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David Duncan
University of Northern Iowa

Bonnie Litwiller
University of Northern Iowa

FILMS, FRACTIONS, RATIOS AND REELS

David Duncan and Bonnie Litwiller
Department of Mathematics and Computer Science
University of Northern Iowa
Cedar Falls, Iowa 50614

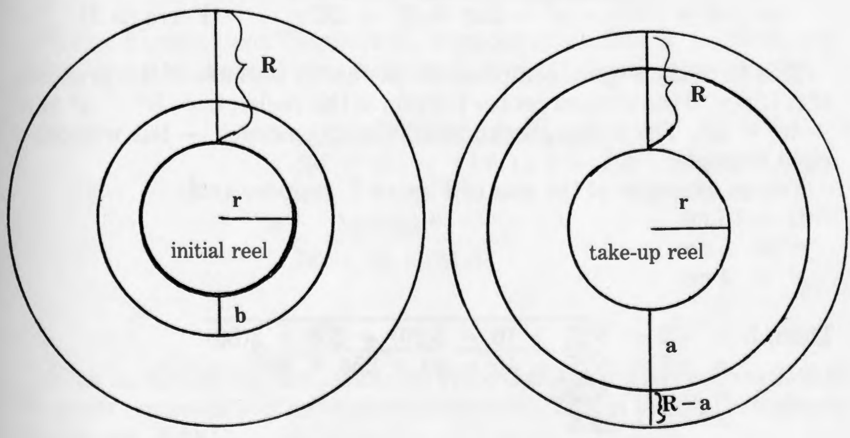
Many teachers show films in their classes to supplement scientific concepts. In addition to learning from the *content* of the film, significant mathematical ideas may be found by observing the motions of the projector itself.

In a typical reel-to-reel projector the film is stored on an initial reel. It is then threaded through the machine. When the film is being shown, it is usually driven by a sprocket within the projector so that a fixed number of frames per second pass through the machine. The film is attached to a take-up reel where it is wound in reverse order.

The take-up reel usually has a type of spring attachment so that it turns at whatever rate is necessary to receive the film leaving the projector; however, the take-up reel does not turn at a fixed number of revolutions per unit of time.

As a consequence of the organization of a movie projector, the initial reel turns very slowly at the beginning of the film while the take-up reel turns very rapidly. When the film is nearly complete the reverse is true.

We shall develop formulas to relate the fraction of the film shown, the cross-sectional width of the film on each of the two reels, and the angular velocity of the two reels.



r = radius of the unused center portions of the reels.
R = cross-sectional width of the complete film when it is on a single reel.
a = cross-sectional width of film remaining on the initial reel at a given time.
b = cross-sectional width of film that has been transferred to the take-up reel at that same time.

Figure 1

Figure I displays the initial reel and the take-up reel during the showing of a film. To simplify Figure I we have not shown the projector itself nor the strand of film that passes from the initial to the take-up reel.

Activity I: Relate a to b.

$$[\pi (a+r)^2 - \pi r^2] + [\pi (b+r)^2 - \pi r^2] = [\pi (R+r)^2 - \pi r^2]$$

cross-sectional area on initial reel	cross-sectional area on take-up reel	total film area

$$(a+r)^2 - r^2 + (b+r)^2 - r^2 = (R+r)^2 - r^2$$
$$a^2 + 2ar + r^2 - r^2 + b^2 + 2br + r^2 - r^2 = R^2 + 2Rr + r^2 - r^2$$
$$b^2 + 2rb + a(a+2r) - R(R+2r) = 0$$

Solve for b.

$$b = \frac{-2r + \sqrt{4r^2 - 4\,[a(a+2r) - R(R+2r)]}}{2}$$
$$= -r + \sqrt{r^2 - [a(a+2r) - R(R+2r)]}$$
$$= -r + \sqrt{r^2 - a^2 - 2ar + R^2 + 2Rr} \qquad \text{(Formula I)}$$

This formula is quite complicated, primarily because of the presence of r. If $r = 0$ (no unused center portion of the reels), $b = R^2 - a^2$ or $a^2 + b^2 = R^2$. The Pythagorean relationship emerges — but without a right triangle!

For an example of the use of Figure I, suppose that:
$R = 16$ cm
$r = 5$ cm
$a = 4$ cm

$$\text{Then, } b = -5 + \sqrt{25 - 16 - 2(20) + 256 + 2(80)}$$
$$= -5 + \sqrt{25 - 16 - 40 + 256 + 160}$$
$$= -5 + \sqrt{385}$$
$$= -5 + 19.6$$
$$= 14.6 \text{ cm}$$

Note that the initial reel has only $4/16$ or 25% of its cross-sectional width remaining. On the other hand the take-up reel has $14.6/16$ or 91% of its cross-sectional width used. Why do these percents not sum to 100%?

Activity II: Relate a, b, and x.

Let x = fraction of film which has been shown at a point in time.

$$\frac{x}{1-x} = \frac{\text{cross-sectional area of film on take-up reel}}{\text{cross-sectional area of film on initial reel}}$$

$$\frac{x}{1-x} = \frac{\pi(b+r)^2 - \pi r^2}{\pi(a+r)^2 - \pi r^2}$$

$$\frac{x}{1-x} = \frac{\pi(b^2 + 2br + r^2) - \pi r^2}{\pi(a^2 + 2ar + r^2) - \pi r^2}$$

$$\frac{x}{1-x} = \frac{\pi(b^2 + 2br)}{\pi(a^2 + 2ar)}$$

$$\frac{x}{1-x} = \frac{b(b + 2r)}{a(a + 2r)}$$

$$x \cdot a \cdot (a + 2r) = (1 - x) \cdot b \cdot (b + 2r)$$

$$x\,[a(a + 2r) + b(b + 2r)] = b(b + 2r)$$

$$x = \frac{b(b + 2r)}{a(a + 2r) + b(b + 2r)} \qquad \text{(Formula II)}$$

Using Figure I and Formula II, suppose again that R = 16cm, r = 5 cm, and a = 4 cm. From Activity I, b = 14.6 cm, consequently,

$$x = \frac{14.6(14.6 + 10)}{4(4 + 10) + 14.6(14.6 + 10)}$$

$$= \frac{14.6(24.6)}{56 + 14.6\,(24.6)}$$

$$= .87$$

That is, 87% of the film is shown. Note that the initial reel has 25% of its cross-sectional width remaining while only 13% of the film remains to be shown. Why?

Activity III: Relate the angular velocities of the two reels.

Suppose at a given time the outside film circumference of the initial reel is three times that of the take-up reel. If the initial reel then makes one revolution, the take-up reel must make three revolutions to receive the film which has been processed. In general,

$$\frac{\text{Circumference of film on initial reel}}{\text{Circumference of film on take-up reel}} = \frac{\text{angular velocity of take-up reel}}{\text{angular velocity of initial reel}}$$

Since the circumference of the film on the initial reel is $2\pi(r + a)$ and the circumference of the film on the take-up reel is $2\pi(r + b)$, it may be concluded that:

$$\frac{\text{angular velocity of take-up reel}}{\text{angular velocity of initial reel}} = \frac{V_t}{V_i} = \frac{2\pi(r + a)}{2\pi(r + b)}$$

$$= \frac{r + a}{r + b} \quad \text{(Formula III)}$$

For instance, assume again that $R = 16$ cm, $r = 5$ cm, and $a = 4$ cm. From Formula I, $b = 14.6$ cm.

Hence $$\frac{V_t}{V_i} = \frac{5 + 4}{5 + 14.6}$$

$$\frac{V_t}{V_i} = \frac{9}{19.6}$$

$$\frac{V_t}{V_i} = .46$$

$$V_t = .46V_i$$

In other words, the take-up reel is revolving at a rate of 46% of that of the initial reel. Since 91% of the film has been shown (using Formula II), one might suspect that the take-up reel would be moving very slowly in relation to the initial reel. Why is the ratio as large as 46%?

Challenge to the reader: Find further relationships among the variables described in this article.