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A Brief Study of Real-Valued Continuous Functions on Various Spaces

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A BRIEF STUDY OF REAL-VALUED CONTINUOUS FUNCTIONS ON VARIOUS SPACES

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1. Abstract

The intent of this study is to find sufficient criteria on a space X in order to bound the cardinality of the real-valued continuous functions on X by 2^ω. The desired result is known for X separable and for X first-countable, Hausdorff, and either Lindelof or ccc, but these are all very strong properties on a space. It is the goal of this study to find properties that are weaker, yet sufficient in bounding the number of real-valued continuous functions on a space by the size of the continuum.

2. Introduction

The road to reaching the topic of this paper was rather long, containing many parallels and sometimes the feeling of getting right back to where it began. Ultimately, through the study of cardinal functions, the focus of the paper was turned towards the bounds on continuous real-valued functions on various spaces. After investigating a few particular spaces, the focus became more specific and it turned towards the question of what properties of a space are necessary in order for the number of continuous real-valued functions on that space to be bounded by the size of the continuum. In order to get a full grasp of the process involved, we will begin with a discussion of cardinal functions and topological definitions, move into some discussion of a few particular theorems bounding continuous real-valued functions, and then begin there with the hopes to come up with a few specific properties that will bound the continuous real-valued functions on a space by 2^ω.

A ***cardinal function*** is a mapping from the class of topological spaces into the class of cardinal numbers. It is one of the most important concepts in set-theoretic topology. Cardinal functions allow one to generalize topological properties, such as the countability axioms and compactness, in order to formulate and prove results concerning different characteristics of a topological space. For example, cardinal functions help prove that 2^ω is a bound on both the number of open sets and the cardinality of a space X when X is Hausdorff with a countable base.

The study of cardinal functions has been a recent development in Set-Theoretic Topology. Many of the concepts that form the foundation of cardinal functions were laid down previously by mathematicians such as Cantor, Alexandroff, Urysohn, and Hausdorff, but the systematic study did not begin until the mid-1960's. Some basic uses for cardinal functions are to find bounds on the cardinality of a space X, the number of open sets in X, the number of compact subsets of X, and the number of continuous, real-valued functions on X. It is with the last one that we are most interested, but we will also touch on some of the others.

3. The Basics

Before we begin, it is important to define some ideas that are very important in the study of Topology. These definitions can be found in any introductory Topology text (see Munkres). As these concepts are introduced, we will define the corresponding cardinal functions.

Definition 1. *A* **topology** *on a set X is a collection $\mathcal{T}$ of subsets of X with the following properties:*

1. *$\emptyset$ and X are in $\mathcal{T}$.*
2. *The union of the elements of any subcollection of $\mathcal{T}$ is in $\mathcal{T}$.*
3. *The intersection of the elements of any finite subcollection of $\mathcal{T}$ is in $\mathcal{T}$.*

A set X for which a topology $\mathcal{T}$ has been specified is called a **topological space**. *A subset of X is an* open set *if it belongs to the collection $\mathcal{T}$. A subset of X is* closed *if its complement is open.*

Example. If we let X be the set of all real numbers, $\mathbb{R}$, then the set of all arbitrary unions of open intervals (i.e. $\{(a,b) : a,b \in \mathbb{R}, a \leq b\}$) forms a topology on $\mathbb{R}$. We call this topology the usual topology on $\mathbb{R}$.

The *cardinality* of a space X is the most basic cardinal function defined as $|X|$ = number of points in $X + \omega$. Thus, in the previous example, the cardinality of the space was 2^ω.

In most cases, it is not necessary to use the entire topology to characterize a space. It is sufficient to consider only the "building blocks" of open sets in order to know exactly what the topology looks like. The next definition gives a formal description of these "building blocks".

Definition 2. *If X is a set, a* **base** *for a topology on X is a collection $\mathcal{B}$ of subsets of X (called* **base elements***) such that*

1. *For each $x \in X$, there is at least one base element B containing x.*
2. *If x belongs to the intersection of two base elements B_1 and B_2, then there is a base element B_3 containing x such that $B_3 \subseteq B_1 \cap B_2$.*

If $\mathcal{B}$, satisfies these two conditions, then we define the **topology $\mathcal{T}$ generated by $\mathcal{B}$** *as follows: A subset U of X is said to be open in X if for each $x \in U$, there is a base element $B \in \mathcal{B}$ such that $x \in B$ and $B \subset U$.*

For a space X, we define the cardinal function *weight* of X as:
$w(X) = \min\{|\mathcal{B}| : \mathcal{B} \text{ a base for } X\} + \omega$.

In our previous example with the usual topology on $\mathbb{R}$, it is easy to see that $w(\mathbb{R}) = \omega$ since the countable set $\{(a,b) : a,b \in \mathbb{Q}, a \leq b\}$ forms a base for the usual topology. This countable base motivates the next definition.

Definition 3. *A topological space X is said to be* **second countable** *if X has a countable base for its topology. Thus, a space is second countable if and only if it has countable weight.*

Second countability is a very strong property with many implications. Often, we do not have such a global countability. A local version of countability is given in the following definition.

Definition 4. *A topological space X is said to be* **first countable** *if each point $x \in X$ has a countable base. More specifically, there is a countable collection $\mathcal{N}$ of open neighborhoods of x so that each neighborhood of x contains at least one element of $\mathcal{N}$.*

For spaces that are not first countable, we can generalize the notion of the cardinality of a local base using the cardinal function called the *character* of a space. Character is defined as follows:

$\chi(p, X) = \min\{|\mathcal{V}| : \mathcal{V}$ is a local base for $p\}$;

$\chi(X) = \sup\{\chi(p, X) : p \in X\} + \omega$.

If X is first-countable, $\chi(X) = \omega$.

Definition 5. *A space is said to be* **compact** *if every open covering of X has a finite subcover. A space is considered* **countably compact** *if every countable open covering of X has a finite subcover. A space X is* **σ-compact** *if X can be expressed as a countable union of compact sets. A space is said to be* **Lindelof** *if every open cover of X has a countable subcover.*

Compactness and the Lindelof property are very strong characteristics of a topological space. Often, one deals with spaces that are neither compact nor Lindelof. In order to generalize the notion of compactness and the Lindelof property to all spaces, we can define the cardinal function called the *Lindelof Degree* of a space. For a set X, the Lindelof Degree is defined as follows:

$L(X) = min\{\kappa :$ every open cover of X has a subcover of cardinality $\leq \kappa\} + \omega$;

If X is Lindelof, then $L(X) = \omega$.

Definition 6. *A space X is* **Hausdorff** *if for every $x, y \in X$, there exists open sets U, V such that $x \in U$, $y \in V$, and $U \cap V = \emptyset$.*

Definition 7. *A space X is said to be* **separable** *if it contains a countable dense set. That is, a set $D \subseteq X$ such that $|D| \leq \omega$ and $\bar{D} = X$.*

Separability is another very strong property of a space. Again, the notion of the cardinality of a dense subset can be generalized to any space using the cardinal function *density*. We define

density as follows:
$d(X) = \min\{|D| : D \subseteq X \wedge \bar{D} = X\}$;
$d(X) = \omega$ if and only if X is separable.

Definition 8. *A cover $\mathcal{A}$ of a set X is said to be a* **separating cover** *if for each $x \in X$, $\cap\{A : A \in \mathcal{A}, x \in A\} = \{x\}$.*

Definition 9. *A function from a space X to a space Y is said to be* **continuous** *if the inverse image of every open set is open. A* **continuous real-valued function** *on a space X is a continuous function from X into $\mathbb{R}$ with the usual topology. We use the notation $C(X)$ to refer to the set of real-valued continuous functions on a space X.*

4. Bounds on Real-Valued Continuous Functions

After achieving a general knowledge of some of the basic cardinal functions, one can begin to use them to place bounds on the number of continuous real-valued functions on a particular space. We can begin by noticing that $|C(X)| \geq 2^\omega$ for any space X. Since $X \neq \emptyset$ and every constant function is continuous, there is at least one function for every $y \in \mathbb{R}$. Thus, $|C(X)| \geq 2^\omega$. The next theorem, although fairly trivial, is also worth noting.

Theorem (1): For any separable space X, $|C(X)| = 2^\omega$.

Proof: Let S be a countable dense subset of X. The number of functions from S into $\mathbb{R}$ is at most $(2^\omega)^\omega = 2^\omega$. Suppose that f and g are two continuous functions that agree on S, yet there exists $x \in D - S$ such that $f(x) \neq g(x)$. Since $\mathbb{R}$ is Hausdorff, let U, V be disjoint open neighborhoods of $f(x), g(x)$, respectively. Then $f^{-1}(U)$ and $g^{-1}(V)$ are both open neighborhoods of x. Since S is dense, let $s \in S$ so that $s \in f^{-1}(U) \cap g^{-1}(V)$. But then $f(s) \in U$ and $g(s) \in V$, contradicting our assumption that f and g agree on S. Thus, any two functions that agree on S also agree on all of X. □

The next theorem is one of the best bounds that we can get on $C(X)$ using cardinal functions. Before one can continue, however, another cardinal function must be introduced. The *weak covering number* of a space X can be defined as follows:
$wc(X) = \min\{\kappa :$ every open cover of X has a subcollection of cardinality $\leq \kappa$ whose union is dense in $X\} + \omega$.

Theorem (2): For any space X, $|C(X)| \leq w(X)^{wc(X)}$.

In order to prove the theorem, we must first prove a related lemma.

Lemma (1): Let D and E be sets, let $\mathcal{F}$ be a collection of functions from D into E, let $\mathcal{S}$ be a separating cover of E. Suppose there is a collection $\mathcal{A}$ of subsets of D so that $f^{-1}(S) \in \mathcal{A}$ whenever $f \in \mathcal{F}$ and $S \in \mathcal{S}$. Then $|\mathcal{F}| \leq |\mathcal{A}|^{|\mathcal{S}|}$.

Proof:(of lemma) Define $\Phi : \mathcal{F} \rightarrow {}^{\mathcal{S}}\mathcal{A}$ by $\Phi(f) = f^*$, where $f^*(S) = f^{-1}(S)$. Then Φ is one-one since $\mathcal{S}$ is separating. Thus, $|\mathcal{F}| \leq |\mathcal{A}|^{|\mathcal{S}|}$. □

Proof:(of theorem) Let $wc(X) = \kappa$. Let $\mathcal{B}$ be a base for X with $|\mathcal{B}| \leq w(X)$. Let $\mathcal{G} = \{\bar{G} : G$ is the union of $\leq \kappa$ elements of $\mathcal{B}$ $\}$, and let $\mathcal{A}$ be all countable unions of elements of $\mathcal{G}$. Then $|\mathcal{A}| \leq w(X)^\kappa$. Using the notation of the lemma, let $\mathcal{S} = \{(-\infty, r) : r \in \mathbb{Q}\} \cup \{(r, \infty : r \in \mathbb{Q})\}$, let $D = X$, $E = \mathbb{R}$, $\mathcal{F} = C(X)$, and $\mathcal{A} = \mathcal{A}$. Using the lemma, it suffices to show that $f^{-1}(S) \in \mathcal{A}$ whenever $f \in C(X)$ and $S \in \mathcal{S}$ for then we get the result that $C(X) \leq |\mathcal{A}|^{|\mathcal{S}|} = [w(X)^\kappa]^\omega = w(X)^\kappa$.

We will show that $f^{-1}((-\infty, r)) \in \mathcal{A}$ for all $f \in C(X)$ and $r \in \mathbb{Q}$. The argument for the intervals (r, ∞) is almost identical. So let $r \in \mathbb{Q}$. Let $V = f^{-1}(-\infty, r)$. We can assume that $V = \bigcup_{n\in\omega} W_n$, where each W_n is open in X and $\bar{W}_n \subseteq W_{n+1}$ for all $n \in \omega$. The previous statement can be seen by taking the inverse images of the countable set $\{(-\infty, q) : q \in \mathbb{Q} \wedge q < r\}$. For each $n \in \omega$, $\{B \in \mathcal{B} : B \subseteq W_{n+1}\}$, together with $(X - \bar{W}_n)$ is an open cover of X, so there is a subcollection of cardinality $\leq \kappa$ whose union is dense in X. Let $\mathcal{B}_n$ be all the elements of this subcollection except $(X - \bar{W}_n)$. Let $G_n = \bigcup \mathcal{B}_n$. Then $W_n \subseteq \bar{G}_n \subseteq \bar{W}_{n+1} \subseteq W_{n+2}$. Thus, $V = \bigcup_{n\in\omega} \bar{G}_n$ and so $V \in \mathcal{A}$. □

Before presenting the next corollary, one should be introduced to the cardinal function *cellularity*. A cellular family of a space X is a pairwise disjoint collection of non-empty open sets in X. The cellularity of a space X can be defined as $c(X) = \sup\{|\mathcal{V}| : \mathcal{V}$ a cellular family in $X\} + \omega$. If $c(X) = \omega$, then X is said to be a *ccc space* (countable chain condition). This implies that every disjoint family of open sets is countable.

Corollary (1): Let X be a first-countable, Hausdorff space so that X is either Lindelof or ccc. Then $|C(X)| = 2^\omega$.

Proof: We know that if X is Hausdorff, $|X| \leq 2^{L(X)\cdot\chi(X)}$ and $|X| \leq 2^{c(X)\cdot\chi(X)}$. Since X is first-countable and either Lindelof or ccc, then $|X| \leq 2^\omega$. First-countability then implies that $w(X) \leq 2^\omega$. Since X is either Lindelof or ccc, then $wc(X) \leq \omega$. Thus, $w(X)^{wc(X)} \leq (2^\omega)^\omega = 2^\omega$. □

These results naturally led to the discussion of whether it was possible or not to tighten the theorem in order to get a better bound on continuous real-valued functions. It was discovered, however, that without the assumption of the generalized continuum hypothesis, $w(x)^{wc(X)}$ was the best bound we can get. With that in mind, let us investigate the number of continuous real-valued functions on a few specific spaces in order to see if any conclusions can be drawn. The ω_1 space, where ω_1 is the first uncountable ordinal and the topology is given by the order topology, is as good a place as any to start. In fact, we already know something very significant about the ω_1 space.

Lemma (2): Every continuous real-valued function of ω_1 is eventually constant.

Proof: Let $f : \omega_1 \longrightarrow \mathbb{R}$ be continuous. Suppose that f is not eventually constant. Then for every $\alpha \in \omega_1$, let $\beta_\alpha \in \omega_1$ and $n_\alpha \in \mathbb{Z}^+$ so that $\beta_\alpha > \alpha$ and

$$|f(\alpha) - f(\beta_\alpha)| > \frac{1}{n_\alpha}.$$

Since ω_1 is uncountable and $\{n_\alpha : \alpha < \omega_1\} \subseteq \mathbb{Z}^+$ is countable, let $n \in \mathbb{Z}^+$ so that $S = \{\alpha \in \omega_1 : n_\alpha = n\}$ is uncountable. Then S is cofinal in ω_1 since every uncountable set in ω_1 is cofinal.

Let $\alpha_0 \in S$. Let $\alpha_1 > \beta_{\alpha_0}$ so that $\alpha_1 \in S$. Continue by recursively building a sequence $(\alpha_n)_{n\in\omega} \subset \omega_1$ so that

(1) $\alpha_n \in S \ \forall n \in \omega$ and

(2) $\alpha_{n+1} > \beta_{\alpha_n} \ \forall n \in \omega$.

Let $\alpha = \sup\{\alpha_n : n \in \omega\} < \omega_1$. Consider the set $T = f^{-1}(f(\alpha) - \frac{1}{2n}, f(\alpha) + \frac{1}{2n})$ which is open in ω_1. By construction, T does not contain any elements of $(\alpha_n)_{n\in\omega}$ but since every open neighborhood of α contains infinitely many points of the sequence and $\alpha \in T$, then we have a contradiction. Thus, f is eventually constant. □

Corollary (2): $|C(\omega_1)| = 2^\omega$.

Proof: Since every continuous real-valued function on ω_1 is eventually constant, we can consider two parts of the domain, the constant part and the non-constant part. Let $\alpha \in \omega_1$. We will consider the cardinality of continuous real-valued functions in which α is the first ordinal where the function becomes constant. Since $\alpha < \omega_1$, $|\alpha| = \omega$ and so the non-constant part is a countable set being mapped into $\mathbb{R}$. The constant part is simply mapping all of the points greater than or equal to α into one point in $\mathbb{R}$. Thus, the cardinality of continuous real-valued functions pertaining to any given α is $(2^\omega)^\omega \cdot 2^\omega = 2^\omega$. Since there are at most ω_1 many α's,

$$|C(\omega_1)| = 2^\omega \cdot \omega_1 = 2^\omega. \ \Box$$

The results of the preceding corollary are rather interesting. Although ω_1 is both first countable and Hausdorff, it is neither Lindelof nor ccc. Thus, Corollary (1) does not apply to the ω_1 space. Additionally, the ω_1 space is not separable, and so Theorem (1) is not applicable. Which properties of the ω_1 space, then, are sufficient to bound the continuous real-valued functions on ω_1 by 2^ω. Aside from satisfying all of the separation axioms except for perfect normality, and being first countable, an important property of the ω_1 space is that it is countably compact. So what can we say about continuous real-valued functions on a countably compact space. One important implication of countable compactness is the following proposition.

Proposition (1): Let X be a countably compact space. Every continuous real-valued function on X has a compact image.

Proof: Let $f : X \to \mathbb{R}$ be continuous. Consider the set $\mathcal{S} = \{(-q, q) : q \in \mathbb{Q}^+\}$. Clearly, $\mathcal{S}$ is a countable open cover of $\mathbb{R}$. Thus, $f^{-1}(\mathcal{S})$ is a countable open cover of X. Since X is countably compact, let $\{q_1, .., q_n\} \subset \mathbb{Q}^+$ so that $f^{-1}(\{(-q_i, q_i) : i = 1, ..., n\})$ is an open cover of X. Let $q = \max\{q_i : i = 1, ..., n\}$. Then $f(X) \subseteq (-q, q)$. Thus, $f(X)$ is bounded.

Let $a \in f(X)'$. Towards a contradiction, suppose that $a \notin f(X)$. Let

$$\mathcal{S} = \{\mathbb{R} - [a - 1/n, a + 1/n] : n \in \mathbb{Z}^+\}.$$

Then $f^{-1}(\mathcal{S})$ is a countable open covering of X with no finite subcover, contradicting the hypothesis that X is countably compact. Thus, $a \in f(X)$. Therefore, X is both closed and bounded. So X is compact. □

This is a very significant step towards our goal, since the set of compact subsets of $\mathbb{R}$ is bounded above by 2^ω. If there is a way to bound the number of continuous real-valued functions from a space onto any particular compact subset of $\mathbb{R}$ by 2^ω, then we will have found the answer to our question. In the meantime, let us notice another result with interesting implications.

Lemma (3): The continuous image of any connected space is connected.

Proof: Let X, Y be spaces. Let X be connected. Let $f : X \to Y$ be continuous. Suppose that $U, V \subset f(X)$ form a separation of $f(X)$. Then $U \cup V = f(X)$, $U \cap \bar{V} = \emptyset$ and $\bar{U} \cap V = \emptyset$. Clearly, $f^{-1}(U)$ and $f^{-1}(V)$ will form a separation of X. Thus, $f(X)$ must be connected. □

When considered together, the last two results tell us a great deal about the image of any continuous real-valued function on a countably compact, connected space. It tells us exactly what the image looks like. Since the image of every continuous real-valued function is both connected and compact, it will be of the form $[a, b]$ with $a, b \in \mathbb{R}$, $a \leq b$.

Now that we know a good deal of information about the image of any continuous real-valued space, it is natural to investigate two continuous functions with the same image and find restrictions on the number of points in their domain where they can differ. After the discovery that every real-valued continuous function on ω_1 is eventually constant, surely, if we assume the correct properties, there is a parallel argument that can give a similar characteristic to any space satisfying the given properties. By assuming that the character of a space is ω_1, we get the desired result.

Proposition (2): Let X be a linearly ordered topological space with $\chi(X) = \omega_1$. Let $f \in C(X)$. Then there exists $x \in X$ such that either there exists some $r < x$ so that

$f([r,x]) = f(x)$ or there exists some $r > x$ so that $f([x,r]) = f(x)$.

Proof: Let X be a linearly ordered topological space with $\chi(X) = \omega_1$. Let $x \in X$ so that $\chi(x, X) = \omega_1$. Let $f : X \to \mathbb{R}$ be continuous. By way of contradiction, suppose that for every element $t \in X$ with $t \neq x$, there exists some x' between x and t so that $f(x') \neq f(x)$.

Let $(x_\alpha)_{\alpha \in \omega_1}$ be a sequence in X so that $(x_\alpha)_{\alpha \in \omega_1} \to x$. For each $\alpha \in \omega_1$, let $\hat{x}_\alpha$ be between x and x_α and $n_\alpha \in \mathbb{Z}^+$ so that $|f(\hat{x}_\alpha) - f(x)| > \frac{1}{n_\alpha}$. Then $(\hat{x}_\alpha)_{\alpha \in \omega_1} \to x$. Let $n \in \mathbb{Z}^+$ so that $S = \{\alpha \in \omega : |f(\hat{x}_\alpha) - f(x)| > \frac{1}{n}\}$ has cardinality ω_1. Then S is cofinal in ω_1 so $(\hat{x}_\alpha)_{\alpha \in S} \to x$.

Consider the set $T = f^{-1}(f(x) - \frac{1}{2n}, f(x) + \frac{1}{2n})$. Since f is continuous, T is an open neighborhood of x. Thus, T contains infinitely many members of the sequence $(\hat{x}_\alpha)_{\alpha \in S}$. By the way that we constructed the sequence, however, T can contain no elements of the sequence. Thus, a contradiction. Therefore, there either exists some $r < x$ so that $f([r,x]) = f(x)$ or there exists some $r > x$ so that $f([x,r]) = f(x)$. □

It should be noted here that this result generalizes for any space of uncountable character. This is easy to see as any uncountable cardinal possesses the same properties of ω_1 that were utilized in the proof. This result tells us a great deal about real-valued continuous functions on linearly ordered topological spaces that are not first countable. At any point x in such a space that has an uncountable base, there is going to be a portion of the function, either directly before or directly after x, that is constant.

One trivial result of the proposition is that any real-valued continuous function on a space $[0, \kappa)$, where κ is an ordinal with uncountable cofinality, is going to be eventually constant. Further, every ordinal $\sigma < \kappa$ with uncountable cofinality will be preceded by a constant section in every real-valued continuous function. For example, every real-valued continuous function on ω_2 will have uncountably many constant sections since there are uncountably many ordinals less than ω_2 with uncountable cofinality. Another interesting result of the proposition is that the cardinality of the real-valued continuous functions on a space $[0, \omega_{n+1})$ is going to be bounded by $2^{\omega_n} \cdot \omega_{n+1}$. Ultimately, this proposition shows us an important similarity between the images of any two real-valued continuous functions on a linearly ordered topological space.

5. Conclusion

Throughout this study, significant steps towards bounding functions have been made, and we have discovered a great deal about the images of real-valued continuous functions on spaces with various properties. Separability is sufficient in bounding the number of real valued continuous functions on a space by 2^ω, as well as if a space is first countable, Hausdorff, and either Lindelof or ccc. Countable compactness and connectedness of a space have been shown to result in a compact, connected image in $\mathbb{R}$. We've shown that when a space is linearly ordered and has uncountable character, this implies that there are sections in any real-valued continuous function that are constant. All of these properties of topological spaces have brought us much closer to finding new criteria that are sufficient in bounding the real-valued continuous functions by 2^ω.

6. Open Questions

In many cases, it feels like the properties we have assumed above will be sufficient in bounding the real-valued continuous functions on a space by 2^ω, but it is yet to be proved. Thus, there are some questions to be proposed. Are the properties assumed above sufficient in bounding the real-valued continuous functions by 2^ω? If not, what properties of a space X, other than the ones previously proved, are sufficient in order to get $|C(X)| \leq 2^\omega$? And is it possible to bound the real-valued continuous functions on a space by 2^ω without putting a bound on the cardinality of the space (i.e. Corollaries 1 and 2) or utilizing a countable dense set (i.e. Theorem 1)?

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