
- *"What do we know?"*
- *"What do we want to know?"*

- *"How might more members change the amount of hours needed to complete the work? Why?"*
- *"If we were told new members came in and it took 10 hours to finish the work, do you think that there were more members or fewer members? Why?"*

Allowing time for students to think critically about the problem before solving it would help them to recognize that it is an inverse relationship, causing them to pause on the potential desire to jump into solving every problem like a direct proportion.

Another idea for the betterment of both direct and inverse problem-solving is working with multiplicative relationships. Helping students to see the reasoning behind scale factor and unit rate will bring a concrete understanding of both proportional relationships. Asking students questions like, *"What would it look like to have just 1 of a quantity? How would that affect the other quantity?"* This would guide students into multiplicative thinking. For the teachers of the 409 midwestern students that were tested, they can use that data to shape their future instruction. Finding where students currently are with proportional reasoning can give an educator the information needed to push students to the next level of mathematical reasoning. I believe that with the information gathered from this data, many teachers can broaden their teaching to include inverse proportional reasoning using effective proportion-solving strategies.

## VIII. Conclusion

After all of the research and analysis of student thinking with direct and inverse proportions, I have found the most common strategies used to solve inverse proportion problems. The "build-up" method and "within-measure space" relationship paralleled the direct proportion strategies that were previously found in other research. The "rate per one" method and the "cross multiplication and between-measure space hybrid" were unique in the data collected from an inverse proportion problem. The common errors found in inverse proportion problems were additive reasoning errors and solving the problem using a direct proportional relationship. Finally, I compared my inverse proportion strategies with a direct proportion problem to see the similarities and differences between the two. The "build-up" method was the most commonly used among the students for both problems. In addition, inverse proportions are more difficult for students than direct proportions. With about a 30% difference in correct answers between the direct proportion problem and the inverse proportion problem, the data proves that inverse proportion problems tend to be more difficult for students to solve. These results show the importance for educators to be including more inverse proportion problems in their teaching so that students can become more familiar with them. Teachers should also provide experiences with inverse proportion problems that develop their mathematical thinking, focusing on the conceptual understanding of proportions. This will help students in the future as they progress in their educational goals and careers. Deeply understanding mathematical concepts over procedural work is what creates bright minds and effective problem-solvers in the future. Throughout my research, I was better able to understand students' proportional reasoning strategies for inverse and direct proportions.

## VII. References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