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Error Analysis for Picard-Chebyshev Iterations

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Error Analysis for Picard-Chebyshev Iterations

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In this paper an iterative technique for solving initial value problems is presented. The technique is based on Picard iterations and the Chebyshev polynomials and has been referred to by this author as the Picard-Chebyshev iteration method for initial value problems, although no such name for it appears in the literature. Its basic strength lies in the fact that it ignores completely the standard distinction between the linear and nonlinear initial value problem and that it produces a Chebyshev series solution which is easily evaluated. Its basic weakness lies in the fact that one must assume a finite series before computation can begin, and the accuracy of the solution depends completely on this assumption. In the literature this problem is overcome by simply repeating the solution with some appropriate increase in the length of the series until the accuracy desired is obtained. This proves to be an extremely lengthy and time-consuming procedure for even the simplest of initial value problems.

A considerable improvement can be obtained by specifying the accuracy desired as well as the length of the series and then determine the interval over which the solution can be constructed which will meet the accuracy requirements. This is not only a workable approach to the problem but also has the added advantage that for certain accuracy requirements a much larger interval than the standard Chebyshev intervals of (0,1) and (−1,1) can be used.

In this paper we shall be concerned with the solution to an initial value problem. The differential equation may be of any order and of any degree. As is usually the case, the theoretical considerations will be directed toward the first order equation since higher order equations can be reduced to a system of first order equations. However, the iterative method which we shall use will produce a Chebyshev series solution and can be easily modified to handle equations of order greater than 1 without the usual reduction to a system of first order equations. The method requires that the number of terms in the Chebyshev series solution be known a priori. When expanding a given function in its Chebyshev series, the same problem occurs. However, some extremely interesting techniques have been developed by Elliot which can evaluate and estimate the coefficients in the series (2, 274-284). This information provides a way to determine how long the series should be. The number of terms is, of course, a question of accuracy; but Elliot's techniques refer to a known function whereas the function we are interested in is known onlyimplicitly through the initial value problem for which it is the solution. Elliot's techniques, and others like them, are therefore not applicable to our problem.

A review of the literature indicates that the best solution to the problem is the following suggestion by Clenshaw and Norton:

> The minimum value of R (we are using R for the number of terms in the Chebyshev series whereas the article being cited here uses N) which is necessary to represent both y and F(x,y) to the desired accuracy will not, in general, be known in advance. Although no harm would result from a value larger than necessary, this would clearly be uneconomic. Indeed, during the early iterations when the approximation is poor, it may be desirable to use a value of R smaller than ultimately required to achieve full accuracy.
>
> One possibility is to start with a moderate or small value, say R = 4, and then introduce further coefficients only when their inclusion appears necessary for further improvement of the solution (1, 88-92).

The article goes on to suggest the quantitative ways one could use to determine when it was necessary to increase the value of R and by how much. However, once R has been increased, the entire solution must be rerun and the same criterion for increasing R must be applied again. The article suggests that R be increased by 2 each time. In the following problem,

$$y' = y^2,\ y(-1) = 0.4,$$

one does not obtain ten places of accuracy until the value of R has reached 26. Starting R at 4 and increasing by 2 each time the criterion requires it would mean that the solution would have to be run in its entirety a total of 12 times. Clearly this is at least as uneconomic as selecting an initial value for R which was larger than necessary.

It is the purpose of this paper to develop a method which will determine the domain of the independent variable which for a given value of R is necessary and sufficient for achieving some specified degree of accuracy. This shift in emphasis from the number of terms to the size of the domain is one of the main points in this work.

Rather than the symbol N for the number of terms, we will use the symbol R and use N instead to refer to the number of leading zeros in the final coefficient of the Chebyshev series. In other words, we shall use N to refer to N-place accuracy.

We shall now turn our attention to the iterative method which we shall use to obtain the Chebyshev series solution to an initial value problem. Since it is based on the Picard iterates as well as the Chebyshev series, we have named it the Picard-Chebyshev iteration method for initial value problems.

THE PICARD-CHEBYSHEV ITERATION METHOD FOR INITIAL VALUE PROBLEMS

There exists in the literature several iteration methoods whose purpose is to iteratively converge to the first R + 1 Chebyshev coefficients in the Chebyshev series solution to an ordinary differential equation. We will focus our attention primarily on the technique based on Picard iteration. This technique will illustrate in the most straightforward way the problem which this paper seeks to solve.

The existence and uniqueness theroem for the initial value problem

$$y' = F(x,y),\ y(x_0) = y_0 \tag{1}$$

is usually stated in the following way:

If: $F(x,y)$ is bounded in E^2
$F(x,y)$ is continuous in E^2
$F(x,y)$ satisfies a Lipschitz condition in E^2
Where E^2 is some region of the xy-plane,
Then: For $|x-x_0| < H$ there exists a unique function $y(x)$ such that $y' = F(x,y)$ and $y(x_0) = y_0$

The proof is somewhat varied from author to author but can generally be thought of as consisting of the following four steps:

Step 1: Determine H.
Step 2: Prove that the Picard iterates are:
(a) Continous.
(b) $\leq MH$ where $|F(x,y)| \leq M$

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Step 3: Prove that the Picard iterates converge uniformly to y(x).
Step 4: Prove that y(x) is unique.

The method which we are about to describe can be applied to any initial value problem for which the above theorem is valid and is basically a matter of constructing the Picard iterates in terms of their Chebyshev series. The method is outlined in the following six steps:

Step 1: After determining the domain for a given value of R and a given value of N by the method to be outlined in this paper, express the initial condition y_0 in its Chebyshev series. When this is done, all the coefficients except the first one, a_0, will be zero. a_0 will be $2y_0$.

Step 2: Evaluate the current Chebyshev series for y(x) at x(i) where $x(i) = \cos(\pi i/R)$, $i = 0,1, \ldots, R$.

Step 3: Evaluate $F(x(i), y(i))$, $i = 0,1,2, \ldots, R$. Note: It is at this point in the method where we avoid the usual difficulties associated with the nonlinear forms of F(x,y).

Step 4: Compute the Chebyshev series coefficients for F(x,y) using discrete least squares formulae (3, 31).

Step 5: The Chebyshev series for F(x,y) can now be easily integrated (3, 61). This will produce all but a_0 in the series for y(x) and a_0 can be determined from the initial values.

Step 6: If the accuracy criteria is satisfied, the solution is complete. If not, return to Step 2.

The computer program, written in a subset of Fortran IV, which carries out the above steps for a 28-term Chebyshev series solution to an initial value problem satisfying the conditions of the Picard-dependent existence and uniqueness theorem is provided in Appendix A.

AN INTERVAL DEPENDENT NECESSARY AND SUFFICIENT CONDITION FOR N-PLACE ACCURACY IN A PICARD-CHEBYSHEV SOLUTION TO AN INITIAL VALUE PROBLEM

We will now formulate and prove a theorem which will relate the following:

(1) N, where N is the number of places of decimal accuracy in the last coefficient of a finite Chebyshev series,
(2) R, where R is the index of the last coefficient in a finite Chebyshev series,
(3) δ, where δ is the length of the interval measured from $x_0 < n < x_0 + \delta$.

Before turning to the proof, let us first point out that the question of accuracy can be dealt with in terms of the magnitude of the coefficient which appears last in the series. This is due to the fact that each term in the Chebyshev series consists of a coefficient multiplied by a Chebyshev polynomial. The Chebyshev polynomials have a maximum absolute value equal to one. Thus, the largest contribution any term can make is equal to the absolute value of its coefficient. Furthermore, the value of the coefficients decreases in proportion to the reciprocal of $R!2^{R-1}$. Therefore, the last term will provide an accurate indication of the accuracy afforded by the finite series in question. For example, if a_R has N leading zeros, then the series which terminates with $a_R T_R(x)$ will provide N-place accuracy.

Experience with actual solutions provides ample illustrations of the above, although there are some exceptions. For example, the initial value problem

$$y = 1 - \text{sqrt}(y) + \cos(\pi x),\ y(-1) = 0.962556070550,$$

when solved by the Picard-Chebyshev iteration method yields the following:

$$a_{14} = -0.000000034904$$

If one assumed that the series terminating with $a_{14}T_{14}(x)$ would yield 7-place accuracy, he would err because

$$a_{15} = -0.000000237369.$$

This value implies by the above considerations only 6-place accuracy. It should be pointed out, however, that none of the test case problems solved by this author have ever yielded a decrease in accuracy by more than one decimal place when considering the next coefficient, and most of the time the coefficients are monotone decreasing after the first few terms.

Finally, we should point out that in an initial value problem's solution we do not solve for the infinite series and then truncate it to reflect the desired accuracy. We truncate it before the solution is started. As a result, truncation error could conceivably contribute enough to the last term in the finite series so as to make our accuracy considerations, which depend on that last term, invalid. It turns out, fortunately, that this is not a serious problem (3,68).

The Accuracy Theorem

$$a_R = 10^{-N}$$

If and only if:

The Picard-Chebyshev iteration method is carried out in the domain $(x_0, x_0+\delta)$ where δ satisfies

$$\delta = \left(\frac{R!\,2^{2R-1}}{|F^{(R)}(\eta)| \cdot 10^N} \right)^{1/R}$$

and where:

(a) R + 1 is the number of terms in the finite Chebyshev series for y = F(x), and,
(b) η is some number such that $x_0 < \eta < x_0 + \delta$, and,
(c) N is the number of leading zeros desired in a_R, and,
(d) F is now being used as the solution to the initial value problem.

Proof of the Accuracy Theorem

The Chebyshev series for a function F (x), $a \leq x \leq b$, is produced by first transforming the domain [a,b] to the domain [-1,1]. The formula for the R^{th} coefficient of the Chebyshev series is then written as follows:

$$a_R = \frac{2}{\pi} \int_{-1}^{1} (1-\xi^2)^{-1/2} F(\alpha\xi+\beta) T_R(\xi)\, d\xi \tag{2}$$

where

$$\alpha = \frac{b-a}{2} \text{ and } \beta = \frac{b+a}{2}.$$

Since our concern is with derivatives of F (x), it is convenient to revise (2) by expanding $F(\alpha\xi+\beta)$ in its Taylor series about the point $\xi = 0$. We obtain

$$F(\alpha\xi+\beta) = \sum_{K=0}^{K=R-1} \frac{(\alpha\xi)^K}{K!} \frac{d^{(K)}F(\beta)}{du} + \frac{(\alpha\xi)^R}{R!} \frac{d^{(R)}F(\eta)}{du^R} \tag{3}$$

where

$$u = \alpha\xi + \beta$$

and

$$a < \eta < b.$$

Since all powers of ξ can be expressed as a finite summation of Chebyshev polynomials each of which is of the same degree or less as the power of ξ and since the Chebyshev polynomials are orthogonal on [-1,1], we may substitute (3) into (2) and obtain

$$a_R = \frac{2}{\pi}\int_{-1}^{1}(1-\xi^2)^{-1/2}\,\frac{(\alpha\xi)^R}{R!}\,\frac{d^{(R)}F(\eta)}{du^R}T_R(\xi)\,d\xi$$

(4)

$$= \frac{\alpha^R}{R!}\,\frac{2}{\pi}\,\frac{d^{(R)}F(\eta)}{du^R}\int_{-1}^{1}(1-\xi^2)^{-1/2}\xi^R T_R(\xi)\,d\xi.$$

It should now be noticed that

$$\frac{2}{\pi}\int_{-1}^{1}(1-\xi^2)^{-1/2}\xi^R T_R(\xi)\,d\xi$$

is simply the coefficient of $T_R(\xi)$ in the Chebyshev expansion of ξ^R, which is of course simply 2^{1-R}. Thus equation (5) follows immediately.

(5) $$a_R = \frac{\alpha^R}{R!\,2^{R-1}}F^{(R)}(\eta)$$

Let us now assume that the interval beginning at x_0 is $[x_0, x_0+\delta]$ and transform $[x_0, x_0+\delta]$ to $[-1,1]$. The equations for doing so are given in (2). We have

(6) $$\begin{cases} \alpha = \dfrac{(x_0+\delta)-x_0}{2} = \dfrac{\delta}{2} \\ \beta = \dfrac{(x_0+\delta)+x_0}{2} = \dfrac{\delta+2x_0}{2}. \end{cases}$$

Substituting (6) in (5) we obtain

(7) $$a_R = \frac{(\delta/2)^R}{R!\,2^{R-1}}F^{(R)}(\eta) = \frac{\delta^R F^{(R)}(\eta)}{R!\,2^{2R-1}} \qquad x_0 < \eta < x_0+\delta$$

It is now a simple matter to show that if we would like the series for $y = F(x)$ to be

$$y = \sum_{r=0}^{r=R\,\prime} a_r T_r(x)$$

with N-place accuracy in a_R, that is,

(8) $$|a_R| = 10^{-N},$$

then by substitution of (7) into (8) we obtain

(9) $$\delta = \left(\frac{R!\,2^{2R-1}}{|F^{(R)}(\eta)|\cdot 10^N}\right)^{1/R}$$

Applying the theorem we have just proved does present some difficulties, and we will discuss each of them in the next section.

APPLICATIONS OF THE ACCURACY THEOREM

Using (7) just as it stands is impossible because the values of δ and η are not known. There is a way, however, to make significant use of (7), and this is accomplished by abandoning the following question:

(1) How many terms must we use in the Chebyshev series solution to obtain N-place accuracy?

in favor of the question:

(2) How large a domain, beginning at x_0, can we use in order to obtain N-place accuracy in a Chebyshev series solution to $y = F(x,y)$, $y(x_0) = y_0$ which contains $R+1$ terms?

This shift in emphasis from *number of terms* to *size of domain* can be facilitated by (9) and avoids the unworkable aspects of (7).

There are two problems which arise in the applications of the accuracy theorem which we will now discuss prior to the actual applications themselves. First, equation (7) is true only for the appropriate value of η, and we do not have access to this value. We can, however, by considering the weighting function for Chebyshev polynomials, namely $(1-\xi^2)^{-1/2}$, which is designed to make the error small at -1 and +1, argue that the best value for η is the one which minimizes the weight function and therefore maximizes the error. This value occurs at $\xi = 0$. $\xi = 0$ maps to the midpoint of $[x_0, x_0+\delta]$. Therefore we take η to be

(10) $$\eta = \frac{\delta+2x_0}{2}$$

Secondly, equation (9) must now be solved iteratively for δ since (10) introduces δ on the right of (9). Theoretically, we are assured of the existence of a value for η which would satisfy the following:

(11) $$|a_R| = \frac{\delta^R|F^{(R)}(\eta)|}{R!\,2^{2R-1}} = 10^{-N}$$

We are also assured on theoretical grounds that for a given η there is an appropriate δ. Since our interpretation of η would alter (11) as follows:

(12) $$\frac{\delta^R\left|F^{(R)}\left(\dfrac{\delta+2x_0}{2}\right)\right|}{R!\,2^{2R-1}} \approx 10^{-N}$$

we can proceed to solve this approximate equality for δ and obtain the following iteration:

(13) $$\delta_{i+1} = \left\{\frac{R!\,2^{2R-1}}{\left|F^{(R)}\left(\dfrac{\delta_i+2x_0}{2}\right)\right|\cdot 10^N}\right\}^{1/R}$$

Appendix B contains a program which performs the above iteration and describes the technique for computing

$$\left|F^{(R)}\left(\frac{\delta_i+2x_0}{2}\right)\right|.$$

Once we have a value for δ we can proceed with the Picard-Chebyshev solution with the assurance that

$$|a_R| \approx 10^{-N}.$$

There are several ways one could illustrate the applications of the accuracy theorem. Our approach shall be based on the following six problems:

(1) $y' = -y$ $\quad y(-1) = 2.718281828459$ (1,90)
(2) $y' = y^2$ $\quad y(-1) = 0.4$ (4, 79-81)
(3) $y' = \exp(-y)$ $\quad y(-1) = 0.0$ (5, 360)
(4) $y' = \sin y$ $\quad y(-1) = 0.705026843560$ (4)
(5) $y' = x - y^2$ $\quad y(-1) = -0.018971824750$ (4)
(6) $y' = 1 - \text{sqrt}(y) + \cos \pi x$ $\quad y(-1) = 0.962556070550$ (4)

These were selected as being representative and are taken from the literature with only slight modifications.

The first three problems have simple analytic solutions and thus offer an opportunity to compare the results of the Picard-Chebyshev method to the true solution. The fourth and fifth problems also have analytic solutions which are provided in Norton's article (4), but these analytic solutions are much more difficult and complicated. The sixth problem has no finite analytical solution.

For each of these problems we shall do the following:

(1) Solve the problem using 28 terms in the Picard-Chebyshev method with $\delta = 2$.
(2) Compute a δ which will produce N-place accuracy in a_R where N is different from the accuracy of a_R that was obtained in the 28-term solution.
(3) Solve the problem using R + 3 terms and check a_R for N-place accuracy. (The use of R + 3 terms rather than R + 1 terms assures us that truncation error will not affect our results (3, 68).

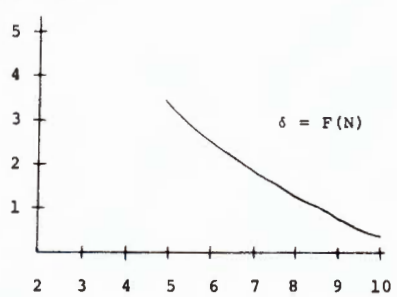
Selected delta values are shown in tabular and graphic form for each problem. The coefficients of the series produced by doing the above are arranged in two tables, the first of which shows the solution for $\delta = 2$ and R set arbitrarily at 27. The second shows the solution for a selected value of R and one of the computed values of δ. The accuracy of a_R should be greater than or equal to the value of N which corresponds to the delta which was used in the solution. The values which need to be compared are enclosed in the table by two horizontal lines.

Problem 1

$y' = -y$ $\quad y(-1) = 2.718281828459$

Results of Delta Iteration for R = 8

N	δ
5	3.595772782445
6	2.533073503352
7	1.818237611597
8	1.321527984811
9	0.992974132835
10	0.266249922131



Results of Picard-Chebyshev Iteration

δ = 2 R = 27	
r	a_r
0	2.532131755504
1	-1.130318207985
2	0.271495339534
3	-0.044336849849
4	0.005474240442
5	-0.000542926312
6	0.000044977323
7	-0.000003198436
8	0.000000199212
9	-0.000000011037
10	0.000000000551

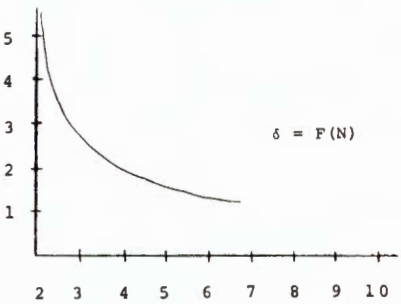
δ = 1.818... R = 10	
r	a_r
0	2.449351976420
1	-1.171295655775
2	0.277449347777
3	-0.040509022276
4	0.004042340585
5	-0.000292765653
6	0.000016005658
7	-0.000000679079
8	0.000000022825
9	-0.000000000617
10	0.000000000014

Problem 2

$y' = y^2$ $\quad y(-1) = 0.4$

Results of Delta Iteration for R = 8

N	δ
2	7.764639685523
3	2.707620636292
4	1.891638484579
5	1.574932007775
6	1.299565543998
7	1.064295194214



Results of Picard-Chebyshev Iteration

δ = 2 R = 27	
r	a_r
0	1.788854382118
1	0.683281573079
2	0.260990337042
3	0.099689438019
4	0.038077977006
5	0.014544492994
6	0.005555501975
7	0.002122012930
8	0.000810536815
9	0.000309597514
10	0.000118255727

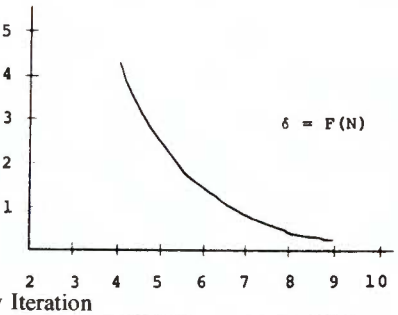
δ = 1.574... R = 10	
r	a_r
0	1.386573322853
1	0.350527712160
2	0.066505723388
3	0.010581390045
4	0.001486276452
5	0.000189735157
6	0.000022423459
7	0.000002484639
8	0.000000260492
9	0.000000025747
10	0.000000002531

Problem 3

$y' = \exp(-y)$ $\quad y(-1) = 0.0$

Results of Delta Iteration for R = 8

N	δ
4	4.366377770383
5	2.477757718250
6	1.391791146061
7	0.916878431466
8	0.349698333258
9	0.289871721778



Results of Picard-Chebyshev Iteration

δ = 2 R = 27	
r	a_r
0	1.247621432730
1	0.535898384862
2	-0.071796769724
3	0.012825257645
4	-0.002577388071
5	0.000552487242
6	-0.000123365425
7	0.000028333428
8	-0.000006642929
9	0.000001582193
10	-0.000000381553

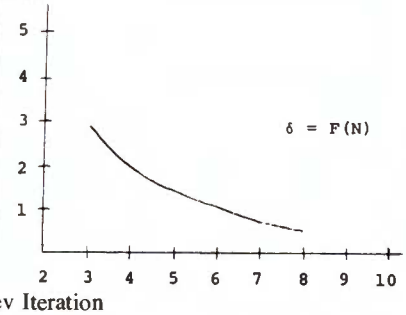
δ = 1.391... R = 8	
r	a_r
0	1.360639985137
1	0.597901290964
2	-0.070836347854
3	0.009896185842
4	-0.001438703403
5	0.000211188353
6	-0.000030986048
7	0.000004526470
8	-0.000000656953
9	0.000000092473
10	-0.000000014208

Problem 4

$y' = \sin y$ $\quad y(-1) = 0.705026843560$

Results of Delta Iteration for R = 7

N	δ
3	2.967528139860
4	1.847133735378
5	1.450160842701
6	1.184471706612
7	0.747525901063
8	0.509896862474



Results of Picard-Chebyshev Iteration

δ = 2 R = 27	
r	a_r
0	3.141592653602
1	0.895867258385
2	-0.000000000001
3	-0.031670934242
4	0.000000000000
5	0.001668508992
6	-0.000000000000
7	-0.000101626744
8	-0.000000000000
9	0.000006711693
10	-0.000000000000

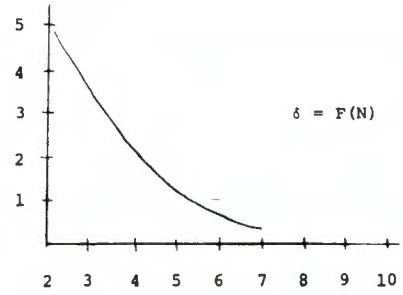
δ = 1.847... R = 9	
r	a_r
0	3.145776682772
1	0.906419146494
2	0.013022139346
3	-0.027966828162
4	-0.001487294118
5	0.001091712193
6	0.000113872373
7	-0.000043281969
8	-0.000007978616
9	0.000001488254
10	*.**********

Problem 5

$y' = x - y^2$ $\quad y(-1) = -0.018971824750$

Results of Delta Iteration for R = 6

N	δ
2	4.951959393674
3	3.854483684697
4	2.057951866760
5	1.060429589678
6	0.631841353289
7	0.365642349412



Results of Picard-Chebyshev Iteration

δ = 2 R = 27	
r	a_r
0	-1.331820135602
1	-0.565774570107
2	0.065558056960
3	-0.012311677977
4	0.002574869425
5	-0.000559796863
6	0.000123750083
7	-0.000027572206
8	0.000006167271
9	-0.000001382249
10	0.000000310126

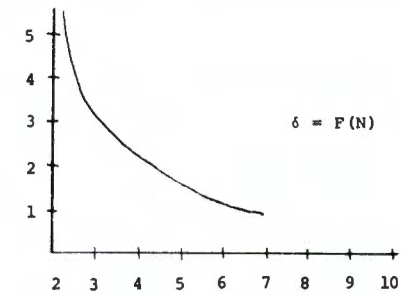
δ = 0.631... R = 8	
r	a_r
0	-1.729980428781
1	-0.806005774616
2	0.033462705741
3	-0.006339482535
4	0.000201643085
5	-0.000008596640
6	0.000000183131
7	-0.000000003837
8	0.000000000056
9	*.**********
10	*.**********

Problem 6

$y' = 1 - \text{sqrt}(y) + \cos \pi x$ $\quad y(-1) = 0.962556070550$

Results of Delta Iteration for R = 10

N	δ
2	10.664323112324
3	3.076876327097
4	2.341341458739
5	1.488840646665
6	1.087264432934
7	0.885596343309



Results of Picard-Chebyshev Iteration

δ = 2 R = 27	
r	a_r
0	1.994588224861
1	0.177079655786
2	-0.048309625947
3	-0.206944113248
4	0.014789026766
5	0.031677253343
6	-0.001217395448
7	-0.001851489116
8	-0.000015381966
9	0.000040342982
10	0.000017960517

δ = 1.488... R = 12	
r	a_r
0	2.150480750532
1	0.434373719737
2	0.228176220938
3	-0.135011014041
4	-0.033409236910
5	0.009972019386
6	0.001546023401
7	-0.000367648389
8	-0.000018327785
9	0.000009750956
10	-0.000002586350

SUMMARY AND CONCLUSIONS

The most general statement that can be made about what we have said is that the question of accuracy, that is, the question of error analysis, is not to be thought of in the usual way when constructing the solution to an initial value problem in the form of a finite Chebyshev series. The usual way has been to fix the length of the interval and concentrate on the number of terms one should use in the series. Instead we have suggested fixing the length of the series and concentrate on the length of the interval in which the solution fits the accuracy requirements. This allows the use of a relatively short Chebyshev series while at the same time meeting the demands of accuracy. In the event that the interval is smaller than desired, the series can be evaluated at the end of that interval and a new series constructed from the "initial value" thus obtained for the next segment of the desired interval. For example, in Problem #2, 16 terms were required for six-place accuracy over the interval $[-1,1]$ whereas only 8 terms were required for a solution over $[-1.0,0.574932007775]$. If the remainder of the interval from 0.574932007775 to 1.0 could be described with six-place accuracy by a Chebyshev series containing 8 or less terms, then we would be better off with a two-series solution over $[-1,1]$ than with the one-series solution over $[-1,1]$. The possibility of fewer terms in an N-series solution than in a one-series solution seems quite high to this author, although further research would be necessary to establish this from an analytical point of view. It is suggested strongly in actual practice.

Several general conclusions are possible. First, working with the Picard-Chebyshev method has convinced this author that it is superior to most of the methods commonly used. It avoids the problem of matrix inversion altogether, and the nonlinear case is handled exactly as the linear, whereas in other methods the nonlinear case requires special treatment. Secondly, the form of the solution is superior to methods which only produce tables of data as a solution. The form, a finite Chebyshev series, is easily differentiated, integrated, and evaluated. No interpolation is ever necessary. And, finally, with reference to the basic contribution of this paper, it is possible to construct a solution and know that all accuracy requirements will be met without having to either form an extremely lengthy Chebyshev series with the hope that one has made it "long enough," or rerun the solution with several additional terms each time until the accuracy requirements are satisfied. The method is limited as far as initial value problems are concerned to problems satisfying the hypotheses of the existence theorem.

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APPENDICES

A. Picard-Chebyshev iteration program

```
C*************************************************************
C     PROGRAM TO SOLVE ORDINARY DIFFERENTIAL EQUATIONS      *
C     IN TERMS OF CHEBYSHEV SERIES.  EQUATION MUST BE       *
C     FIRST ORDER, LINEAR, OR NONLINEAR.  THE NUMBER        *
C     OF TERMS IS SET AT R+1 IN THIS LISTING.  R.LE.27.     *
C*************************************************************
      INTEGER R
      DIMENSION X(28), A(28), Y(28), F(28), B(28)
C*************************************************************
C     THE COEFFICIENTS OF THE INITIAL CHEBYSHEV SERIES      *
C     ARE THOSE WHICH REFLECT THE INITIAL CONDITION.        *
C     THEREFORE, THEY WILL ALL EQUAL ZERO EXCEPT A(1).      *
C     THIS PROGRAM REQUIRES THE INITIAL CONDITION AT X=-1.  *
C*************************************************************
      DATA A/1.925112141100,27*0.0/
      R=12
      RR=R
C*************************************************************
C     DELTA PRODUCES A SOLUTION ON (-1,-1+DELTA)            *
C     THIS DELTA GUARANTEES 5 LEADING ZEROS IN A(10).       *
C*************************************************************
      DELTA=1.488840646665
      PI=3.141592653589
      ITER=12
C*************************************************************
C     EVALUATION OF THE CURRENT CHEBYSHEV SERIES FOR        *
C     Y AT X(I).                                            *
C*************************************************************
    1 DO 3  I=1,R+1
      X(I)=DELTA/2.0*(COS((I-1)*PI/RR)+1.0)-1.0
      AT=ATAN(SQRT(1.0-X(I)*X(I))/X(I))
      IF(X(I))11,12,12
   11 AT=AT+PI
   12 Y(I)=A(1)/2.0
      DO 2 J=2,R+1
    2 Y(I)=Y(I)+A(J)*COS((J-1)*AT)
C*************************************************************
C     EVALUATION OF F(X(I),Y(X(I)))                         *
C*************************************************************
    3 F(I)=1.0-SQRT(Y(I))+COS(PI*X(I))
C*************************************************************
C     EVALUATION OF THE COEFFICIENTS IN THE CHEBYSHEV
C     SERIES FOR F(X,Y).  FOX AND PARKER, PAGE 31.
C*************************************************************
      DO 5  I=1,R+1
      B(I)=F(1)/2.0
      DO 4  J=2,R
    4 B(I)=B(I)+F(J)*COS(PI*(I-1)*(J-1)/RR)
    5 B(I)=(B(I)+(-1)**(I-1)*F(R+1)/2.0)*2.0/RR
C*************************************************************
C     EVALUATION OF THE COEFFICIENTS IN THE CHEBYSHEV
C     SERIES FOR Y.  FOX AND PARKER, PAGE 61.
C*************************************************************
      DO 6  I=2,R
    6 A(I)=(B(I-1)-B(I+1))/(2.0*(I-1))
      A(R+1)=B(R)/(2.0*RR)
      T=A(1)
      A(1)=1.925112141100
      DO 7  I=2,R+1
    7 A(1)=A(1)+2.0*A(I)*(-1.0)**I
      ITER=ITER+1
      IF(ITER-12)9,8,8
    8 WRITE(6,20)
      ITER=0
    9 WRITE(6,30)(A(I),I=1,25,4),(A(I),I=2,26,4),(A(I),
     CI=3,27,4),(A(I),I=4,28,4)
C*************************************************************
C     THE ITERATION WILL TERMINATE WHEN THE INITIAL
C     COEFFICIENT IS ACCURATE TO TEN DECIMAL PLACES.
C*************************************************************
      D=ABS(T-A(1))
      IF(D-0.00000000001)10,10,1
   10 STOP
   20 FORMAT(1H1,4X,'A(0,1,2,3)',8X,'A(4,5,6,7)',8X,
     C'A(8,9,10,11)',6X,'A(12,13,14,15)',4X,'A(16,17,18,19)
     C4X,'A(20,21,22,23)',4X,'A(24,25,26,27)')
   30 FORMAT(/(1X,7F18.12))
      END
      FUNCTION F(X,Y)
      PI=3.141592653589
      F=1.0-SQRT(Y)+COS(PI*X)
      RETURN
      END
C*************************************************************
C     INITIALIZATION DATA SAMPLE                            *
C*************************************************************
     -1.0                0.962556070550    0.05    41 10  7
```

B. Delta iteration program

```
C*************************************************************
C     PROGRAM FOR ITERATION TO OBTAIN DELTA AS DEFINED      *
C     IN EQUATION (4.8).  R AND N MUST BE PROVIDED          *
C     BY THE USER.                                          *
C     X(1) AND Y(1) ARE INITIAL VALUES                      *
C     H=STEP SIZE                                           *
C     NS=NUMBER OF POINTS IN (-1,+1)                        *
C     R=INDEX ON FINAL CHEBYSHEV COEFFICIENT                *
C     N=NUMBER OF PLACES OF ACCURACY IN RTH COEFFICIENT     *
C*************************************************************
      INTEGER E,R
      DIMENSION X(41),Y(41),D(10,40)
C*************************************************************
C     THE ITERATION BEGINS BY SOLVING THE INITIAL VALUE     *
C     PROBLEM USING THE RUNGE KUTTA METHOD TO OBTAIN THE    *
C     FIRST THREE VALUES AND THEN PROCEEDS USING THE        *
C     ADAMS-BASHFORTH, ADAMS-MOULTON PREDICTOR-CORRECTOR.   *
C*************************************************************
      READ(5,1)X(1),Y(1),H,NS,R,N
    1 FORMAT(3F20.12,3I3)
      DO 2 I=1,3
      RKC1=H*F(X(I),Y(I))
      RKC2=H*F(X(I)+H/2.0,Y(I)+RKC1/2.0)
      RKC3=H*F(X(I)+H/2.0,Y(I)+RKC2/2.0)
      RKC4=H*F(X(I)+H,Y(I)+RKC3)
      Y(I+1)=Y(I)+(RKC1+2.0*RKC2+2.0*RKC3+RKC4)/6.0
    2 X(I+1)=X(I)+H
      DO 3 I=4,NS-1
      Y(I+1)=Y(I)+H/24.0*(55.0*F(X(I),Y(I))-59.0*F(X(I-1),
     CY(I-1))+37.0*F(X(I-2),Y(I-2))-9.0*F(X(I-3),Y(I-3)))
      X(I+1)=X(I)+H
    3 Y(I+1)=Y(I)+H/24.0*(9.0*F(X(I+1),Y(I+1))+19.0*F(X(I),
     CY(I))-5.0*F(X(I-1),Y(I-1))+F(X(I-2),Y(I-2)))
C*************************************************************
C     THE UPPER LIMIT OF DO-LOOP 4 IS THE NUMBER OF STEPS   *
C     REQUIRED TO TRAVERSE THE INTERVAL (-1,1) BY H.        *
C*************************************************************
      DO 4 I=1,NS-1
    4 D(1,I)=(Y(I+1)-Y(I))/H
C*************************************************************
C     THE UPPER LIMIT OF DO-LOOP 5 DETERMINES THE HIGHEST   *
C     ORDER DERIVATIVE APPROXIMATION.  THE APPROXIMATION    *
C     IS ACCOMPLISHED USING DIVIDED DIFFERENCES.  R.LE.NS.  *
C*************************************************************
      DO 5 J=2,R
      DO 5 I=1,NS-J
    5 D(J,I)=(D(J-1,I+1)-D(J-1,I))/H
C*************************************************************
C     CALCULATION OF THE CONSTANT PORTION OF EQUATION 20    *
C*************************************************************
      CON=1.0
      DO 6 I=1,R
    6 CON=CON*2.0*I/10.0
      CON=CON*2.0**(R-1)/(10.0**(N-R))
C*************************************************************
C     BEGINNING OF DELTA ITERATION                          *
C*************************************************************
      WRITE(6,66)
   66 FORMAT('1',8X,'DELTA',15X,'X(1)',16X,'Y(1)'
     C,16X,'H    NS   R   N')
      DELTA=1.0
      WRITE(6,67)DELTA,X(1),Y(1),H,NS,R,N
   67 FORMAT(3F20.12,F12.4,3I4)
    7 T=(DELTA+2.0*X(1))/2.0
C*************************************************************
C     LOCATION OF (DELTA+2*X(1))/2) IN X VECTOR             *
C*************************************************************
      DO 8 I=1,NS
      IF(T-X(I))9,8,8
    8 CONTINUE
    9 J=I-R/2-1
      FR=ABS(D(R,J))
      DELTAN=(CON/FR)**(1.0/R)
      WRITE(6,10)DELTAN
   10 FORMAT(F20.12)
C*************************************************************
C     TEST FOR CONVERGENCE                                  *
C*************************************************************
      DIFF=ABS(DELTA-DELTAN)
      IF(DIFF-0.000001)12,11,11
   11 DELTA=DELTAN
      GO TO 7
   12 STOP
      END
```