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Herbert H. Snyder
*Southern Illinois University*

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**HERBERT H. SNYDER[2]**

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SYNOPSIS: Guided wave propagation is studied for the interior of circularly cylindrical domains; on the (cylindrical) boundary the ratios $E_z/H_\phi$ and $H_z/E_\phi$ are assigned functions of frequency and position, called surface impedances. Even in the case where the impedances are independent of position on the wall, the normal modes of propagation (expressed in terms of certain Bessel functions) do not form an orthogonal set on sections of the guide normal to the propagation direction. It follows that mode relative amplitudes cannot be determined by a Fourier-Bessel expansion of steady-state fields given on an input plane. To determine those amplitudes, the author introduces the Laplace transforms of Maxwell's equations, which are solved for the transforms of the field components **E** and **H** in the case that the impedances are (at most) functions of frequency. However, this method of analysis shows that the transform provides a method of attacking the variable impedance case. An example is then given, in which the transform of the axial component $E_z$ is inverted explicitly in the axially symmetric case for a simple choice of input field. The author concludes with some remarks on representation of electromagnetic fields by contour integrals, of which the Laplace inversion integral is a special case.

INDEX DESCRIPTORS: contour integrals; Laplace transform; microwave tubes; modes of propagation; surface impedance; wave guides.

We have been engaged for some time in the development of a *field theory* of phenomena in certain electron beam devices, e.g., traveling wave amplifiers and backward wave oscillators. (For details concerning such tubes, we refer to such monographs as those of Beck (1958), Hutter (1960), Pierce (1950) and Slater (1950).) From the standpoint of electromagnetic theory, there are just two basic components in such devices, namely, a drifting *beam* of electrons (usually in the shape of a right circular cylinder), and coaxial with which is a *wave guide* of rather complicated geometry (usually a periodic structure). One of the problems standing in the way of the desired field-theoretical development is the complexity and diversity of the various wave guide types of actual or potential technical interest. (For the sake of demonstrating this diversity, we sketch two of them in Fig. 1.) Since the element common to the class of such devices is the cylindrical beam of electrons, it is reasonable to seek a mathematical model in which properties of the wave guide are all subsumed under some simple type of boundary conditions which, moreover, only need to be applied *at the cylindrical boundary of the beam.* Otherwise, one must carry out a special analysis for each one of the many interesting and potentially useful wave guide geometries—an almost hopelessly difficult program. The first task is to investigate the character of propagation under the influence of such boundary conditions, but in the *absence* of the beam. If these conditions yield fields in the beam region of the same nature as those of an actual wave guide, then the possibly artificial quality of them is a matter of indifference.

On the basis of results published elsewhere by Unger (1963), Birdsall and Whinnery (1953), and Snyder (1971), it seems that the notion of *surface impedance* leads to boundary conditions which show considerable promise for our purpose. We were led to pose the following boundary value problem: in the cylindrical

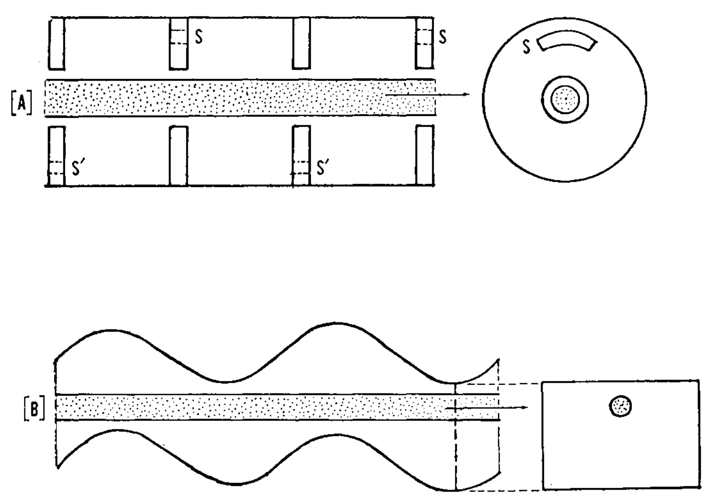


Fig. 1. Longitudinal and cross-sections of two periodic wave guides. (The dotted regions represent cylindrical beams of electrons—or other charged particles—in which the arrows indicate direction of their drift velocities.) In (A), the guide consists of a cylindrical pipe loaded with periodically spaced punctured discs, in each of which there is an off-center slot opening (*S* or *S′*) between adjacent cavity regions. (The slot position in a given disc, e.g., *S*, is rotated 180 degrees from the slot positions *S*' of its neighboring discs. In (B), the guide is a hollow pipe of rectangular section, the walls of which are deformed sinusoidally along its whole length. In both, all guide surfaces are supposed perfectly conducting. Information concerning propagation in these two guide types may be found in Pierce (1950) and Snyder (1967), respectively.

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[2]Department of Mathematics, Southern Illinois University, Carbondale, Illinois 62901.

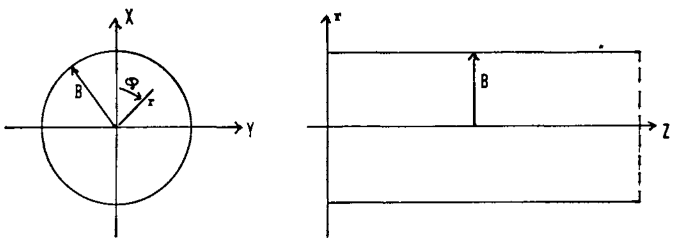


Fig. 2. Sections of the cylindrical region $V$, referred to cylindrical coordinates $r$, $\phi$, $z$. The central idea is to replace the study of the various special beam-and-guide systems by the study of propagation in the beam region V under the influence of assigned impedances at the lateral boundary $r = B$ of $V$.

domain $V$ (Fig. 2) defined by the inequalities

$$0 \leq r < B, \; |z| < \infty$$

(in the usual cylindrical coordinates $r$, $\phi$, $z$), find solutions of Maxwell's steady-state homogeneous equations which (a) have the character of waves propagating without attenuation in the direction $z > 0$, and (b) at the boundary $r = B$, they satisfy the conditions

$$\frac{E_z}{H_\phi} = -Z_1, \text{ and } \frac{E_\phi}{H_z} = Z_2, \tag{1}$$

where $Z_1$, $Z_2$ are assigned functions of frequency and of position ($\phi$, $z$) on $r = B$. The quantities $Z_1$ and $Z_2$ are called *impedances* of the bounding wall $r = B$. (It is not possible at this early stage to give an exact description of the impedance functions $Z_1$, $Z_2$ and, indeed, it is one of the objectives of analysis to discover physically meaningful criteria for their specification.) The left-hand sides in (1) are, of course, quotients of those components of the electric and magnetic vectors **E** and **H** which are tangent to the boundary.

In this paper, we shall begin by setting forth a brief but indispensable summary of some results published elsewhere (Snyder (1971)). From this summary, it will be seen why knowledge of the individual modes of propagation in $V$ nevertheless fails to provide the complete determination of the field. We then attack the problem with the aid of a technique whose power is adequate to the task—the Laplace transform—obtaining therefrom certain contour integral representations of the fields. These representations will then be specialized to the case of axially symmetric fields, with the aid of which a particular example is then solved completely. Finally, we conclude with some observations concerning complex contour integral field representations, with the aim of suggesting how it might be possible to avoid some of the analytical labor involved in the inversion of Laplace transforms.

## SUMMARY OF SOME PROPERTIES OF MODES OF PROPAGATION WITH CONSTANT IMPEDANCE WALLS

The problem stated above has been solved by Snyder (1971) for the case of arbitrary but *constant* wall impedances in the axially symmetric case. (All of the expressions we now write are in MKS units, in the notation and terminology of the treatise of Stratton (1941).)

When the fields **E** and **H** are axially symmetric, then their non-identically-vanishing components split into E- and H-waves. Let $k = \omega/c$, where $\omega = 2\pi f$ ($f > 0$ being an arbitrary but fixed frequency), and $c$ ($\simeq 3 \times 10^8$ meters/sec.) the velocity of phase propagation of plane waves in vacuum. Define

$$\gamma = \sqrt{\beta^2 - k^2}, \; \beta > k, \text{ and}$$
$$\gamma = \sqrt{k^2 - \beta^2}, \; \beta \leq k, \tag{2}$$

where $\beta$ is a (real) constant to be determined. Then axially symmetric solutions of Maxwell's equations, valid in $V$, are given by

$\beta \geq k$

| *E-waves* | *H-waves* |
|---|---|
| $E_r = -\frac{i\beta}{\gamma} A_0 I_1(\gamma r)$ | $H_r = -\frac{i\beta}{\gamma} B_0 I_1(\gamma r)$ |
| $H_\phi = -\frac{i\omega\epsilon}{\gamma} A_0 I_1(\gamma r)$ | $E_\phi = \frac{i\omega\mu}{\gamma} B_0 I_1(\gamma r)$ (3) |
| $E_z = A_0 I_0(\gamma r)$ | $H_z = B_0 I_0(\gamma r)$ |

$\beta \leq k$

| *E-waves* | *H-waves* |
|---|---|
| $E_r = -\frac{i\beta}{\gamma} C_0 J_1(\gamma r)$ | $H_r = -\frac{i\beta}{\gamma} D_0 J_1(\gamma r)$ |
| $H_\phi = -\frac{i\omega\epsilon}{\gamma} C_0 J_1(\gamma r)$ | $E_\phi = \frac{i\omega\mu}{\gamma} D_0 J_1(\gamma r)$ (4) |
| $E_z = C_0 J_0(\gamma r)$ | $H_z = D_0 J_0(\gamma r)$ |

in which each expression on the right is multiplied by the phase factor, $\exp i(\beta z - \omega t)$; $A_0$, $B_0$, $C_0$, $D_0$ are constants of integration; $\epsilon$ and $\mu$ the usual permittivities of vacuum; and $J_n(x)$, $I_n(x)$ are the ordinary and modified Bessel functions of order $n$, respectively.

Now let $Z_1$ and $Z_2$ in (1) both be constant (i.e., independent of wall position $\phi$, $z$ but possibly functions of the frequency parameter, $k$). When we apply to field components (3) and (4) the boundary conditions (1), we are led to the following equations: for (3),

$$\frac{xI_0(x)}{I_1(x)} = kB\frac{iZ_1}{Z_0} \text{ (E-waves) (a)} \tag{5}$$
$$= kB\frac{iZ_0}{Z_2} \text{ (H-waves) (b)}$$

and for (4),

$$\frac{xJ_0(x)}{J_1(x)} = kB\frac{iZ_1}{Z_0} \text{ (E-waves) (a)} \tag{6}$$
$$= kB\frac{iZ_0}{Z^2} \text{ (H-waves) (b)}$$

in which $Z_0 = \sqrt{\mu/\epsilon}$, and $x = \gamma B$. For given $Z_1$ and $Z_2$, and any $kB > 0$, (5) and (6) are thus equations to determine $\gamma$ and hence the phase-shift constant $\beta$, from (2). For the remainder of our summary, it will suffice to concentrate on the E-waves, so that we can write

$$kB\frac{iZ_1}{Z_0} = \frac{xI_0(x)}{I_1(x)}, \; \beta > k, \text{ (a) and}$$
$$kB\frac{iZ_1}{Z_0} = \frac{xJ_0(x)}{J_1(x)}, \; \beta \leq k. \text{ (b)} \tag{7}$$

It may be shown that (7) has real solutions, $x$, only for real values of the left-hand side, hence of $iZ_1$, so that $Z_1$ itself must be purely imaginary. (Then $\beta$ is real and positive, and the solutions (3) and (4) do indeed represent waves propagating without attenuation in the direction $z > 0$.) Hence let $kB(iZ_1/Z_0)$ be real. Then there are two cases. If $kB(iZ_1/Z_0) > 2$, then (7a) has exactly one real positive solution $X_0$, while (7b) has a countable unbounded infinite sequence of real positive solutions $X_1, X_2, X_3, \ldots$, where $X_j$ lies between the $j$-th and $(j+1)$-*st* positive zeros of $J_1(x)$ $(j = 1,2,3,\ldots)$. If $kB(iZ_1/Z_0) \leq 2$, then (7a) has no real solutions at all; however (7b), in addition to its solutions $X_j$ of the type just mentioned, also has a real solution $U_0$ satisfying $0 < U_0 \leq x_1$, $x_1$ being the first positive zero of $J_1(x)$. We may thus regard the solution $X_0$ of (7a) as transforming continuously into the solution $U_0$ as $kB(iZ_1/Z_0)$ decreases from values $>2$. (Since the functions on the right in (7) are both even, it follows that $-X_0$ and $-U_0$, and $-X_1, -X_2, \ldots$ are also solutions; however, as will be seen immediately, these negative solutions do not lead to different values of $\beta$.)

Thus far, our results are formally similar to the classical theory of modes of propagation in cylindrical perfectly conducting guides (see, e.g., Collin (1960)). To see that our solutions do possess certain essentially new qualities, it is sufficient to examine the field component $E_z$ (for example) in somewhat more detail. Suppose that $kB(iZ_1/Z_0) > 2$; then from the solution $\gamma B = X_0$ of (7a), we have from (2) the corresponding value of $\beta$, $\beta_0 = \sqrt{k^2 + (X_0/B)^2}$. In the same way, for the solutions $\gamma B = X_n$ of (7b), we have

$$\beta_n = \sqrt{k^2 - (X_n/B)^2},\ k \geq X_n/B \quad \text{(a), and} \qquad (8)$$
$$\beta_n = i\sqrt{(X_n/B)^2 - k^2},\ k < X_n/B \quad \text{(b).}$$

Then the *total* component $E_z$ is given by a linear combination of all these modes of propagation; from (3) and (4), we have

$$E_z = A_0 I_0\left(X_0 \frac{r}{B}\right)e^{i\beta_0 z} + \sum_{n=1}^{m} C_n J_0\left(X_n \frac{r}{B}\right)e^{i\beta_n z} \qquad (9)$$
$$+ \sum_{n=m+1}^{\infty} C_n J_0\left(X_n \frac{r}{B}\right)e^{-|\beta_n| z}$$

in which $m$ is the largest positive integer for which 8(a) holds. (If $kB(iZ_1/Z_0 \leq 2$, then the first term in (9) is replaced by $C_0 J_0\left(U_0 \frac{r}{B}\right)\exp(i\beta_0 z)$ with $\beta_0 = \sqrt{k^2 - (U_0/B)^2}$.) Just as in the classical problem, we see that all but a finite number of the modes with $\beta \leq k$ are cut off. How are the relative mode amplitudes, the constants $A_0$, $C_0$, $C_1, \ldots$, to be determined? If we evaluate (9) on the plane $z = 0$, we obtain

$$E_z(r,0,\omega) = A_0 I_0\left(X_0 \frac{r}{B}\right) + \sum_{n=1}^{\infty} C_n J_0\left(X_n \frac{r}{B}\right),\ kB(iZ_1/Z_0) > 2 \qquad (10)$$
$$= C_0 J_0\left(U_0 \frac{r}{B}\right) + \sum_{n=1}^{\infty} C_n J_0\left(X_n \frac{r}{B}\right),\ kB(iZ_1/Z_0) \leq 2$$

Now, for perfectly conducting guides, the series (10) (in which case $Z_1 = 0$, and we would have the second series) is a series of *orthogonal* Bessel functions (then $U_0, X_1, X_2, \ldots$ reduce to zeros of $J_0(x)$). Since, in that case, the Bessel sequence is also complete, the constants $C_0, C_1, C_2, \ldots$ are simply the expansion coefficients of an assigned value of $E_z(r,0,\omega) = F(r)$ (say) in its Fourier-Bessel series on $0 \leq r < B$. But it is not difficult to show that the sequences of Bessel functions appearing in (10) are orthogonal on $0 \leq r < B$ in neither case, and therefore the relative amplitudes cannot be determined in such a simple and obvious way. With these remarks, our summary is concluded and we proceed with our analysis.

## LAPLACE TRANSFORMS OF THE FIELD EQUATIONS AND THEIR SOLUTION

For propagation in cylindrical domains with general (and constant) impedance walls, we have shown that the modes of propagation do not constitute a set of complete orthogonal functions on the plane $z = 0$. To obtain the relative mode amplitudes, or expansion coefficients—and therewith, the unique determination of the field—it is thus necessary to employ a technique which introduces the boundary data at $z = 0$ at the outset. Such a method is furnished by the Laplace transform.

First, let $F(x_1, x_2, \ldots, x_p)$ $(p \geq 1)$ be any function of the $p$ arguments $x_1, x_2, \ldots, x_p$ which—for all $x_2, x_3, \ldots, x_p$—is bounded and measurable on $x_1 \geq 0$. Then the Laplace transform $\bar{F} = L(F)$, of $F$ with respect to $x_1$ is given by

$$L(F) = \bar{F}(s, x_2, \ldots, x_p) = \int_0^{\infty} e^{-sx_1} F(x_1, x_2, \ldots, x_p)\,dx_1$$

if the integral on the right exists for all sufficiently large values of the real part of the complex parameter $s = \sigma + i\tau$. If, furthermore, $F$ is also at least twice-differentiable with respect to $x_1$, then its Laplace transform enjoys also the well-known operational properties

$$L\left(\frac{\partial F}{\partial x_1}\right) = s\bar{F} - F(0, x_2, x_3, \ldots, x_p)$$
$$L\left(\frac{\partial^2 F}{\partial x_1^2}\right) = s^2 \bar{F} - sF(0, x_2, \ldots, x_p) - F_{x1}(0, x_2, \ldots, x_p)$$

(with $F_{x1}(0, x_2, \ldots, x_p)$ denoting the value of $\partial F/\partial x_1$ at $x_1 = 0$).

We now take Laplace transforms, with respect to the $z$-coordinate, of the steady-state Maxwell's homogeneous equations,

$$\nabla \times \mathbf{H} + i\omega\epsilon\,\mathbf{E} = 0,\ \nabla \times \mathbf{E} - i\omega\mu\,\mathbf{H} = 0 \qquad (11)$$

when the latter are written in the cylindrical coordinates $r, \phi, z$ appropriate to the open domain $V'$ for which $0 \leq r < B$ and $z > 0$. There is the usual ambiguity in the angular coordinate, $\phi$: in order that both $\mathbf{E}$ and $\mathbf{H}$, together with their transforms, be one-valued functions of position on $V'$, they must be periodic functions of $\phi$ with period $2\pi$. Thus we may suppose that the components of $\mathbf{E}$ and $\mathbf{H}$ in (11) are multiples of $exp(in\phi)$, and we obtain from (11) Laplace transforms of the Fourier $\phi$-components of $\mathbf{E}$ and $\mathbf{H}$. In particular, for the transforms of $E_r$, $E_\phi$, $H_r$, $H_\phi$, computation yields

$$\bar{E}_r = \frac{1}{s^2 + k^2}\left[s\frac{\partial \bar{E}_z}{\partial r} - \frac{n\omega\mu}{r}\bar{H}_z + sE_r(0) + i\omega\mu H_\phi(0)\right],$$
$$\bar{E}_\phi = \frac{1}{s^2 + k^2}\left[\frac{ins}{r}\bar{E}_z - i\omega\mu\frac{\partial \bar{H}_z}{\partial r} + sE_\phi(0) - i\omega\mu H_r(0)\right], \qquad (12)$$
$$\bar{H}_r = \frac{1}{s^2 + k^2}\left[\frac{n\omega\epsilon}{r}\bar{E}_z + s\frac{\partial \bar{H}_z}{\partial r} - i\omega\epsilon E_\phi(0) + sH_r(0)\right], \text{ and}$$
$$\bar{H}_\phi = \frac{1}{s^2 + k^2}\left[i\omega\epsilon\frac{\partial \bar{E}_z}{\partial r} + \frac{ins}{r}\bar{H}_z + i\omega\epsilon E_r(0) + sH_\phi(0)\right]$$

(in which $E_r(0)$ denotes $E_r(r,n,0,\omega)$—the value at $z = 0$ of the $n$-th Fourier $\phi$-component of $E_r$; similarly for the other terms in (12) with zero argument). Formulas (12) show that the transforms of the transverse components are determined if we know the transforms $\bar{E}_z$, $\bar{H}_z$ of the axial components. But it is well-known that $E_z, H_z$ themselves both satisfy the scalar Helmholtz equation (see, e.g., Stratton (1941), p. 33)

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial G}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 G}{\partial \phi^2}+\frac{\partial^2 G}{\partial z^2}+k^2 G=0$$

whose Laplace transform must therefore satisfy

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\bar{G}}{dr}\right)+\left(s^2+k^2-\frac{n^2}{r^2}\right)\bar{G}=sG(r,n,0,\omega)+G_z(r,n,0,\omega) \quad (13)$$

which is a non-homogeneous Bessel equation. Letting $G$ be either $E_z$ or $H_z$, we find for the general solutions

$$\begin{aligned}\bar{E}_z &= A_n J_n(\gamma r)+F_n(r)\\ \bar{H}_z &= B_n J_n(\gamma r)+G_n(r)\end{aligned} \quad (14)$$

where

$$F_n(r)=-\frac{\pi i}{2}\left[J_n(\gamma r)\int_r^B H_n^{(1)}(\gamma x)E_1(x,s)x\,dx + H_n^{(1)}(\gamma r)\int_0^r J_n(\gamma x)E_1(x,s)x\,dx\right] \quad (15)$$

$$G_n(r)=-\frac{\pi i}{2}\left[J_n(\gamma r)\int_r^B H_n^{(1)}(\gamma x)H_1(x,s)x\,dx + H_n^{(1)}(\gamma r)\int_0^r J_n(\gamma x)H_1(x,s)x\,dx\right]$$

In (14) and (15), $\gamma^2 = s^2 + k^2$; $H_n^{(1)}(x) = J_n(x) + iY_n(x)$, Hankel's function of the first kind; and $E_1(r,s)$, $H_1(r,s)$ are functions of the initial values of $E_z$, $H_z$ given by the right-hand side of (13):

$$E_1(r,s)=sE_z(0)+\left.\frac{\partial E_z}{\partial z}\right|_{z=0},\quad H_1(r,s)=sH_z(0)+\left.\frac{\partial H_z}{\partial z}\right|_{z=0} \quad (16)$$

To find the components of $\bar{\mathbf{E}}$, $\bar{\mathbf{H}}$ tangent to the boundary $r = B$, substitute (14) into (12) and obtain for the angular components

$$\bar{E}_\phi=\frac{1}{s^2+k^2}\left[\frac{ins}{r}A_nJ_n(\gamma r)-i\omega\mu\gamma B_nJ_n'(\gamma r)+\frac{ins}{r}F_n(r)-i\omega\mu\frac{d}{dr}G_n(r)+sE_\phi(0)-i\omega\mu H_r(0)\right] \quad (17)$$

$$\bar{H}_\phi=\frac{1}{s^2+k^2}\left[i\omega\epsilon\gamma A_nJ_n'(\gamma r)+\frac{ins}{r}B_nJ_n(\gamma r)+i\omega\epsilon\frac{d}{dr}F_n(r)+\frac{ins}{r}G_n(r)+i\omega\epsilon E_r(0)+sH_\phi(0)\right]$$

We are now ready to consider the boundary conditions (1), which we re-write here in the form

$$E_z+Z_1H_\phi=0,\quad H_z-Y_2E_\phi=0\quad \text{at } r=B \quad (18)$$

where $Y_2 = Z_2^{-1}$ (replacing $Z_2$ by the corresponding *admittance* $Y_2$ will give our expressions a certain symmetry they would otherwise lack). How are these conditions to be understood in terms of the Laplace *transforms* of the field components, rather than the field components themselves? In case $Z_1$, $Y_2$ are constant, such reinterpretation presents no difficulty; the Laplace transform of $E_z + Z_1H_\phi$ (for $r < B$) is simply $\bar{E}_z + Z_1\bar{H}_\phi$, and similarly for $H_z - Y_2E_\phi$. We then evaluate the latter quantities at $r = B$, and require that both vanish. This is equivalent to simply replacing the components in (1) by their transforms, i.e., (1) holds for the transforms of the field components when the impedances are constant. However, (1) and (18) clearly fail—in general—in case the impedances are functions of $z$. On the other hand, the transforms are analytic functions of the complex transform variable, $s$; and therefore (18) remain well-defined if the field components are replaced by their Laplace transforms and the impedances *by any analytic functions of* s. We shall interpret the boundary conditions in this sense in all of the *general* formulas to be derived below. We therefore write

$$\bar{E}_z+\bar{Z}_1\bar{H}_\phi=0,\quad \bar{H}_z-\bar{Y}_2\bar{E}_\phi=0 \text{ at } r=B \quad (19)$$

where $\bar{Z}_1 = \bar{Z}_1(s)$, $\bar{Y}_2 = \bar{Y}_2(s)$ ( $= (\bar{Z}_2(s))^{-1}$) and call the latter quantities *transform-impedances*. Our application of the Laplace transform thus pays an extra dividend: not only will it allow us to overcome the lack of orthogonality of the modes in the constant impedance case, but it also furnishes a new and unexpectedly simple approach to situations where the impedances are not constant.

To the transforms (14) and (17), we now apply the boundary conditions (19). Denote by $\bar{Z}_1^{(n)}(s)$, $\bar{Z}_2^{(n)}(s)$ the transform-impedances associated with the n-th Fourier component of the field. Then we obtain the equations

$$\left[J_n(\gamma B)+\frac{i\omega\epsilon}{\gamma}\bar{Z}_1^{(n)}J_n'(\gamma B)\right]A_n+\frac{ins}{\gamma^2 B}\bar{Z}_1^{(n)}J_n'(\gamma B)B_n=M_n(s)$$

$$\frac{ins}{\gamma^2 B}\bar{Y}_2^{(n)}J_n(\gamma B)\,A_n-\left[J_n(\gamma B)+\frac{i\omega\mu}{\gamma}\bar{Y}_2^{(n)}J_n'(\gamma B)\right]B_n=N_n(s) \quad (20)$$

where

$$M_n(s)=-F_n(B)-\frac{\bar{Z}_1^{(n)}}{s^2+k^2}\left[i\omega\epsilon F_n'(B)+\frac{ins}{B}G_n(B)+i\omega\epsilon E_r(B,n,0,\omega)+sH_\phi(B,n,0,\omega)\right] \quad (21)$$

$$N_n(s)=G_n(B)-\frac{\bar{Y}_2^{(n)}}{s^2+k^2}\left[\frac{ins}{B}F_n(B)-i\omega\mu G_n'(B)+sE_\phi(B,n,0,\omega)-i\omega\mu H_r(B,n,0,\omega)\right]$$

Let $D_n(s)$ be the determinant of coefficients of (20), given by

$$D_n(s)=\left(\frac{ns}{\gamma^2 B}\right)^2\bar{Z}_1^{(n)}\,\bar{Y}_2^{(n)}\,J_n^2(\gamma B)-\left[J_n(\gamma B)+\frac{i\omega\epsilon}{\gamma}\bar{Z}_1^{(n)}J_n'(\gamma B)\right]\left[J_n(\gamma B)+\frac{i\omega\mu}{\gamma}\bar{Y}_2^{(n)}J_n^1(\gamma B)\right] \quad (22)$$

Then the solutions of (20) are

$$A_n(s)=-\frac{\left[J_n(\gamma B)+\frac{i\omega\mu}{\gamma}\bar{Y}_2^{(n)}J_n^1(\gamma B)\right]M_n(s)+\frac{ins}{\gamma^2 B}\bar{Z}_1^{(n)}J_n(\gamma B)N_n(s)}{D_n(s)}$$

$$B_n(s)=-\frac{\frac{ins}{\gamma^2 B}\bar{Y}_2^{(n)}J_n(\gamma B)M_n(s)+\left[J_n(\gamma B)+\frac{i\omega\epsilon}{\gamma}\bar{Z}_1^{(n)}J_n^1(\gamma B)\right]N_n(s)}{D_n(s)} \quad (23)$$

Upon substituting (23) into (14), which we re-write here in the form

$$\begin{aligned}\bar{E}_z(r,n,s,\omega) &= A_n(s)J_n(\gamma r)+F_n(r),\\ \bar{H}_z(r,n,s,\omega) &= B_n(s)J_n(\gamma r)+G_n(r)\quad (24)\\ &(\gamma^2=s^2+k^2)\end{aligned}$$

we have the Laplace transforms of the n-th Fourier component of $E_z$ and $H_z$. To recover $E_z$ and $H_z$ themselves from (24), we have, using well-known theorems on the inversion of Laplace

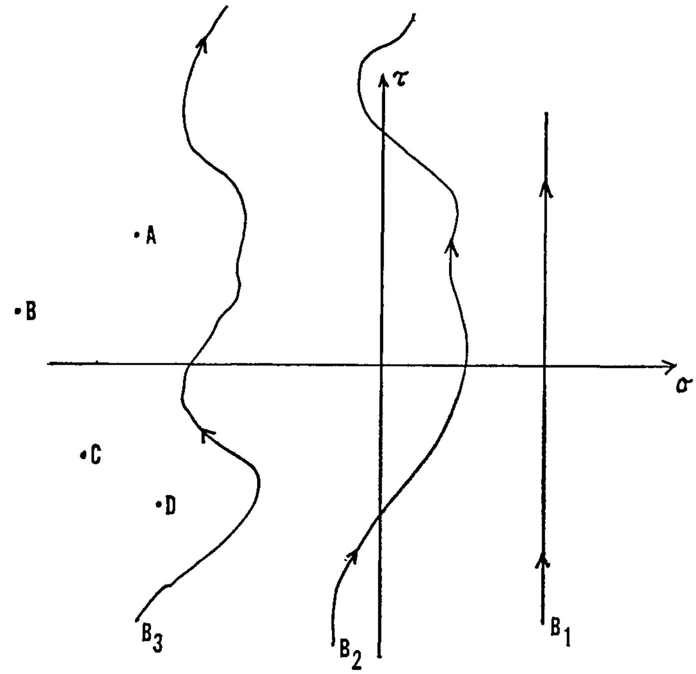


Fig. 3. The contours $B_1$, $B_2$ and $B_3$ are equivalent Bromwich paths for an analytic function $F(s)$ having singular points $A$, $B$, $C$, $D$, since the latter lie to the left of each of the paths.

transforms (see, e.g., McLachlan (1939), Ch. IV; and/or Dettman (1965), p. 400f.),

$$E_z(r,n,z,\omega) = \frac{1}{2\pi i}\int_{B_1} \bar{E}_z(r,n,s,\omega)e^{sz}ds \tag{25}$$

and similarly for $H_z$. In (25), $B_1$ denotes a *Bromwich contour* $Re(s) = h =$ const., where $h$ is chosen so that those singular points of $\bar{E}_z$ which are physically relevant to the field lie in the half-plane $Re(s) < h$ (Fig. 3). (By the term "Bromwich contour", we also understand any Jordan path $C$ which is topologically equivalent to $B_1$ with respect to the singularities, i.e., so that $C$ may be deformed continuously into $B_1$, and conversely, without crossing any singular point.) Once the transforms are inverted, we multiply by $exp(in\phi)$ and sum over $n$, obtaining finally the Fourier developments

$$E_z = \sum_{n=-\infty}^{\infty} E_z(r,n,z,\omega)e^{in\phi},\quad H_z = \sum_{n=-\infty}^{\infty} H_z(r,n,z,\omega)e^{in\phi} \tag{26}$$

If one substitutes the transforms (24) into (12), then one would obtain formulas for the Laplace transforms of the transverse components. However, it is generally a simpler matter to obtain those components directly, using Maxwell's equations and the formulas (26).

## LAPLACE TRANSFORMS OF AXIALLY SYMMETRIC FIELDS; AN EXAMPLE FOR WHICH THE INVERSION IS EXPLICITLY CALCULABLE

For fields which are axially symmetric, i.e., independent of the angular coordinate, the preceding formulas simplify radically. Setting $n = 0$ and dropping subscripts wherever appropriate, we have

$$\bar{E}_z(r,s,\omega) = A(s)J_0(\gamma r)+F(r),\quad \bar{H}_z(r,s,\omega) = B(s)J_0(\gamma r)+G(r) \tag{27}$$

in which

$$A(s) = \frac{M(s)}{J_0(\gamma B) - \frac{i\omega\epsilon}{\gamma}\bar{Z}_1 J_1(\gamma B)}, \tag{28}$$

$$B(s) = -\frac{N(s)}{J_0(\gamma B) - \frac{i\omega\mu}{\gamma}\bar{Y}_2 J_1(\gamma B)}$$

where

$$M(s) = -F(B) - \frac{\bar{Z}_1}{s^2+k^2} \times [\, i\omega\epsilon F'(B) + i\omega\epsilon E_r(B,0,\omega) + sH_\phi(B,0,\omega)] \tag{29}$$

$$N(s) = G(B) - \frac{\bar{Y}_2}{s^2+k^2} \times [\, -i\omega\mu G'(B) + sE_\phi(B,0,\omega) - i\omega\mu H_r(B,0,\omega)]$$

and where $F,G,F',G'$ are given by

$$\begin{aligned} F(B) &= -\frac{\pi i}{2} H_0^{(1)}(\gamma B)\int_0^B J_0(\gamma x)E_1(x,s)xdx \\ G(B) &= -\frac{\pi i}{2} H_0^{(1)}(\gamma B)\int_0^B J_0(\gamma x)H_1(x,s)xdx \\ F'(B) &= \frac{\pi i}{2}\gamma\, H_1^{(1)}(\gamma B)\int_0^B J_0(\gamma x)E_1(x,s)xdx \\ G'(B) &= \frac{\pi i}{2}\gamma\, H_1^{(1)}(\gamma B)\int_0^B J_0(\gamma x)H_1(x,s)xdx \end{aligned} \tag{30}$$

The functions $F(r)$, $G(r)$ in (27) are given by (15) with $n = 0$.

It will suffice for the remainder of our exposition to discuss only $E_z$ and its transform, $\bar{E}_z$. Evaluating $M(s)$ in (29) with the aid of (30), computation gives, upon setting $z_1 = i\bar{Z}_1/Z_0$,

$$M(s) = \frac{\pi i}{2}\left[H_0^{(1)}(\gamma B) - \frac{kz_1}{\gamma}H_1^{(1)}(\gamma B)\right]\int_0^B J_0(\gamma x)E_1(x,s)xdx - \frac{\bar{Z}_1}{\gamma^2}\left[i\omega\epsilon E_r(B,0,\omega) + sH_\phi(B,0,\omega)\right]$$

which enables us to compute $A(s)$ in (28). When the resulting expression for $A(s)$ is substituted into the first of formulas (27), we obtain for the transform $\bar{E}_z$,

$$E_z(r,s,\omega) =$$

$$\frac{\pi i}{2}\left[\int_0^B J_0(\gamma x)E_1(x,s)x\,dx\right]\frac{H_0^{(1)}(\gamma B)-\frac{k}{\gamma}z_1H_1^{(1)}(\gamma B)}{J_0(\gamma B)-\frac{k}{\gamma}z_1J_1(\gamma B)}J_0(\gamma r)$$

$$-\frac{Z_1}{\gamma^2}\frac{sH_\phi(B,0,\omega)+i\omega\epsilon E_r(B,0,\omega)}{J_0(\gamma B)-\frac{k}{\gamma}z_1J_1(\gamma B)}J_0(\gamma r) \tag{31}$$

$$-\frac{\pi i}{2}\left[J_0(\gamma r)\int_r^B H_0^{(1)}(\gamma x)E_1(x,s)x\,dx + H_0^{(1)}(\gamma r)\int_0^r J_0(\gamma x)E_1(x,s)x\,dx\right]$$

Notice that data concerning the fields on the input plane $z = 0$ appears in (31) explicitly, through the quantities

$$H_\phi(B,0,\omega),\ E_r(B,0,\omega) \text{ and } E_1(r,s) = sE_z(r,0,\omega) + \left.\frac{\partial E_z}{\partial z}\right|_{z=0} \tag{32}$$

$E_z$ itself now follows from (31) upon applying to it the inversion formula (27) with $n = 0$.

The inversion of (31) as it stands, i.e., with only general restrictions of a physically reasonable but otherwise unspecified nature on the quantity $E_1(r,s)$, would be a somewhat formidable task. Instead, we shall give an *example* for which most of the initial quantities vanish, is such that the integrals in (31) are all explicitly evaluable, and which is of some technical interest in its own right. At $z = 0$, suppose that all components of **E** and **H** vanish, together with their derivatives, except for $E_z$; and that, for $z < 0$, $E_z = E_0\, e^{i\beta z}$, where $E_0 = \text{const.} \neq 0$, and $\beta$ is some fixed but arbitrary positive real number. We have thus supposed the fields in our impedance-wall guide to be excited by a simple plane wave of $E_z$. Then $E_1(r,s) = E_0(s + i\beta)$ and $sH_\phi(B,0,\omega) + i\omega\epsilon E_r(B,0,\omega) = 0$. For the integrals in (31), we have the theorem (Watson (1958), p. 133)

$$\int x^{\nu+1}C_\nu(x)\,dx = x^{\nu+1}C_{\nu+1}(x)$$

where $C_\nu(x)$ is any solution of Bessel's equation of order $\nu$. Then the integrals in (31) are given by

$$\int_0^r J_0(\gamma x)x\,dx = \frac{\gamma r J_1(\gamma r)}{\gamma^2}$$

$$\int_r^B H_0^{(1)}(\gamma x)x\,dx = \frac{1}{\gamma^2}\left(\gamma B H_1^{(1)}(\gamma B) - \gamma r H_1^{(1)}(\gamma r)\right)$$

and when these expressions are substituted into (31), one obtains

$$\bar{E}_z(r,s,\omega) =$$

$$\frac{\pi i}{2}E_0B^2\frac{J_1(\gamma B)}{\gamma B}\,\frac{\gamma BH_0^{(1)}(\gamma B)-kBz_1H_1^{(1)}(\gamma B)}{\gamma BJ_0(\gamma B)-kBz_1J_1(\gamma B)}J_0(\gamma r)(s+i\beta)$$

$$+E_0\frac{s+i\beta}{\gamma^2}-\frac{\pi i}{2}E_0B^2\frac{H_1^{(1)}(\gamma B)}{\gamma B}J_0(\gamma r)(s+i\beta) \tag{33}$$

Then $E_z$ itself is given by

$$E_z(r,z,\omega) =$$

$$\frac{E_0B^2}{2\pi i}\int_C\left\{\frac{\pi i}{2}\frac{J_1(\gamma B)}{\gamma B}\,\frac{\gamma BH_0^{(1)}(\gamma B)-kBz_1H_1^{(1)}(\gamma B)}{\gamma BJ_0(\gamma B)-kBz_1J_1(\gamma B)}J_0(\gamma r)(s+i\beta)e^{sz}\right.$$

$$\left.+\frac{s+i\beta}{(\gamma B)^2}e^{sz}-\frac{\pi i}{2}\frac{H_1^{(1)}(\gamma B)}{\gamma B}J_0(\gamma r)(s+i\beta)e^{sz}\right\}ds \tag{34}$$

To evaluate (34), we require the following data: (a) the location of all singular points of the integrand in the plane of the complex variable, $s = \sigma + i\tau$; (b) specification of a branch of the multivalent function $\gamma(s) = (s^2 + k^2)^{1/2}$; (c) determination of a suitable Bromwich path, $B$; and finally (d) a sequence of bounded and closed Jordan paths, associated with B in such a way that we may apply the residue theorem to evaluation of the integral.

Let us now also suppose that $z_1 = iZ_1/Z_0 = \text{const.} > 2$; this will not only simplify the remainder of our analysis, but will allow us to exploit fully our knowledge of the roots of (7) discussed in the summary given above.

For the singular points of $E_z(r,s,\omega)$ in (34), we remark that both $J_0(s)$ and $J_1(s)/s$ are *even entire functions*; hence $J_0(\gamma r)$ and $J_1(\gamma B)/\gamma B$ possess no singularities on any bounded subset of the $s$-plane. On the other hand, $s = 0$ is a *branch point* of $H_0^{(1)}(s)$ and $H_1^{(1)}(s)$; hence the points $s = \pm ik$ where $\gamma = 0$, are branch points of $H_0^{(1)}(\gamma B)$ and $H_1^{(1)}(\gamma B)$, and thus of the transform. (The points $\pm ik$ are also simple poles of the second summand in (34).) The only remaining singular points are the zeros of $\gamma BJ_0(\gamma B) - kBz_1J_1(\gamma B)$, i.e., the solutions of

$$\frac{\gamma B\ J_0(\gamma B)}{J_1(\gamma B)} = kBz_1. \tag{35}$$

But (35) reduces to (7a) when $\gamma B$ is purely imaginary, and to (7b) when $\gamma B$ is real. Hence, in the notation already introduced, the roots of (35) are

$$\gamma = \pm iX_0/B, \pm X_1/B, \pm X_2/B, \ldots \tag{36}$$

and therefore the values of $s$ corresponding to these roots satisfy

$$s^2 = -(k^2 + (X_0/B)^2), -(k^2 - (X_1/B)^2), \ldots, -(k^2 - (X_m/B)^2),$$
$$(X_{m+1}/B)^2 - k^2, (X_{m+2}/B)^2 - k^2, \ldots \tag{37}$$

and so the values of $s$ themselves are

$$s = \pm i\sqrt{k^2+(X_0/B)^2}, \pm i\sqrt{k^2-(X_1/B)^2}, \ldots, \pm i\sqrt{k^2-(X_m/B)^2},$$
$$\pm\sqrt{(X_{m+1}/B)^2-k^2}, \pm\sqrt{(X_{m+2}/B)^2-k^2}, \ldots \tag{38}$$

Comparing (37) and (38) with the formulas (8) for the phases $\beta_j$, we see that

$$s = \pm i\beta_0, \pm i\beta_1, \ldots, \pm i\beta_m, \pm\beta_{m+1}, \pm\beta_{m+2} \tag{39}$$

(so that $j \geq m + 1$ corresponds to all those modes which are cut off, as the notation (37) is intended to suggest). It is not difficult to show that the quantities (39) are simple zeros of $\gamma BJ_0(\gamma B) - kBz_1J_1(\gamma B)$, and therefore they are *simple poles* of the transform. With this remark, the determination of the singular points of the transform is complete, and their disposition in the complex $s$-plane is sketched in Fig. 4.

The expressions just obtained for the poles of the transform also furnish criteria sufficient for the determination of a branch of $\gamma(s) = (s^2 + k^2)^{1/2}$. With the exception of $\pm i\beta_0$, all of the poles (38) satisfy $|Im(s)| < k$ and correspond to *real* values of $\gamma$, while $\pm i\beta_0$ correspond to the purely imaginary values $\gamma = \pm iX_0/B$. If we first set $s = i\tau$, and then $s = \sigma$, in the radicand of $\gamma$, it follows that $\gamma(s)$ must be a branch with the properties

$$\gamma(i\tau) = \pm i\sqrt{\tau^2 - k^2},\ |\tau| > k,$$
$$= \pm\sqrt{k^2 - \tau^2},\ 0 < |\tau| < k,$$
$$\gamma(\pm\sigma) = \pm\sqrt{k^2 + \sigma^2}$$

the ambiguous sign being unimportant, since either one of the values $\pm iX_0/B$ or $\pm X_j/B$ in (36) leads to the same expressions (39) for the poles. If we choose, say, the upper sign, then a

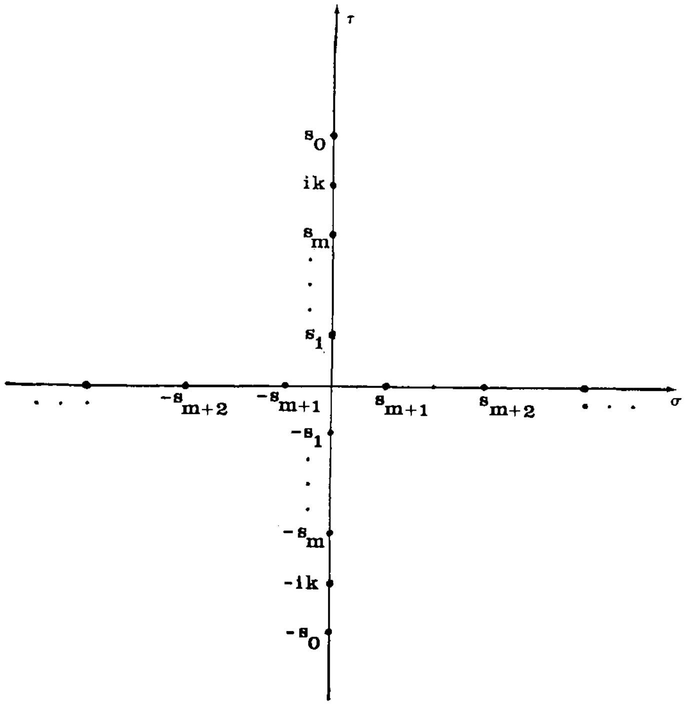


Fig. 4. Disposition of the branch points $\pm\ ik$ and the poles $s_j = \pm\ i\beta_j$ ($j = 0,1,2, \ldots, m$) and $s_{m+j} = \pm\ \beta_{m+j}$ ($j = 1,2,3, \ldots$) of the transform $\bar{E}_z(r,s,\omega)$ given by (33).

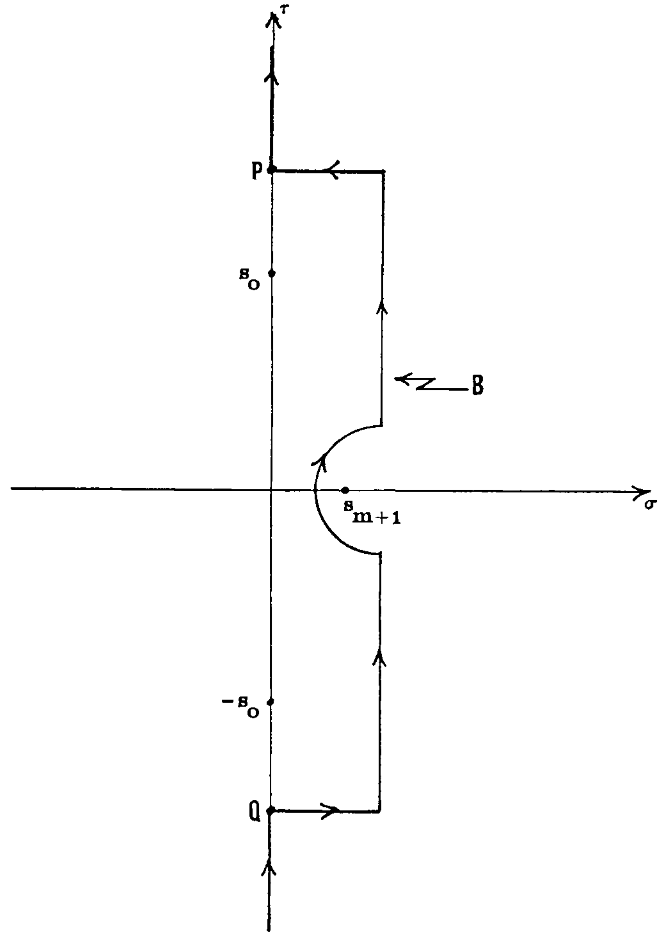


Fig. 5. The Bromwich contour B for the inversion of $\bar{E}_z(r,s,w)$ consists of the entire $\tau$-axis except for a rectangular portion connecting the points $P$ and $Q$, and lying to the right of the purely imaginary singular points, but avoiding all of the positive real poles $s_{m+j}$ (Fig. 4). The presence of the branch points $\pm\ ik$ on the left of $B$ is dealt with by excluding them from the interior of the closed paths $B_A$ associated with $B$, as described in Fig. 6.

determination of $\gamma(s)$ in the whole plane which reduces on the axes to the values last written may be made in many ways; one such determination adequate to our needs is as follows: write $s$ in polar coordinates, $s = \rho e^{i\Theta}$, and let $R$ be the real positive fourth root of the quantity $\rho^4 + k^4 + 2k^2\rho^2\cos2\Theta$; let $\alpha$ be the principal value of

$$\alpha = \tan^{-1}\left[\frac{\rho^2\sin2\Theta}{\rho^2\cos2\Theta + k^2}\right]$$

and, finally, define

$$\begin{aligned}\gamma(s) &= iRe^{i\alpha},\ Im(s) > k,\ Re(s) \leq 0 \qquad (40)\\ &= Re^{i\alpha},\ Im(s) < k\end{aligned}$$

(Some such explicit representation of $\gamma$ is needed, especially for study of the behavior of (34) on portions of the paths associated with the Bromwich path, e.g., the semicircle $C_A$ described below.)

Next, we consider how to choose a suitable Bromwich path $B$ for the inversion integral (34). Inasmuch as our original aim was to find, in particular, the amplitudes of the propagating waves, then we must choose $B$ so that the purely imaginary poles in (39) lie to the left of $B$. Then, of the remaining poles, those which are *negative* real will also lie to the left of $B$, and will represent the modes which are cut off. Finally, we must exclude from the left of $B$ all the positive real poles, since otherwise there would be a contribution to $E_z$ which grows without bound in the direction $z > 0$. A Bromwich path which meets these requirements is the contour $B$ sketched in Fig. 5.

Finally, we specify a collection of *closed* Jordan paths associated with $B$. Referring to (39), let $A$ be any real number satisfying $A > \beta_0$ but $A \neq \beta_j$ for $j = m + 1, m + 2, \ldots$, and locate the points $\pm iA$ (Fig. 6). With $s = 0$ as center, describe the semi-circle $C_A$, of radius $A$, in the left half-plane, and also circles of small radius $\rho$ about the branch points $\pm ik$. Finally, on either side of, parallel to and at small distance $\epsilon$ from, the lines $\sigma \pm ik$ ($\sigma \leq 0$), draw the segments $BC, \ldots, HK$. Finally, deleting the small arcs $BE$ and $FK$ from $C_A$, and $CD$ and $GH$ from the small circles $|s \pm ik| = \rho$, we obtain the closed path, $B_A$. By construction, $B_A$ encloses neither of the branch points $\pm\ ik$ and passes through no singular point. Hence, the only singularities of $\bar{E}_z(r,s,\omega)$ within $B_A$, for any such $A$, are all of the purely imaginary simple poles and some finite number of the negative real poles. Except at these poles, $\bar{E}_z(r,s,\omega)$ is thus *a one-valued analytic function of* s *within and on* $B_A$.

Letting $A$ be as restricted above, take (34) along $B_A$. Then,

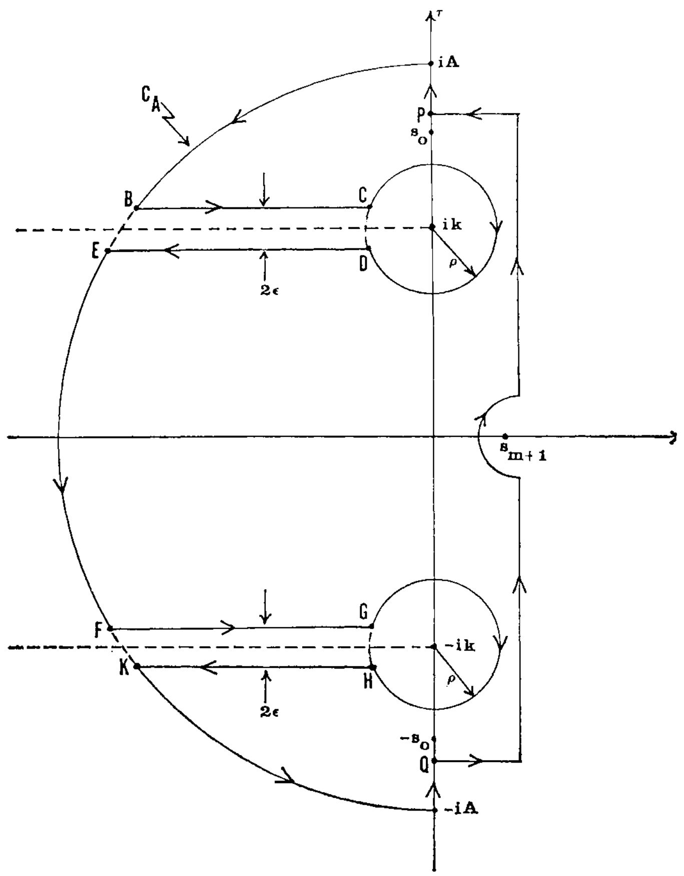


Fig. 6. A typical one of the closed Jordan paths $B_A$ associated with the Bromwich path B (Fig. 5), for inversion of the transform $E_z$ (r,s,k). (The positive sense of traversal of $B_A$ is indicated by the arrows.) Choose a radius $A$ which is larger than the modulus $|s_0| = \max(|s_j|)$, but which does not pass through any of the negative real poles, and join the points $\pm iA$ by the semi-circle $C_A$ in the left half-plane. To avoid enclosing the branch points $\pm ik$, exclude them from the interior of $B_A$ by the path portions $BCDE$ and $FGHK$, as shown. As $A$ is made to grow without bound through a suitable increasing sequence $A_1, A_2, A_3, \ldots$ the portions lying on $C_A$ recede indefinitely from the origin, and the portion on and to the right of the $\tau$-axis approach the path $B$. Then one lets the auxiliary parameters $\rho$ and $\epsilon$ tend to zero. The dotted lines emerging from $\pm ik$ are the branch cut lines $-x \pm ik$ $(x \geq 0)$.

by the residue theorem,

$$\frac{E_0B^2}{2\pi i}\int_{B_A} = E_0B^2 \sum \begin{pmatrix}\text{residues of integrand (34)} \\ \text{at the poles within } B_A\end{pmatrix} \quad (41)$$

We notice that the second and third terms of the integrand,

$$\frac{s+i\beta}{(\gamma B)^2}e^{sz} - \frac{\pi i}{2}\frac{H^{(1)}(\gamma B)}{\gamma B}J_0(\gamma r)(s+i\beta)e^{sz}$$

have no singularities within $B_A$. Hence the right-hand side of (41) is just the sum of the residues of

$$\frac{\pi i}{2}\frac{J_1(\gamma B)}{\gamma B}\frac{\gamma B H_0^{(1)}(\gamma B) - kBz_1H_1^{(1)}(\gamma B)}{\gamma B J_0(\gamma B) - kBz_1J_1(\gamma B)}J_0(\gamma r)(s+i\beta)e^{sz} \quad (42)$$

at the poles $\pm i\beta_j$ $(j = 0,1,\ldots,m)$ and some finite number of the poles $-\beta_{m+j}$ $(j = 1,2,3,\ldots)$. The computation of these residues is straightforward and, letting $I_A$ denote the sum on the right in (41), we have

$$I_A(r,z,\omega) = E_0\sum_j \frac{\pi i}{2}\left\{\gamma B J_1(\gamma B)\frac{\gamma B H_0^{(1)}(\gamma B) - kBz_1H^{(1)}(\gamma B)}{(1-kBz_1)\gamma BJ_0(\gamma B) + [kBz_1 - (\gamma B)^2]J_1(\gamma B)}\frac{s+i\beta}{s}J_0\left((\gamma B)\frac{r}{B}\right)e^{sz}\right\}\Bigg|_{s=s_j} \quad (43)$$

where $s_j$ runs through the poles (39) lying within $B_A$; (43) will be computed explicitly below.

We have thus found

$$\frac{E_0B^2}{2\pi i}\int_{B_A} = I_A(r,z,\omega)$$

On the other hand, if we now decompose the integral along $B_A$ into the sum of integrals along its constituient portions (Fig. 6), we have (starting at the point $-iA$ on $B_A$)

$$\int_{B_A} = \int_{-iA}^{iA} + \int_{C_A} + \int_B^C + \int_{C\,(|s-ik|=\rho)}^{D} + \int_D^E + \int_F^G + \int_{G\,(|s+ik|=\rho)}^{H} + \int_H^K \quad (44)$$

where $\int_{C_A}$ denotes the integral taken along the semi-circle $C_A$, with the small arcs $BE$ and $FK$ deleted. (In each integral in (44), the integrand is now the *entire* integrand in (34), not just (42).) For convenience of discussion, rearrange this sum of integrals in the following way:

$$\int_{B_A} = \int_{-iA}^{iA} + \int_{C_A} + \left\{\int_{C\,(|s-ik|=\rho)}^{D} + \int_{G\,(|s+ik|=\rho)}^{H}\right\} + \left\{\int_B^C - \int_E^D\right\} + \left\{\int_F^G - \int_K^H\right\} \quad (44a)$$

(the changes in sign meaning, as usual, that the sense of traversal of the affected arc has been reversed). If now we let $A \longrightarrow \infty$ (say, through a sequence of values $A_1, A_2, A_3, \ldots$ each of which satisfies the conditions imposed on $A$ above), then the first integral on the right in (44a) becomes the integral over the Bromwich contour $B$, specified above. Furthermore, with considerable analytical labor, it may be shown that for all sufficiently large positive values of $z$,

$$\int_{C_A} \longrightarrow 0 \text{ as } A \longrightarrow \infty \quad (44b)$$

(In fact, (44b) holds if $z > 3B$. A brief discussion of the kind of considerations involved in the proof of (44b) is given in the Appendix.) On the other hand, it is not difficult to show that the integrals in the first bracketed sum on the right in (44a) both $\longrightarrow 0$ as the radius $\rho \longrightarrow 0$. It follows that, for large $A$ and small $\rho > 0$, we can write (44a) in the form

$$\frac{E_0B^2}{2\pi i}\int_{-iA}^{iA} = I_A(r,z,\omega) - \frac{E_0B^2}{2\pi i}\left[\left\{\int_B^C - \int_E^D\right\} + \left\{\int_F^G - \int_K^H\right\}\right] + \Gamma(A,\rho) \quad (45)$$

where $\Gamma(A,\rho) \rightarrow 0$ as $A \rightarrow \infty$ and $\rho \rightarrow 0$.

The remaining integrals on the right in (45), taken along either side of the branch cut lines $\sigma \pm ik$ ($\sigma \leq 0$), do not vanish as $A \rightarrow \infty$ and $\epsilon \rightarrow 0$; they therefore represent the effect on the propagation of the multivalent character of $\gamma$. To obtain reasonable explicit expressions for them, first let $\epsilon > 0$ be small and the radius $A$ large but finite. Since the integrals in (44a) along the circular arcs about $\pm ik$ both vanish, extend each of the segments $BC, \ldots, KH$ to within a short distance $b$ of the $\tau$-axis, and also extend them to the left an amount equal to the radius $A$ (we may also make both $b$ and $A$ depend on $\epsilon$ in such a way that $A \rightarrow \infty$ and $b \rightarrow 0$, as $\epsilon \rightarrow 0$, if it should prove convenient to do so). Now, on the segment $BC$ (for example), put $s = -x + i(k+\epsilon)$ ($x \geq 0$) in the integral along $BC$, and write the result as an integral with respect to the real variable $x$. When this is done, we obtain—for all sufficiently small $\epsilon > 0$—

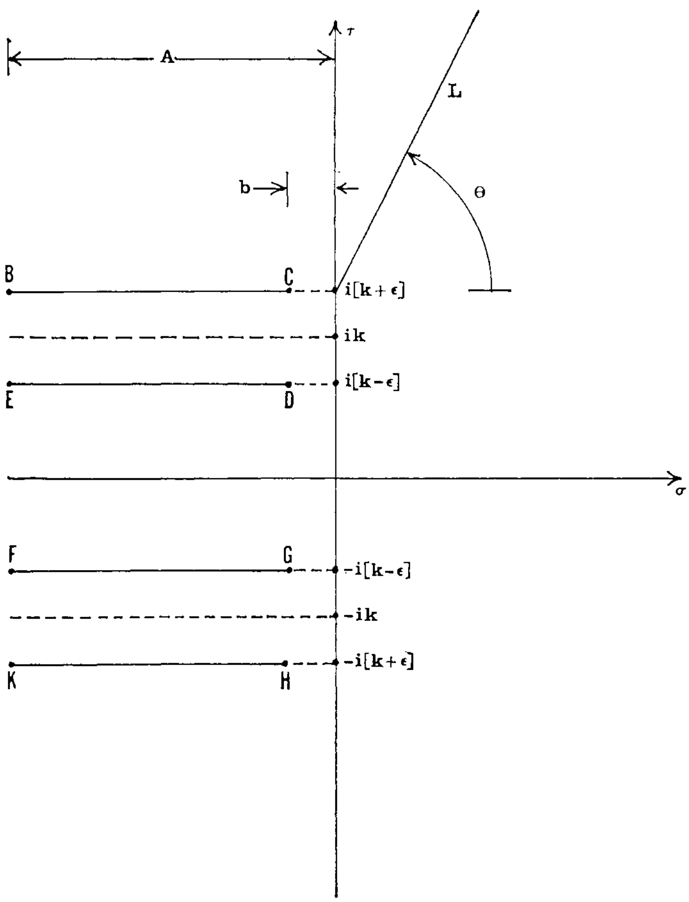


Fig. 7. A diagram illustrating the calculation of the values $\gamma_j$ of $\gamma = \sqrt{s^2+k^2}$, appropriate to the upper and lower sides of the branch cut lines. To compute (for example) $\gamma_1$, draw the half-line $L$ emerging from $i(k+\epsilon)$, the points of which are given by $s = i(k+\epsilon) + xe^{i\Theta}$. Substituting this into $\gamma^2 = s^2 + k^2$ and letting $\Theta \rightarrow \pi$, one obtains $\gamma_1 = e^{\pi i/2}(x - i\epsilon)^{1/2}(-x + i(2k+\epsilon))^{1/2}$. For sufficiently small $\epsilon > 0$, this reduces to the formula given in the text. The other values are calculated in the same way at the points $i(k-\epsilon)$ and $-i(k \pm \epsilon)$.

the integral

$$\frac{E_0B^2}{4}\, e^{ikz} \int_{b(\epsilon)}^{A(\epsilon)} N_1(x)\, J_0(\gamma_1 r)\, [i(\beta+k) - x]\, e^{-zx}\, dx \tag{46}$$

in which

$$N_1(x) = \frac{J_1(\gamma_1 B)}{\gamma_1 B}\, \frac{\gamma_1 B H_0^{(1)}(\gamma_1 B) - kBz_1 H_1^{(1)}(\gamma_1 B)}{\gamma_1 B J_0(\gamma_1 B) - kBz_1 J_1(\gamma_1 B)} - \frac{H_1^{(1)}(\gamma_1 B)}{\gamma_1 B} \tag{47}$$

where $\gamma_1$ is a determination of $\sqrt{s^2+k^2}$ appropriate to the *upper* side of the branch cut line $-x + ik$ ($x \geq 0$) (Fig. 7), as $\epsilon \rightarrow 0$, given by

$$\gamma_1 = X\, e^{i(\pi/2 + \Theta_2 - \Theta_1)},$$
$$X = [(x^2+\epsilon^2)(x^2+4k^2)]^{1/4},$$

and the phase $\Theta_2 - \Theta_1$ is given below. For the integrals in (45) over the remaining three segments, we obtain in the same way expressions entirely similar to (46) and (47), so that the entire bracketed quantity in (45) is given by

$$E_0B^2\, [e^{ikz}T_1(r,z,\omega) + e^{-ikz}T_2(r,z,\omega)], \tag{48}$$

where $T_1$ and $T_2$ are

$$T_1 = \int_{b(\epsilon)}^{A(\epsilon)} [N_1(x)J_0(\gamma_1 r) - N_2(x)J_0(\gamma_2 r)][i(\beta+k) - x]e^{-zx}dx,$$

$$T_2 = \int_{b(\epsilon)}^{A(\epsilon)} [N_3(x)J_0(\gamma_3 r) - N^4(x)J_0(\gamma_4 r)][i(\beta-k) - x]e^{-zx}dx. \tag{49}$$

In (49), $N_j(x)$ ($j = 1,2,3,4$) denotes the expression on the right of (47), with $\gamma_j$ written in place of $\gamma_1$; and $\gamma_j = X\, exp(i\Psi_j)$, where the phases $\Psi_j$ are given by

$$\Psi_1 = \pi/2 + \Theta_2 - \Theta_1, \quad \Psi_2 = -\pi/2 + \Theta_2 + \Theta_1,$$
$$\Psi_3 = \pi/2 - (\Theta_2 + \Theta_1), \quad \Psi_4 = -\pi/2 + \Theta_2 - \Theta_1$$

and in which $\Theta_2 \pm \Theta_1$ is given by (taking the principal values)

$$\Theta_2 \pm \Theta_1 = \tan^{-1}\left[\frac{\sqrt{x^2+4k^2}+x}{\sqrt{x^2+4k^2}-x}\right]^{1/2} \pm \tan^{-1}\left[\frac{\sqrt{x^2+\epsilon^2}-x}{\sqrt{x^2+\epsilon^2}+x}\right]^{1/2}$$

We now set (48) into (45) and pass to the limit as $\epsilon \rightarrow 0$, obtaining

$$E_z(r,z,\omega) = \frac{E_0B^2}{2\pi i}\int_B (\ldots)ds \tag{50}$$

$$= \lim_{A \rightarrow \infty} I_A(r,z,\omega) - \frac{E_0B^2}{4}\left[T_1(r,z,\omega)e^{ikz} + T_2(r,z,\omega)e^{-ikz}\right]$$

(where we have continued to write $T_1$ and $T_2$ for the *limits* of integrals (49)). The first term on the right in (50) is now the sum (43), taken over *all* the poles (39) with $Re(s) \leq 0$. The calculation of the resulting series is not difficult, and we obtain

$$\lim_{A \rightarrow \infty} I_A(r,z,\omega)$$
$$= -E_0\left[X_0 I_1(X_0)\, \frac{X_0K_0(X_0) + kBz_1K_1(X_0)}{(1-kBz_1)X_0I_0(X_0) + (kBz_1 + X_0^2)I_1(X_0)}\right]$$

$$\times I_0\left(X_0 \frac{r}{B}\right)\left(\frac{\beta+\beta_0}{\beta_0} e^{i\beta_0 z} + \frac{\beta-\beta_0}{\beta_0} e^{-i\beta_0 z}\right) \quad (51)$$
$$+\sum_{j=1}^{m} E_0 \frac{\pi i}{2}\left[ X_j J_1(X_j \frac{X_j H_0^{(1)}(X_j) - kBz_1 H_1^{(1)}(X_j)}{(1-kBz_1)X_j J_0(X_j) + (kBz_1 - X_j^2)J_1(X_j)}\right]$$
$$\times J_0\left(X_j \frac{r}{B}\right)\left(\frac{\beta+\beta_j}{\beta_j} e^{i\beta_j z} + \frac{\beta-\beta_j}{\beta_j} e^{-i\beta_j z}\right)$$
$$+\sum_{j=m+1}^{\infty} E_0 \frac{\pi i}{2}\left[\begin{matrix}\text{same as preceding}\\ \text{coefficient}\end{matrix}\right] J_0\left(X_j \frac{r}{B}\right) \frac{|\beta_j| - i\beta}{|\beta_j|} e^{-|\beta_j| z}$$

Comparing (50) and (51) with (9), we see that (51) *contains precisely the development* (9) *of* $E_z$, *with the expansion coefficients* $A_0, C_0, C_1, \ldots$ *in the latter now all uniquely determined.* Of course, (51) also contains waves traveling with the same phase shift factors, but in the direction $z < 0$. The presence of these waves is inevitable, in view of the fact that the plane $z = 0$ is the locus of a discontinuity in the *total field*, i.e., the field for which $E_z$ is given by (50) for $z > 0$, together with $E_z = E_0 exp(i\beta z)$ for $z < 0$. We could have avoided the presence of these backward traveling waves simply by modifying the Bromwich path $B$ so as to exclude from consideration all of the poles (39) lying on the lower half of the $\tau$-axis; or—since we have now actually obtained both sets of waves—we may observe that, in order that the input field $E_z = E_0 exp(i\beta z)$ should match smoothly (i.e., without reflections) into our impedance wall guide, one must superpose upon it at $z = 0$ precisely that portion of (51) consisting of the backward waves (but of opposite sign).

It remains to discuss the last two terms, $T_{1,2} exp(\pm ikz)$ of (50); do they possess physical significance of importance to the propagation? We notice, in the first place, that these terms represent inhomogeneous plane waves traveling in both $z$-directions with phase shift just equal to the free-space wave number $k$ (i.e., with phase velocity equal to $c$) and thus, with phases *entirely unaffected* by the boundary conditions. The boundary conditions only enter in the amplitude factors $T_{1,2}$ (since the integrands (49) contain $z_1 = iZ_1/Z_0$), both of which also depend on $z$. Since we have simple expressions available for the $\gamma_j$, we could—with sufficient labor—obtain approximate representations for $T_{1,2}$. Fortunately, it suffices instead to dispose of them with the aid of simple estimates.

For every $\epsilon > 0$, it may be shown that the functions

$$N_1(x)J_0(\gamma_1 r) - N_2(x)J_0(\gamma_2 r),\ N_3(x)J_0(\gamma_3 r) - N_4(x)J_0(\gamma_4 r)$$

in (49) are *bounded* for all $x \geq 0$. For some fixed $\epsilon$, let $M$ be a common bound. Then, from (49), we have the estimate

$$|T_{1,2}| = \left|\int_b^A (\cdots)dx\right| \leq M\int_b^A (|\beta| + k + x)e^{-zx}dx$$

or

$$|T_{1,2}| \leq M(|\beta| + k + A)\frac{e^{-bz} - e^{-Az}}{z} \quad (52)$$

The right-hand side of (52) is, on the one hand, bounded at $z = 0$; *but*, on the other hand, *it* $\rightarrow 0$ *as* $z \rightarrow \infty$. It follows that the portions of the field represented by these branch cut integrals do not represent an important part of the field at points of the guide which are not too near the input plane. They constitute, indeed, a kind of oscillatory *transient effect*, but in the distance $z$ along the guide instead of in the time. If we then neglect all those terms in (50) and (51) which $\rightarrow 0$ as $z \rightarrow \infty$, together with the backward waves, we finally obtain all *those waves which*, for the chosen input field, *propagate without attenuation in the domain* $0 \leq r < B$, $z > 0$:

$$E_z(r,z,\omega) \approx$$
$$-E_0 \frac{X_0 I_1(X_0)[X_0 K_0(X_0) + kBz_1 K_1(X_0)]}{(1-kBz_1)X_0 I_0(X_0) + (kBz_1 + X_0^2)I_1(X_0)} I_0\left(X_0 \frac{r}{B}\right)$$
$$\times \frac{\beta+\beta_0}{\beta_0} e^{i\beta_0 z} \quad (53)$$
$$+\sum_{j=1}^{m} E_0 \frac{\pi i}{2} \frac{X_j J_1(X_j)[X_j H_0^{(1)}(X_j) - kBz_1 H_1^{(1)}(X_j)]}{(1-kBz_1)X_j J_0(X_j) + (kBz_1 - X_j^2)J_1(X_j)} J_0\left(X_j \frac{r}{B}\right)$$
$$\times \frac{\beta+\beta_j}{\beta_j} e^{i\beta_j z}$$

(The transverse components $E_r$ and $H_\phi$ may now be written down directly from (53) with the aid of (3) and (4).)

## CONCLUSION AND SOME REMARKS ON COMPLEX INTEGRAL REPRESENTATIONS OF FIELDS

In conclusion, we remark that the Laplace transform of an axially symmetric field in a cylindrical impedance wall domain has been inverted explicitly, for a case of constant impedance and an especially simple choice of input field. Moreover, the technique reveals the possibility of making an effective attack in the case where the wall impedance depends on $z$, by allowing the impedance to be an analytic function of the complex transform variable, $s$. In particular, we conjecture that one might simulate the character of propagation in a periodic structure by allowing the impedance to be the Laplace transform of a *periodic function* of $z$.

Finally, we offer a few comments concerning the efficacy of the Laplace transform as a general method for the solution of such problems as we have considered above. The Bromwich integral (34) (with the path $B$) which inverts the transform is a representation of one of the components of **E** by means of a contour integral. But, as a matter of fact, quite general contour integral representations of electromagnetic fields of this character are readily constructed—without explicit reference to any integral transform. Consider, for example, the $E$-wave components $E_r$, $H_\phi$, $E_z$ of expressions (4). Let $\gamma$ and $\delta$ be any complex numbers satisfying $\gamma^2 = k^2 + \delta^2$; then $J_0(\gamma r)exp(\delta z)$ is a solution of the (angularly independent) Helmholtz equation satisfied by $E_z$, and it remains a solution if we multiply it by an arbitrary integrable function $F$ of $\delta$: $F(\delta)J_0(\gamma r)exp(\delta z)$; and we again obtain a solution if we integrate the last expression with respect to $\delta$ along any Jordan path $C$:

$$E_z = \frac{1}{2\pi i}\int_C F(\delta)J_0(\gamma r)e^{\delta z}d\delta$$

(where, of course, $C$ is any path for which the integral exists). The $E$-wave class for which the last expression defines $E_z$ thus has the components

$$E_r = -\frac{1}{2\pi i}\int_C F(\delta)\frac{\delta}{\gamma} J_1(\gamma r)e^{\delta z}d\delta$$
$$H_\phi = -\frac{1}{2\pi i}\int_C \frac{i\omega\epsilon}{\gamma} F(\delta)J_1(\gamma r)e^{\delta z}d\delta \quad (54)$$
$$E_z = \frac{1}{2\pi i}\int_C F(\delta)\ J_0(\gamma r)e^{\delta z}d\delta$$

Let us apply our (constant) impedance wall boundary conditions to expressions (54): we have, at $r = B$,

$$E_z + Z_1 H_\phi = \frac{1}{2\pi i}\int_C F(\delta)\left[J_0(\gamma B) - \frac{i\omega\epsilon}{\gamma} Z_1 J_1(\gamma B)\right] e^{\delta z} d\delta \quad (55)$$

and the right-hand side of (55) vanishes provided that

$$F(\delta) = \frac{G(\delta)}{J_0(\gamma B) - \frac{i\omega\epsilon}{\gamma} Z_1 J_1(\gamma B)} \quad (56)$$

for any choice of an analytic function $G(\delta)$ and path $C$ which neither passes through nor encloses a singularity of $G(\delta)$. If we substitute (56) into (54), we have—for $E_z$ in particular—

$$E_z = \frac{1}{2\pi i}\int_C \frac{G(\delta)}{J_0(\gamma B) - \frac{i\omega\epsilon}{\gamma} Z_1 J_1(\gamma B)} J_0(\gamma r) e^{\delta z} d\delta \quad (57)$$

The similarity of (57) and (31) is evident; in fact, comparing the two expressions in detail, if we set

$$G(\delta) = \frac{\pi i}{2}\left(H_0^{(1)}(\gamma B) - \frac{i\omega\epsilon}{\gamma} Z_1 H_1^{(1)}(\gamma B)\right)\int_0^B J_0(\gamma x) E_1(x,\delta) x dx$$

in (57), then (57) yields (53) *directly* if $C$ is a contour which encloses only the purely imaginary poles in (39) lying on the positive half of the $\tau$-axis. Of course, it would be desirable to be able to determine $G$ in a less *ad hoc* manner; we hope to return to this question in another place.

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## APPENDIX

In taking the inversion integral over the path $B_A$, it was asserted without proof that the portion of the integral over the large semi-circle $C_A$ in the left half-plane $\rightarrow 0$ as $A \rightarrow \infty$. The complete proof is lengthy and tedious. We shall give here a small part of the proof, sufficient to show the nature of the considerations involved. For this purpose, consider the least complicated terms in integrand (34)—the second and third—and form their line integral taken over that portion of $C_A$ joining the points $iA$ and $B$, for vanishingly small $\epsilon$. Let

$$I = \int_{iA}^{B}\left[\frac{s+i\beta}{(\gamma B)^2} - \frac{\pi i}{2}\frac{H_1^{(1)}(\gamma B)}{\gamma B}(s+i\beta) J_0(\gamma r)\right] e^{sz} ds \quad (A)$$

On the arc, we have $s = Ae^{i\Theta}$ and the corresponding values of $\gamma = iRe^{i\alpha}$. When $A$ is large, so is $|\gamma|$, and the appropriate asymptotic formulas for $H_1^{(1)}(\gamma B)$ and $J_0(\gamma r)$ (for $r > 0$) are

$$H_1^{(1)}(\gamma B) \simeq \sqrt{\frac{2}{\pi}}\frac{e^{i(\gamma B - 3\pi/4)}}{\sqrt{\gamma B}}, J_0(\gamma r) \simeq \sqrt{\frac{2}{\pi}}\frac{\cos(\gamma r - \pi/4)}{\sqrt{\gamma r}}$$

(see, e.g., Abramowitz & Stegun (1964), p. 364). Substituting these expressions into (A), we have the corresponding asymptotic representation of $I$:

$$I = \int_{\pi/2}^{\pi - \arcsin(k/A)}\left\{1 - \frac{i}{2}\left(\frac{B}{r}\right)^{1/2}\left[e^{-(B+r)R\cos\alpha} e^{-i(\pi + (B+r)R\sin\alpha)} + e^{-(B-r)R\cos\alpha} e^{-i(\pi/2 + (B-r)R\sin\alpha)}\right]\right\}\frac{Ae^{i\Theta} + i\beta}{A^2 e^{2i\Theta} + k^2} \times iAe^{i\Theta} e^{iAz\sin\Theta} e^{Az\cos\Theta} d\Theta \quad (B)$$

Taking absolute values in $(B)$, we have

$$|I| \leq \frac{A(A + |\beta|)}{|A^2 - k^2|}\int_{\pi/2}^{\pi - \arcsin(k/A)}\left\{1 + \left(\frac{B}{r}\right)^{1/2} \times \left[e^{-(B+r)R\cos\alpha} + e^{(B-r)R\cos\alpha}\right]\right\} e^{Az\cos\Theta} d\Theta \quad (C)$$

From the definition of $R$ and $\alpha$ in (40), it may be shown that, on the interval $[\pi/2, \pi\text{-}\arcsin(k/A)]$, $\cos\alpha$ is positive and satisfies $1/\sqrt{2} \leq \cos\alpha \leq 1$; also, by inspection of the expression for $R$, we see that on $C_A$, $R$ satisfies $\sqrt{A^2 - k^2} \leq R \leq \sqrt{A^2 + k^2}$. It follows that for the bracketed exponential terms in $(C)$, we have the estimate

$$e^{-(B+r)R\cos\alpha} + e^{-(B-r)R\cos\alpha} \leq e^{-(B+r)R/\sqrt{2}} + e^{-(B-r)R/\sqrt{2}} \leq 2e^{-(B-r)\sqrt{A^2-k^2}/\sqrt{2}}$$

Hence $(C)$ may be replaced by

$$|I| \leq \frac{A(A + |\beta|)}{|A^2 - k^2|}\left[1 + \left(\frac{B}{r}\right)^{1/2} 2e^{-(B-r)\sqrt{A^2-k^2}/\sqrt{2}}\right] \times \int_{\pi/2}^{\pi - \arcsin(k/A)} e^{-Az|\cos\Theta|} d\Theta \quad (D)$$

(since, on $[\pi/2, \pi\text{-}\arcsin(k/A)]$, $\cos\Theta \leq 0$). By inspection of the graph of $|\cos\Theta|$ on $[\pi/2, \pi]$, we have $|\cos\Theta| \geq \frac{2}{\pi}\Theta - 1$. The integral in (D) is thus dominated by the elementary integral,

$$\int_{\pi/2}^{\pi - \arcsin(k/A)} e^{-Az(2\Theta/\pi - 1)} d\Theta$$

and when this is evaluated, we finally obtain

$$|I| \leq \frac{A(A + |\beta|)}{|A^2 - k^2|}\left[1 + \left(\frac{B}{r}\right)^{1/2} 2e^{-(B-r)\sqrt{A^2-k^2}/\sqrt{2}}\right] \times \frac{\pi}{2Az}\left(1 - e^{-Az} e^{(2z/\pi)A\arcsin(k/A)}\right) \quad (E)$$

An application of l'Hospital's rule shows that $A\arcsin(k/A) \rightarrow k$ as $A \rightarrow \infty$; and it is thus clear that, for every $z > 0$ and all $r$ with $0 < r < B$, the expression on the right in $(E) \rightarrow 0$ as $A \rightarrow \infty$.

On other portions of $C_A$, it happens that the terms

$$e^{i(B+r)\gamma} e^{-\pi i} + e^{i(B-r)\gamma} e^{-\pi i/2}$$

do not lead to exponential functions with exponents which are negative on the domain of integration; it is then necessary to take $z > 3B$ (at least) to secure convergence, instead of merely $z > 0$. Also, when $r = 0$, then $J_0(\gamma r) = 1$ identically, and the integrals in this case have to be considered separately.

We remark, finally, that when the integral along $C_A$ of the first term of integrand (34) is considered, it is *not* necessary to set the transform impedance function $z_1(s) = \text{const.}$; it remains true that the integral $\rightarrow 0$ if $z_1(s)$ is (for example) any analytic function which is bounded for all sufficiently large $|s|$.

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