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FILMS, FRACTIONS, RATIOS AND REELS

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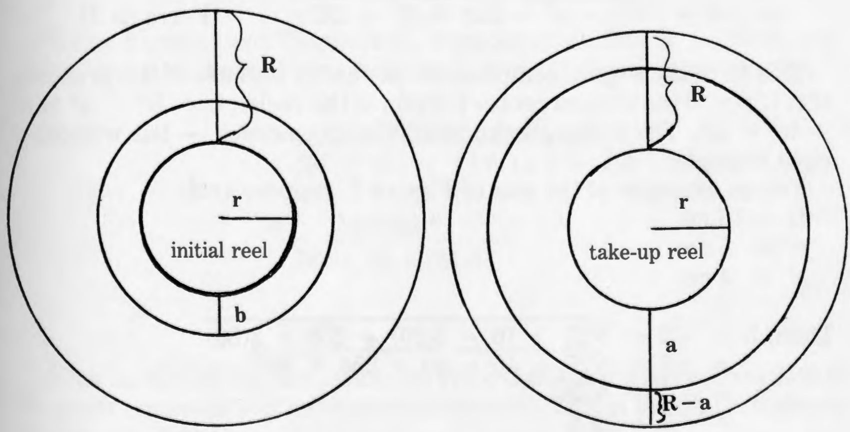
Many teachers show films in their classes to supplement scientific concepts. In addition to learning from the *content* of the film, significant mathematical ideas may be found by observing the motions of the projector itself.

In a typical reel-to-reel projector the film is stored on an initial reel. It is then threaded through the machine. When the film is being shown, it is usually driven by a sprocket within the projector so that a fixed number of frames per second pass through the machine. The film is attached to a take-up reel where it is wound in reverse order.

The take-up reel usually has a type of spring attachment so that it turns at whatever rate is necessary to receive the film leaving the projector; however, the take-up reel does not turn at a fixed number of revolutions per unit of time.

As a consequence of the organization of a movie projector, the initial reel turns very slowly at the beginning of the film while the take-up reel turns very rapidly. When the film is nearly complete the reverse is true.

We shall develop formulas to relate the fraction of the film shown, the cross-sectional width of the film on each of the two reels, and the angular velocity of the two reels.



r = radius of the unused center portions of the reels.
R = cross-sectional width of the complete film when it is on a single reel.
a = cross-sectional width of film remaining on the initial reel at a given time.
b = cross-sectional width of film that has been transferred to the take-up reel at that same time.

Figure 1

Figure I displays the initial reel and the take-up reel during the showing of a film. To simplify Figure I we have not shown the projector itself nor the strand of film that passes from the initial to the take-up reel.

Activity I: Relate a to b.

$$\underbrace{[\pi (a+r)^2 - \pi r^2]}_{\text{cross-sectional area on initial reel}} + \underbrace{[\pi (b+r)^2 - \pi r^2]}_{\text{cross-sectional area on take-up reel}} = \underbrace{[\pi (R+r)^2 - \pi r^2]}_{\text{total film area}}$$

$$(a+r)^2 - r^2 + (b+r)^2 - r^2 = (R+r)^2 - r^2$$

$$a^2 + 2ar + r^2 - r^2 + b^2 + 2br + r^2 - r^2 = R^2 + 2Rr + r^2 - r^2$$

$$b^2 + 2rb + a(a+2r) - R(R+2r) = 0$$

Solve for b.

$$b = \frac{-2r + \sqrt{4r^2 - 4\,[a(a+2r) - R(R+2r)]}}{2}$$

$$= -r + \sqrt{r^2 - [a(a+2r) - R(R+2r)]}$$

$$= -r + \sqrt{r^2 - a^2 - 2ar + R^2 + 2Rr} \qquad \text{(Formula I)}$$

This formula is quite complicated, primarily because of the presence of r. If $r = 0$ (no unused center portion of the reels), $b = R^2 - a^2$ or $a^2 + b^2 = R^2$. The Pythagorean relationship emerges — but without a right triangle!

For an example of the use of Figure I, suppose that:

$R = 16$ cm
$r = 5$ cm
$a = 4$ cm

Then,

$$\begin{aligned} b &= -5 + \sqrt{25 - 16 - 2(20) + 256 + 2(80)} \\ &= -5 + \sqrt{25 - 16 - 40 + 256 + 160} \\ &= -5 + \sqrt{385} \\ &= -5 + 19.6 \\ &= 14.6 \text{ cm} \end{aligned}$$

Note that the initial reel has only $4/16$ or 25% of its cross-sectional width remaining. On the other hand the take-up reel has $14.6/16$ or 91% of its cross-sectional width used. Why do these percents not sum to 100%?

Activity II: Relate a, b, and x.

Let x = fraction of film which has been shown at a point in time.

$$\frac{x}{1-x} = \frac{\text{cross-sectional area of film on take-up reel}}{\text{cross-sectional area of film on initial reel}}$$

$$\frac{x}{1-x} = \frac{\pi(b+r)^2 - \pi r^2}{\pi(a+r)^2 - \pi r^2}$$

$$\frac{x}{1-x} = \frac{\pi(b^2 + 2br + r^2) - \pi r^2}{\pi(a^2 + 2ar + r^2) - \pi r^2}$$

$$\frac{x}{1-x} = \frac{\pi(b^2 + 2br)}{\pi(a^2 + 2ar)}$$

$$\frac{x}{1-x} = \frac{b(b + 2r)}{a(a + 2r)}$$

$$x \cdot a \cdot (a + 2r) = (1 - x) \cdot b \cdot (b + 2r)$$

$$x\,[a(a + 2r) + b(b + 2r)] = b(b + 2r)$$

$$x = \frac{b(b + 2r)}{a(a + 2r) + b(b + 2r)} \qquad \text{(Formula II)}$$

Using Figure I and Formula II, suppose again that R = 16cm, r = 5 cm, and a = 4 cm. From Activity I, b = 14.6 cm, consequently,

$$x = \frac{14.6(14.6 + 10)}{4(4 + 10) + 14.6(14.6 + 10)}$$

$$= \frac{14.6(24.6)}{56 + 14.6\,(24.6)}$$

$$= .87$$

That is, 87% of the film is shown. Note that the initial reel has 25% of its cross-sectional width remaining while only 13% of the film remains to be shown. Why?

Activity III: Relate the angular velocities of the two reels.

Suppose at a given time the outside film circumference of the initial reel is three times that of the take-up reel. If the initial reel then makes one revolution, the take-up reel must make three revolutions to receive the film which has been processed. In general,

$$\frac{\text{Circumference of film on initial reel}}{\text{Circumference of film on take-up reel}} = \frac{\text{angular velocity of take-up reel}}{\text{angular velocity of initial reel}}$$

Since the circumference of the film on the initial reel is $2\pi(r + a)$ and the circumference of the film on the take-up reel is $2\pi(r + b)$, it may be concluded that:

$$\frac{\text{angular velocity of take-up reel}}{\text{angular velocity of initial reel}} = \frac{V_t}{V_i} = \frac{2\pi(r + a)}{2\pi(r + b)}$$

$$= \frac{r + a}{r + b} \quad \text{(Formula III)}$$

For instance, assume again that $R = 16$ cm, $r = 5$ cm, and $a = 4$ cm. From Formula I, $b = 14.6$ cm.

Hence $$\frac{V_t}{V_i} = \frac{5 + 4}{5 + 14.6}$$

$$\frac{V_t}{V_i} = \frac{9}{19.6}$$

$$\frac{V_t}{V_i} = .46$$

$$V_t = .46V_i$$

In other words, the take-up reel is revolving at a rate of 46% of that of the initial reel. Since 91% of the film has been shown (using Formula II), one might suspect that the take-up reel would be moving very slowly in relation to the initial reel. Why is the ratio as large as 46%?

Challenge to the reader: Find further relationships among the variables described in this article.