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Secondary preservice teachers' understanding of Euclidean geometry

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SECONDARY PRESERVICE TEACHERS’ UNDERSTANDING
OF EUCLIDEAN GEOMETRY

A Dissertation
Submitted
in Partial Fulfillment
of the Requirements for the Degree
Doctor of Education
Approved:

Dr. Linda May Fitzgerald, Chair

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May 2009
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ABSTRACT

The lack of mathematical understanding of elementary preservice teachers with respect to many mathematical topics has been well documented. Far fewer studies have been conducted on any content area of knowledge of secondary teachers and even less in the mathematics content area. The assumption by the general population is that mathematics majors are adequately prepared to teach the content of secondary mathematics because they major in mathematics. Mathematics educators, however, note the lack of conceptual understandings in many content areas in these mathematics teaching majors. While mathematics educators are aware of this deficiency, there is little research data at the secondary level to support those beliefs.

Euclidean geometry is regularly taught in high schools, a content area with which U.S. students have difficulty. Perhaps the reason students have difficulty learning geometry is that their teachers have not developed a deep knowledge of the subject themselves and are not able to teach their students with enough understanding to enable their students to gain mathematical proficiency in that content area. Therefore the focus for this study was secondary preservice teachers' knowledge of Euclidean geometry.

The research participants were 15 secondary mathematics education majors who were taking a mathematics methods course at a Midwestern state university. Mixed methods were used to ascertain the mathematical proficiency that secondary preservice teachers have in the content area of Euclidean geometry. A paper-and-pencil test revealed a wide span of mathematical knowledge ranging from the top three participants who correctly answered all 15 questions to the bottom three participants who only answered
eight of the 15 questions correctly. Follow-up, in-depth interviews were conducted with five participants: two from the high group, two from the low group, and one from the middle group.

The participants in the high group demonstrated conceptual understanding, procedural fluency, strategic competence, adaptive reasoning and a productive disposition—the five strands of mathematical proficiency (NRC, 2001). The participants in the low group, on the other hand, showed deficiencies in each of the five strands. The interviewed participant in the middle group shared more closely the characteristics of the participants in the high group than the participants in the low group. This study showed that successful completion of the required mathematics content courses does not guarantee the presence of a thorough understanding of the mathematics that secondary preservice teachers will be expected to teach.

The possible lack in mathematical proficiency in secondary preservice teachers needs to be brought to the attention of the mathematics community and documented so that improvements in teacher education can be made. Implications from this study may prompt teacher educators to integrate conceptually-based content along with pedagogy into the methods courses.
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CHAPTER 1

INTRODUCTION

My journey of professional development has taken me down roads I never anticipated. From the time I was in junior high, I knew I wanted to be a high school mathematics teacher. That looked like a straight path, from tutoring algebra students while in high school to completing my undergraduate mathematics major and student teaching at the secondary level. Mathematics was not always easy for me, but I enjoyed the challenges it presented. When I began teaching, I felt that the struggles I had experienced as a student made me a better teacher.

The master's program I chose was one specifically designed for secondary mathematics teachers, and it exposed me to some new content areas. Coincidentally, this was also about the time the National Council of Teachers of Mathematics (NCTM) published its *Curriculum and Evaluation Standards for School Mathematics* (NCTM, 1989). Up to this point, my journey was straightforward except for a slight detour (or what I perceived to be a detour at that time) when I took an elementary mathematics methods class. Since I knew so much about the content of secondary mathematics, I thought it might be helpful to know how elementary mathematics concepts, such as place value and fractions, were taught (or should be taught). In retrospect, I think my experience in that class—seeing how most of my classmates were struggling—set the stage for my entrance into the doctoral program. I became concerned about the mathematical preparation of elementary teachers. For example, I wondered how my colleagues in this elementary mathematics methods class would be able to teach
mathematics to elementary students and not pass on their own math anxiety to their students. My focus was to find a way to alleviate this lack of mathematical understanding in preservice elementary teachers.

I continued teaching, now at the college level. As an adjunct instructor, my usual assignments were either general education mathematics courses that would satisfy the liberal arts requirement or remedial intermediate algebra courses designed for students who were underprepared for college mathematics. Conversations with many of these students revealed that they first encountered difficulties with mathematics in upper elementary school or in algebra. This reinforced my concerns regarding the mathematical competence of elementary teachers. I renewed my goal of finding ways to improve mathematics education by working to improve the mathematical background of elementary teachers.

After a few years teaching these courses, I was given the opportunity to teach a secondary mathematics methods course. I looked at this course as a wonderful opportunity for professional development as well as a respite from my usual students who often disliked mathematics and lacked conceptual understanding of most mathematical topics.

Most of my preconceived notions about these secondary mathematics preservice teachers were correct. The students loved and appreciated mathematics. They were excited about teaching and were looking forward to student teaching where they could put their education into practice. I was surprised, however, by the lack of mathematical knowledge that students sometimes exhibited. One focus of our course was
communication, in particular effective communication of technical mathematics. I gave the students a routine algebra problem involving averages, designed to launch a more involved extended problem. While most of the students were able to solve the problem correctly using procedures, they were initially unable to explain the algorithms they used. On the board, students displayed different solution methods, and we discussed similarities and differences in the methods and relationships among the methods. The lone student who drew a picture to show his thinking was hesitant to display his method and apologized saying, “I’m sorry, but this is what I have to do to understand a problem.” This made me think, “What are we doing to our mathematics majors to make them feel that one kind of representation is less appropriate than another?”

Throughout this secondary mathematics methods course, I was continually confronted by my students’ seeming lack of conceptual understanding of many topics found in high school curricula. Was I too ready to place the responsibility on elementary teachers for my poorly prepared college students? Was I blind to the possible gaps in knowledge of secondary mathematics education majors because my focus had been on the lack of mathematical knowledge of elementary education majors? Or have secondary mathematics education majors learned enough skills to hide their knowledge gaps? In pondering this issue, I reminded myself that the students who were majoring in elementary education had had the same elementary background as those majoring in mathematics. So it would be logical that both groups would have some of the same gaps in their conceptual understanding. I was puzzled by my secondary mathematics education students’ lack of content knowledge because I thought that, as an instructor of
mathematics majors, I would not have to bother with content in the methods course, that they had been well prepared mathematically. Were they? At this point, my focus changed somewhat. Although my primary focus was still on ways to improve the teaching of mathematics through improving the content knowledge of preservice teachers, I now turned to look at all levels rather than just elementary preservice teachers. In trying to sort out my seemingly mistaken beliefs, I looked at what other researchers studied regarding content knowledge.

Content knowledge encompasses a broad range of disciplines. Of the many choices within mathematical content knowledge, I chose to focus on Euclidean geometry. Not only is it a content area that is regularly taught in high schools, but also, data obtained from the National Assessment of Educational Progress (NAEP; NCES, 2006) and the Trends in International Mathematics and Science Study (TIMSS, 2003) have shown geometry to be one of the areas of weakness for students in the United States. The results of the 2003 TIMSS show that even though the United States is improving in mathematics achievement, measurement and geometry are still areas of concern. U. S. students continue to perform lower in these two content areas than in any other mathematical topics (Clements & Bright, 2003). Perhaps the reason students have difficulty learning geometry is that their teachers have not developed a deep knowledge of the subject themselves and are not able to teach their students with enough understanding to enable their students to gain mathematical proficiency in that content area. Therefore the focus for this study was secondary preservice teachers’ knowledge of Euclidean geometry.
Mathematical Content Knowledge

The definition of "knowing" appears to be multifaceted, encompassing the conscious mind, experience, skills, facts, memory, and awareness. Greeno (1991) gives an all encompassing view of understanding or knowing by comparing deep understanding of subject matter to knowing your way around your city or other environment. This type of understanding allows a person to use what had been previously learned and apply that knowledge to new and unfamiliar topics.

Shulman and colleagues' conceptualization of knowledge serves as a framework for much of the current research on teacher knowledge (Grossman, Wilson, & Shulman, 1989). Shulman (1986) listed three categories of content knowledge: subject matter content knowledge, pedagogical content knowledge, and curricular knowledge. Subject matter content knowledge in itself is multidimensional.

Drawing on Ryle's (1949) distinction between knowing how and knowing that, Mason and Spence (1999) divided knowledge into two types of knowing: "knowing-to" (p. 136) and "knowing-about" (p. 137). Knowing-about is the type of knowledge that generally is seen in schools because it is easier to teach and test. It includes facts, techniques, skills, and explanations. Knowing-to, more difficult to quantify, is knowledge that is readily available when needed to solve a problem. Knowing-to consists of more than simply looking at how much mathematics a person knows but rather focuses on how an individual uses that knowledge. This type of knowing is the deep understanding Greeno (1991) referred to when talking about navigating within an environment. Knowing-about, however, does not always produce knowing-to. Mason and Spence
(1999) have found, instead, that what students know tends to be compartmentalized rather than connected and therefore not at hand when it is needed. This leaves a gap in their understanding making it difficult to gain the deep understanding that is desired for our students.

Levels of Understanding

The authors of *Adding It Up* (National Research Council [NRC], 2001) have done extensive research on the content knowledge of elementary students, referring to mathematical proficiency as the educational goal for elementary students. Even though these authors studied the mathematical proficiency of elementary school children, specifically in the content area of number and operations, they noted interweaving of the five strands of mathematical proficiency across the different domains of mathematics, including geometry. Their model for mathematical proficiency consists of five intertwining strands of proficiency: conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition. This multidimensional view of mathematical understanding is presented as the ideal way to know mathematics. NCTM maintains that "teachers must know and understand deeply the mathematics they are teaching and be able to draw on that knowledge with flexibility in their teaching tasks" (NCTM, 2000, p. 17).

Although researchers focus their attention on different aspects of knowing and understanding, the five intertwining strands of mathematical proficiency (NRC, 2001) encompass many of the categories of knowledge other researchers have identified. For example, Ma (1999), focusing on the conceptual understanding strand (NRC, 2001),
referred to a “Profound Understanding of Fundamental Mathematics” (PUFM). The Chinese teachers in her study showed algorithmic competence (procedural fluency [NRC, 2001]) coupled with conceptual understanding. Their mathematical knowledge had coherence and familiarity with interconnections among mathematical topics. Similarly, Greeno (1991) used a metaphor to compare mathematical knowledge to knowledge about living in an environment. A knowledgeable person in an environment can find his way around, knows the resources that are available, and can use and appreciate those resources in a productive manner. The views of both Ma and Greeno regarding understanding seem to incorporate the strands of mathematical proficiency identified by the authors of *Adding It Up* (NRC, 2001) and which is the framework of the analysis used in this study.

**Conceptual Understanding**

Hiebert and Carpenter (1992) defined conceptual knowledge as knowledge that is understood and knowledge that is part of a network. Conceptual knowledge is knowledge that is rich in relationships and can be thought of as a connected web of knowledge (Hiebert & Lefevre, 1986). Kinach (2002a) considered the difference between content level and concept level understanding (conceptual understanding) as similar to the difference between learning the vocabulary of the concepts and understanding the concepts themselves. Conceptual understanding is shown with the comprehension of mathematical ideas, operations, and understanding relationships among those mathematical ideas. When considering the *conceptual understanding* strand in geometry, a person with mathematical proficiency shows an integrated and functional grasp of
mathematical ideas regarding shape and space. Conceptual understanding requires deeper knowledge and more learning than memorizing formulas and procedures.

**Procedural Fluency**

Procedural knowledge in mathematics is composed of two parts: knowledge of the formal language of mathematics (symbols and syntax) and the rules, algorithms, or procedures used to solve mathematical tasks (Hiebert & Lefevre, 1986). Hiebert and Carpenter (1992) regard procedures as a sequence of actions that allow for the efficient completion of mathematical tasks. Procedural fluency can be compared to Kinach's (2002a) content level understanding, which she contends is superficial understanding. This includes the ability to recall facts, the capability to perform a skill in the same context in which it was learned, and being able to describe the skills using appropriate vocabulary. *Procedural fluency* in geometry would be exhibited by skill in performing flexibly, accurately, and efficiently procedures such as constructing shapes, measuring space, and knowing when and how to use the procedures.

**Strategic Competence**

Strategic competence is the ability to solve a given problem with a variety of techniques such as guess-and-check, looking for patterns, working backwards, and applying arithmetic and/or algebraic notation. Flexibility of approach is a major cognitive requirement for solving non-routine problems. Strategic competence is also shown when students learn to replace cumbersome procedures with more concise and efficient procedures. Kinach (2002a) referred to this level of understanding as the problem-solving level. Students exhibiting this level of understanding are able to recognize how various
strategies relate and lead to a common solution for the same problem. In this study, for
gometry, *strategic competence* would be revealed in the ability to formulate
mathematical problems and to represent and solve a mathematical problem in multiple
ways. For example, students would be able to find area by subdividing irregular shapes
and then reconstructing them in ways that would necessitate using actual measurements.

**Adaptive Reasoning**

"Adaptive reasoning refers to the capacity to think logically about the
relationships among concept and situations. Such reasoning is correct and valid, stems
from careful consideration of alternatives and includes knowledge of how to justify the
conclusions" (NRC, 2001, p. 129). Understanding terminology and being able to apply
ccepts in unfamiliar situations is a sign of adaptive reasoning. The ability to use
ductive reasoning is evident in people displaying adaptive reasoning. Also included in
this strand would be competence with informal explanation, justification; intuitive and
uctive reasoning based on pattern, analogy, and metaphor. Students exhibiting this
level of understanding are able to apply what they learned in different contexts to new
problems. The adaptive reasoning strand of mathematical proficiency aligns with
Kinach's (2002a, 2002b) epistemic level of understanding, in which students are able to
fify their thinking and also understand the logical structure of the discipline being
sidered. NCTM's (2000) Process Standard of reasoning and proof would also fall into
this category of understanding. *Adaptive reasoning*, specifically related to geometry,
would be shown in the capacity to think logically about relationships among concepts and
itations and apply these in new, similar problem situations. Learning about various
shapes, their properties, and how they are related deepens a child's understanding of geometry when the child is also given the opportunity to communicate that knowledge to others through communication using informal proof and justification.

This type of reasoning, at a more simplified level, should begin in the early grades and continue throughout one's education (NCTM, 2000). Reasoning is one of the Processes Standards that is recommended by NCTM (2000) as a way of acquiring knowledge. Adaptive reasoning moves knowledge about one concept to application to another. Adaptive reasoning connects concepts and problem situations.

**Productive Disposition**

The fifth strand of mathematical proficiency is productive disposition. It includes having a belief in the utility and value of mathematics. Coupled with this would be the attitude that learning can be achieved through perseverance. The authors of *Adding It Up* (NRC, 2001) state that productive disposition develops along with the other strands of mathematical proficiency and also helps in the development of those strands.

Mathematical understanding has been studied with and without explicit consideration of attitudes and beliefs. In fact, studies could be devoted solely to attitudes and beliefs because of the extent of this topic. Guided by other researchers who have not included attitudes and beliefs in studies of mathematical understanding, productive disposition was not included in this study.

**Interplay Among the Strands of Mathematical Proficiency**

Much has been written about conceptual and procedural knowledge in mathematics and the relationships between these two types of knowledge (e.g., Hiebert &
Lefevre, 1986). Some schools place an emphasis on procedural knowledge, emphasizing symbol manipulation and valuing the importance of finding answers quickly. When this occurs, procedural knowledge and conceptual knowledge tend to develop on separate tracks, and learning often becomes compartmentalized (Hiebert & Lefevre, 1986). When procedures are taught without a conceptual foundation, those procedures may be forgotten quickly or only partially remembered. These partially remembered procedures are evident in many students, even at the college level. Correcting these misconceptions is often more difficult than teaching new content.

Research has shown that even well-executed procedures will not lead to conceptual understanding and importance needs to be placed on the connections among the procedures and the conceptual ideas (Hiebert & Carpenter, 1992; Van de Walle, 2001). Furthermore, researchers have found that once a learner masters a procedure, it is very difficult to go back with that learner and help him develop the conceptual understanding behind that procedure (Pesek & Kirshner, 2000; Wearne & Hiebert, 1988).

National professional organizations and researchers agree that all teachers should teach for flexible subject matter understanding, draw relationships within the subject as well as across disciplinary fields, make connections to the world outside of school, and teach using different representations (Hiebert & Lefevre, 1986; McDiarmid, Ball, & Anderson, 1989; NCTM, 2000). These four essentials for good teaching are incorporated in the conceptual understanding strand (NRC, 2001) but also include aspects of strategic competence and adaptive reasoning.
Even’s (1993) description of teachers’ subject matter knowledge referred to the mathematical construct of function and incorporated much of what would be included in the conceptual understanding and strategic competence strands of mathematical proficiency (NRC, 2001). Given as small a slice of mathematical content as the concept of function for determining the important aspects of subject matter knowledge for teachers, the strands of mathematical proficiency (NRC, 2001) are still present. Even more so for the larger mathematical content of geometry, the strands of mathematical proficiency would again stand out.

**Teacher Preparation in Mathematics**

The interwoven strands of mathematical proficiency apply not only to elementary students’ understanding, but also extend to all grade levels, including teacher preparation. If elementary students are expected to learn using these strands, they will expect and need to develop these strands while studying mathematics in secondary schools. This, in turn, places a significant demand on secondary mathematics teachers’ content knowledge. Secondary preservice teachers need to possess these strands of mathematical proficiency in order to be prepared to teach in ways that will deepen their students’ understanding.

All mathematics teacher education programs agree that subject matter knowledge is necessary for teaching, but the components of content knowledge and how to measure those components remain elusive. A deep mathematical understanding allows teachers to see relationships among concepts and possess a full collection of strategies for teaching. Are teachers able to interpret student work and analyze any student errors? These abilities are vital in the scaffolding process that help students progress in their learning. Being
able to represent mathematical concepts in a variety of ways can be an indicator of mathematical understanding. Deep understanding of mathematics will result in making connections among concepts, having less material to memorize because of that understanding, enhancing transfer of new knowledge, and influencing positive beliefs about mathematics (Hiebert & Carpenter, 1992; Ma, 1999).

Knowing the types of understanding that are necessary for effective teaching leads us to consider what institutions of higher learning consider necessary to prepare our future teachers. Following is the vision that guided the NCTM in the development of their *Principles and Standards for School Mathematics* (2000).

Imagine a classroom, a school or a school district where all students have access to high-quality, engaging mathematics instruction ... Teachers help students make, refine, and explore conjectures on the basis of evidence and use a variety of reasoning and proof techniques to confirm or disprove those conjectures. (NCTM, 2000, p. 3)

For this vision to become reality, skilled and knowledgeable teachers are needed to implement this plan. It will be important to equip high school mathematics teachers with the knowledge that is necessary to teach for understanding. “The specialized knowledge of mathematics that [mathematics teachers] need is different from the mathematical content contained in most college mathematics courses” (NRC, 2001, p. 375).

**Quality and Depth of Content Knowledge**

A large body of literature documents the lack of quality and depth of mathematical understanding in elementary preservice and inservice teachers with respect to many mathematical topics (see Baturo & Nason, 1996; Hill, Schilling, & Ball, 2004; Ma, 1999; Menon, 1998). Far fewer studies have been conducted on the content
knowledge of secondary preservice and inservice teachers (see Eade, 2003; Even, 1993; Thornton, 2003). This may be because secondary teachers usually have an academic major in their content area and thus are perceived to be knowledgeable in their subject area. Unfortunately, the small body of research that exists suggests otherwise. Could this be because we are looking more at the number and types of courses that secondary mathematics majors take rather than the depth of understanding they gain from these courses?

Much research has focused on conceptual and procedural knowledge, especially related to elementary students and teachers, but less research is available on the other three strands of mathematical proficiency mentioned in *Adding It Up* (NRC, 2001). That research was done entirely at the elementary level but might well apply to knowledge requirements for teaching at other levels as well as in other content areas. Hill et al. (2004) stated:

... rather than focusing on how much mathematics an individual knows, ... researchers must also ask how an individual holds and uses that knowledge—whether a teacher can use mathematical knowledge to generate representations, interpret student work, or analyze student mistakes (p. 27).

This statement reflects what was observed of preservice teachers in mathematics methods classes. The preservice teachers had command of an extensive knowledge base of mathematics and most were adept at using that knowledge. Many of the preservice teachers, however, thought of only one way to solve a problem. They were very interested, though, in learning new ways of approaching problems when peers shared new problems and solutions.
Rationale and Conceptual Framework

An assumption by the general population is that mathematics majors are adequately prepared to teach the content of secondary mathematics as evidenced by their completion of a mathematics major. While mathematics educators are aware that conceptual understanding in many content areas is lacking in these mathematics teaching majors (Bryan, 1999; Eade, 2003; Even, 1993; Grossman, 1989; Thornton, 2003; Wilson, 1994; Wilson & Wineburg, 1988; Zancanella, 1991), little research data exist at the secondary level to support those beliefs.

Secondary mathematics education majors may appear to have adequate procedural fluency, as evidenced by their ACT and PPST scores (both tests of procedural knowledge). Hiebert and LeFevre (1986) state, "Being competent in mathematics involves knowing concepts, knowing symbols and procedures, and knowing how they are related" (p. 16), but students do not automatically connect their conceptual and procedural knowledge or examine relationships among concepts and procedures. While it is important to look at the conceptual understanding and procedural fluency of secondary preservice teachers, it is also important to look at the additional strands of mathematical proficiency (NRC, 2001) and not merely rely on procedural fluency to determine the mathematical proficiency of secondary mathematics education majors.

Focus of Study

Euclidean geometry is the content area of focus in this study because it is an area that is regularly taught in secondary schools and it is an area of weakness for students in the United States. While not a focus of this study, problem solving was used as a vehicle
to discern the presence or absence of the strands of mathematical proficiency in the secondary preservice teachers participating in this study.

Various aspects of problem solving, such as approaches to problem solving (e.g., Brownell, 1942; Polya, 1957) and how problem solving supports student learning (Hiebert & Wearne, 2003), have been researched. Schoenfeld (1992) acknowledged that the meaning attached to problem solving is diverse. On one end of the scale are formulaic exercises that are practiced for fluidity, while at the other end of the scale is problem solving requiring creative thought and application of new strategies. This is sometimes referred to as non-routine problem solving. Effective non-routine problem solving requires a broader, multidimensional knowledge of mathematics. Expert problem solvers have a mathematical point of view. Besides looking at mathematics as a set of skills, they look at its value, how it is done, and they are flexible thinkers (Schoenfeld, 1992).

Because there is a minimal amount of research done on secondary preservice teachers and the quality of content knowledge of that population is often overlooked, the focus of this study was narrowed to studying the mathematical content knowledge of secondary preservice teachers, specifically related to their knowledge of Euclidean geometry, in the context of mathematical proficiency (NRC, 2001). Although Adding It Up (NRC, 2001) considered the mathematical understanding of children, their framework for mathematical proficiency could be applied in a study of the content knowledge of secondary preservice teachers. In addition, the conceptual framework used by the NAEP, which consists of items in the categories of conceptual understanding, procedural knowledge, and problem solving, was considered.
Conceptual understanding, procedural fluency, strategic competence, and adaptive reasoning in secondary preservice teachers needs to be brought to the attention of the mathematics community and documented so that improvements in teacher education can be made. The following research questions were posed in this study:

1. Do secondary preservice teachers have mathematical proficiency in topics commonly taught in high school geometry courses?

2. What interplay is seen among the strands of mathematical proficiency (NRC, 2001) related to geometric content knowledge of secondary mathematics preservice teachers?

These questions were explored by examining the geometry content knowledge of mathematics teaching majors in a teacher education program.
CHAPTER 2
REVIEW OF LITERATURE

The purpose of this study was to look at the mathematical understanding of secondary preservice teachers. Mathematical understanding in this population has been studied in mathematical topics such as number and operations (Bryan, 1999; Kinach, 2002a, 2002b; Van Dooren, Verschaffel, & Onghena, 2002), algebra (Bryan, 1999; Van Dooren et al., 2002), geometry (Bryan, 1999), functions (Even, 1993; Wilson, 1994), and problem solving (Van Dooren et al., 2002). The multidimensional nature of mathematical understanding is well documented (Ball, 1990a, 1990b; Ball & McDiarmid, 1990; Ma, 1999; NRC, 2001). While the mathematical understanding investigated in this study is a necessary component for teaching effectively, it is certainly not the only dimension. Research has also focused on pedagogical content knowledge which is specifically related to strategies and techniques used while teaching (Hill et al., 2004; Izsak & Sherin, 2003; Kinach, 2002a, 2002b; Shulman, 1986; Thornton, 2003; Ward, Anhalt, & Vinson, 2004). This study, however, was limited to investigating the mathematical content knowledge of secondary preservice teachers.

The research in this chapter will be divided into three major sections. The first section will present literature related to the multidimensional nature of mathematical knowledge. More specifically, using the framework of the intertwining strands of mathematical proficiency (NRC, 2001), I present research that has focused on the individual strands and their interconnectedness. The second section will show
mathematical understanding related to teaching. Finally, the ways of assessing mathematical proficiency will be discussed in light of the available research.

Quality and Depth of Mathematical Knowledge

The conceptual framework for studying mathematical content knowledge in this study was the intertwining strands of mathematical proficiency as described in *Adding It Up* (NRC, 2001). This view of mathematical proficiency was applied to the understanding of elementary school children, but the same type of understanding is necessary for teachers (Ball, 1990a; Ball & McDiarmid, 1990; Kinach, 2002a, 2002b; Ma, 1999). Kinach (2002b), using a modification of Skemp's (1978) levels of understanding, similarly looked at five levels of understanding that closely align with the five strands of mathematical proficiency described in *Adding It Up* (NRC, 2001). Kinach found that although the secondary preservice teachers were able to do mathematics, they lacked the deeper understanding necessary to explain and really know mathematics. The mathematical topic of study for Kinach was number and operations, specifically integer subtraction, while this study looked at a different topic, namely geometry.

Ball (1990a) compared elementary and secondary preservice teachers. She found that secondary mathematics majors did not have a substantial advantage over elementary majors in articulating and connecting underlying mathematical concepts, principles, and meanings. Secondary majors, nonetheless, believed they knew the mathematics and realized that there were explanations but often could not give an explanation or simply restated the rule. Also, secondary preservice teachers did not give much consideration to their own subject matter knowledge because they felt they understood the content they
would teach (NRC, 2001). In general, the secondary preservice teachers felt that they were familiar with the high school material since it was easy for them when they were students and as mathematics majors they had taken much more advanced courses than they would be teaching. This study will strive to determine whether secondary preservice teachers are accurate in their personal appraisals of their knowledge.

Researchers recognize that knowledge is multidimensional and can be classified into different levels of understanding (Ball, 1990a, 1990b; Grossman, 1990; Hill et al., 2004; Leinhardt & Smith, 1985; Ma, 1999; Shulman, 1986). This multidimensional nature of understanding compares to the five intertwining strands of mathematical proficiency (NRC, 2001), four of which are being considered in this study. We will now look at what researchers have found related to the strands of mathematical proficiency being considered in this study.

Conceptual Understanding

Conceptual understanding has been extensively studied, especially in relation to procedural knowledge (Ball, 1990a, 1990b; Baturo & Nason, 1996; Bryan, 1999; Eade, 2003; Kinach, 2002b; Ma, 1999; Wilson, 1994). Some of the major studies related to this research are summarized below.

Ma (1999) studied the conceptual understanding of elementary teachers from China and the United States. She found that teachers from China had a deeper conceptual understanding of mathematics, specifically subtraction, multi-digit multiplication, division by fractions, and the relationship between area and perimeter. She contends that the Chinese teachers' deeper understanding comes from the way they learn mathematics.
Even in the early elementary grades, Chinese students are taught about mathematics in a conceptual way as opposed to the procedural techniques that are often found in American schools. In addition, Ma (1999) found that although Chinese teachers generally had fewer years of schooling (11 to 12 years, as opposed to between 16 and 17 years for American teachers), the Chinese teachers began their teaching careers with a better understanding of elementary mathematics than was seen in the American teachers.

Kinach (2002b) provided a study at the secondary level equivalent to Ma’s (1999) study at the elementary level. Kinach’s concept level of disciplinary understanding most closely aligns to the conceptual understanding strand used in this study. In Kinach’s study, conceptual explanations using number tiles for integer subtraction were modeled for the participants; then participants practiced using those conceptual explanations. Participants reverted back to procedural explanations (tricks or rules), however, when a number line model was introduced for the same integer subtraction problem, 5 – (-3). While participants were able to perform the computation required for integer subtraction, conceptual understanding was shown to be lacking when participants were not able to apply their conceptual explanations using a slightly different model. This study will examine a slightly larger piece of mathematical content--topics from Euclidean geometry.

The number of university mathematics courses studied does not necessarily lead to increased conceptual understanding as evidenced by the following studies. Eade (2003) studied secondary preservice teachers to determine their conceptual understanding of convergence and continuity, important topics in calculus. He noted participants who misinterpreted questions and used partially remembered theorems. He found that even
participants with a significant amount of mathematics in their degrees had difficulties with elementary mathematical concepts. Likewise, Cooney (1995) found that preservice teachers lacked a fundamental understanding of school mathematics despite their success in studying advanced levels of university mathematics. Bryan (1999) also found that even though the participants successfully completed the required university coursework for a mathematics teaching major, they still only had a surface knowledge of the mathematics content they would be teaching. This study will also be looking at the mathematical understanding of secondary mathematics teaching majors.

Baturo and Nason (1996) studied elementary preservice teachers in their first year of a teacher education program to determine their subject matter knowledge in the topic of area measurement. The participants seemed to have the understanding that area must be enclosed, but confused area and volume when shown a three-dimensional shape. This is evidence of the participants' lack of conceptual understanding in problems involving area. Baturo and Nason used a modification of Leinhardt and Smith's (1985) theory of understanding in their study, specifically substantive knowledge, which focuses on the correctness of knowledge, the understanding of underlying meanings, and the degree of connectiveness between measurement concepts and processes. This description of substantive knowledge aligns with both the conceptual understanding and procedural fluency strands of mathematical proficiency (NRC, 2001) being used in this study.

Other researchers also drew attention to the interplay among various aspects of understanding. Rittle-Johnson and Alibali (1999) studied elementary students to examine the relationships between their conceptual understanding and their procedures for solving
equivalence problems. While that study was on a different population, it suggested causal connections between these two strands and also suggested that conceptual knowledge may have a greater influence on procedural knowledge than the reverse. For example, having a deep conceptual understanding of a topic may prevent children from using incorrect procedures.

Likewise, Wilson (1994) found that the participant in his study had a weak conceptual understanding of function and therefore preferred mathematics that emphasized procedures and concrete answers. Ball (1990b) also found that most preservice teachers looked for a rule they could state rather than focusing on the underlying meaning associated with the problem when asked to explain their answers related to their understanding of division. That indicated a lack of conceptual understanding. This study also looked at the interplay of the strands of mathematical proficiency (NRC, 2001) but related to geometric understanding rather than division.

Procedural Fluency

Procedural fluency is often assumed when studying secondary preservice teachers since they generally have a major in their content area. Contrary to that assumption, however, Bryan (1999) found that procedural fluency in topics that are commonly taught in the high school curriculum was not always demonstrated by the participants in his study. This study focused only on the topic of geometry, but delved more deeply into that topic.

Content-level understanding was a category used in Kinach’s (2002b) study of secondary preservice teachers. That level is consistent with the procedural knowledge
strand used in this study. It is Kinach's belief that this level of understanding is the most superficial kind of knowledge anyone can have in a discipline. Procedural fluency was found in participants' abilities to correctly complete a procedure for integer subtraction problems.

Ball (1990b) studied elementary and secondary preservice teachers regarding their understanding of division algorithms. Seventeen out of the 19 participants were able to give the correct answer to a division of fractions problem, most by using a rule they had been taught. Some participants could not remember the rules. Seven preservice teachers explained why division by zero is impossible by stating a rule. When manipulating algebraic equations, 14 preservice teachers focused on the mechanics, while four preservice teachers had no idea how to solve the equation. Only one preservice teacher—an elementary major—was able to talk about the meaning behind solving the algebraic equation.

Graeber and Tirosh (1988) studied elementary preservice teachers, and like Ball (1990b), found that preservice teachers held misconceptions about division procedures. For example, some participants thought that division by a decimal was not possible. Another misconception was that one cannot divide by a fraction—instead, one must invert and multiply. The participants seemed to only be familiar with rote procedures, such as moving the decimal point, and they had no understanding behind that procedure. They would also over-generalize algorithms (since \( \frac{1}{2} \) of 70 is \( 2 \sqrt[70]{ } \), then \( .65 \) of 30 is \( .65 \sqrt[30]{ } \)).
Likewise, Ma (1999), in studying division of fractions, found that only 43% of the American teachers were successful in calculating a division of fractions problem. None of them showed an understanding of the rationale for the traditional algorithm and many reported their anxiety in having to do these calculations. The Chinese teachers, on the other hand, were all successful in their calculations.

Menon (1998) studied elementary preservice teachers regarding their knowledge of perimeter and area. The tasks were categorized as Lo (low), Me (medium), and Hi (high) using Skemp's (1978) types of understanding. The levels, Lo and Me (Skemp's instrumental understanding), would correspond to what would be expected in the procedural fluency strand of mathematical proficiency. Menon found that slightly more than 80% of the participants exhibited instrumental understanding that required generally procedural and well-practiced skills.

**Strategic Competence**

There has not been extensive study in the area of strategic competence for secondary preservice teachers. A facet of strategic competence is the ability to formulate, represent, and solve problems. When Ball (1990b) studied elementary and secondary preservice teachers regarding their understanding of division, only five out of 19 participants (all secondary preservice teachers) were able to give an appropriate representation for a division of fractions problem \((1\frac{3}{4} \div \frac{1}{2})\). The most frequent incorrect representation showed division by two rather than division by one-half.

Borko et al. (1992), in their study of elementary preservice teachers, found most participants were unable to create a word problem for a mixed number divided by a
fraction and often the problem created represented a multiplication situation instead. Ma (1999) had similar findings when studying American and Chinese elementary teachers. Unlike the American teachers, the Chinese teachers were not only successful when dividing fractions, they also enthusiastically presented several ways of finding the answer. Ma (1999) asked her participants to formulate word problems that would represent division of fraction situations. While the American teachers had more difficulty with this and often wrote word problems that would be solved by dividing by two rather than dividing by one-half, the Chinese teachers were able to generate multiple representations for the division of fractions problems.

During interviews with elementary preservice teachers in their first year of a teacher education program, Baturo and Nason (1996) looked for knowledge about the nature and discourse of mathematics. For example, they wanted to see what participants would consider to be an appropriate answer, whether it was reasonable, and what was involved in doing mathematics. Additionally, they wanted to find out the participants' knowledge of the standard measurement of area. They found that most students felt they were done with a problem once they found an answer. Participants had to be urged to check their answers for reasonableness. When asked to justify their answers, they attempted to redo the paper-and-pencil calculations, perhaps checking their computations with a calculator, rather than trying some other strategy to check their answers. This demonstrated that they were not aware of multiple strategies.

The problem-solving level of understanding mentioned in Kinach's (2002b) study corresponds to the strategic competence strand (NRC, 2001), specifically when
considering using multiple representations. Participants in her study worked with algebra tile unit squares when learning how to use conceptual explanations for integer addition and subtraction. Being able to transfer the ideas for conceptual explanations used with algebra tile unit squares to a different model, the number line, would have demonstrated strategic competence. The participants in her study reverted back to procedural explanations, however, when faced with using the number line as a model.

**Adaptive Reasoning**

Adaptive reasoning requires understanding relationships among concepts and situations and applying prior knowledge in new situations. Bryan (1999) studied the mathematical understanding of secondary preservice teachers in content areas that are commonly taught in the high school curriculum. The participants were given a task and asked to demonstrate procedural knowledge regarding the task. Similar to Ball’s (1990a) study, the preservice teachers were then asked whether the mathematical idea could be explained conceptually and how they could justify the conceptual explanation. Bryan did not let the participant’s lack of procedural knowledge prevent him from asking follow-up questions of a conceptual nature that would illuminate adaptive reasoning. If the participant, however, stated that the procedure just had to be memorized and that there was no conceptual basis for the procedure, then further conceptual questioning was not continued. The participants’ knowledge was determined to be primarily memorized rather than deeply understood. However, when participants were able to give conceptually sound explanations, it was usually possible only after some false starts, requests for
clarification, and prodding questions from the interviewer. This showed evidence of a lack of adaptive reasoning.

Pesek and Kirshner (2000) studied the interference of procedural instruction with subsequent conceptual instruction. The participants in that study were elementary students in the 5th grade, but the study illustrated the relationship between procedural and conceptual instruction. The students in the group receiving procedural instruction before the conceptual instruction were less able to determine whether a question would require the use of area or perimeter than the group receiving the conceptual instruction alone. Also the group receiving procedural instruction before the conceptual instruction were more likely to refer to and use formulas, operations, and fixed procedures; whereas the group receiving only the conceptual instruction used more flexible solution methods with which they had had concrete experiences, thus, demonstrating adaptive reasoning. Contrary to common expectations, more time was spent on procedural instruction, but this did not lead to more learning.

Leinhardt and Smith (1985) found that teachers with moderate or low content knowledge presented algorithms and gave misinformation about multiplication and division of fractions because they based their assumptions on multiplication and division of whole numbers (multiplication makes the answer larger and division makes the answer smaller—failing to understand relationships between multiplication and division). Also in that study, a teacher with low content knowledge presented an inefficient algorithm for simplifying fractions that was only effective if both numerator and denominator were
divisible by two. This teacher also said that “reducing fractions” resulted in a “smaller” fraction. These incorrect statements show evidence of the lack of adaptive reasoning.

In Baturo and Nason’s (1996) study of elementary preservice teachers, substantive knowledge was subdivided into three knowledge types—concrete, computational, and principled conceptual. The concrete and computational types most closely align with the procedural fluency strand. Out of 13 participants in that study, only two showed an understanding of the relationship between the area of a triangle and the area of a rectangle. The rules they had learned were disconnected and thus, lacked meaning. Most could not remember the formulas for area. One participant did not appear to know basic multiplication facts. Students looked at mathematics as mechanical—using a known procedure to calculate an answer. Overall, participants showed a lack of adaptive reasoning in problems involving area because they were not able to apply what knew about the area of one figure to the area of a different figure.

Likewise, Reinke’s (1997) study of elementary preservice teachers found a lack of adaptive reasoning when working with a non-routine area and perimeter problem. The figure was a rectangle with a semicircle at one end. When attempting to find the perimeter, some participants gave the perimeter of the rectangle ignoring the fact that one end consisted of a semicircle. Similarly, when finding the area of the figure, the semicircle was again ignored. Participants were not able to think logically about the relationship among the concepts related to area and perimeter problems.

Eade (2003) interviewed secondary preservice teachers in his study to determine their reasoning about convergence and continuity. He noted that participants
misinterpreted questions and used partially remembered theorems, thus showing a lack of adaptive reasoning. He found that even participants with a significant amount of mathematics in their degrees had difficulties with elementary mathematical concepts.

**Mathematical Knowledge for Teaching**

Researchers have found that preservice and inservice teachers make decisions about how they will teach based on how deeply they understand their content (Floden & Meniketti, 2005; Kinach, 2002b; Ma, 1999; Menon, 1998; Thornton, 2003). Other researchers have looked at the relationship between the depth of teacher knowledge and student achievement (e.g., Begle, 1979; Floden & Meniketti, 2005; Ma, 1999). Researchers also acknowledge the multidimensional nature of knowledge, most commonly referring to conceptual understanding and procedural knowledge (Ball, 1990a, 1990b; Carpenter, 1986; Eisenhart et al., 1993; Even, 1993; Graeber, 1999; Hiebert & Lefevre, 1986; Ma, 1999; Rittle-Johnson & Alibali, 1999). Although some amount of content knowledge is necessary for effective teaching, there are differing opinions on how much knowledge is necessary and how to assess that knowledge (Ball, 1990a; Bryan, 1999; Fernandez, 1997; Floden & Meniketti, 2005; Hill et al., 2004; Lubinski, Otto, Rich, & Jaberg, 1998; Mason & Spence, 1999; Mewborn, 2000; Thornton, 2003; Van Dooren et al., 2002).

**Teaching for Conceptual Understanding**

Eisenhart et al. (1993) looked at how to teach for conceptual understanding, that is, knowing the connections and relationships among ideas that give meaning to mathematical procedures. They found that although the preservice teachers realized the
importance of teaching for understanding, including both procedural and conceptual understanding, their knowledge of both content and pedagogy prevented them from being able to teach conceptually. The observed classroom teaching was more skills-oriented and this was also reflected in the classroom testing. Eisenhart et al. found a pattern in which the preservice teacher, the mathematics methods course instructor, the cooperating teachers, and the administrators of the cooperating school all expressed strong commitment to teaching for both procedural and conceptual knowledge, but the preservice teacher, nevertheless, taught more consistently for procedural knowledge—mastery of computational skills, knowledge of procedures, algorithms, and definitions—than for conceptual understanding.

In contrast, Leinhardt and Smith (1985) looked at experienced mathematics teachers with different levels of mathematical understanding in the content area of fractions. The teachers with the strongest mathematics content knowledge were able to provide students with a rich body of conceptual information without giving algorithms and then were able to help the students know when and how to simplify fractions. The teachers with moderate to low content knowledge presented procedural algorithms, and also presented some misinformation to their students.

Even (1993) found that the preservice teachers in her study had a limited conceptual understanding of function. When teaching about functions, many of the preservice teachers in her study made the pedagogical decision to simply provide students with a rule to follow rather than focusing on student understanding of the concept of function because they had a limited conceptual understanding of function themselves.
Likewise in his study, Menon (1998) concluded that teachers with a poor conceptual understanding of mathematics felt more comfortable teaching for procedural understanding and would not be willing or able to provide opportunities for students to do problems that would require a deeper (conceptual) understanding. Testing in those classroom environments would also emphasize procedures rather than concepts.

Grossman (1989) and Zancanella (1991) had similar findings in their studies of English teachers and Thornton (2003) in his study of history teachers. For example, Grossman (1989) and Zancanella (1991) both researched secondary English teachers’ approaches to literature depended upon their levels of content knowledge expertise. Wilson and Wineburg’s (1988) research was in the social studies content area. These teachers’ approaches to their subject matter were closely tied to their areas of expertise. Rather than having a deep understanding across their domain and having the ability to be flexible in their teaching, these teachers dealt with their inadequacies by using an approach that was more familiar to them.

Student Thinking

Ball (1990b) found that when preservice teachers' mathematics is not sufficiently connected, they do not have the deep understanding of the principles underlying mathematical procedures that is adequate for teaching. They would not be able to capitalize on student thinking. The secondary mathematics teachers with strong credentials that Fernandez (1997) studied were able to model adaptive reasoning by applying a student’s solution method to a related problem, and to follow through on a student’s comment and uncover errors in the student’s thinking by providing a
counterexample. This requires depth of understanding in all the strands of mathematical proficiency (NRC, 2001).

In contrast, Lubinski et al. (1998) found in their study that a novice teacher even with a strong mathematical background did not use student thinking to guide his lesson. Instead, the novice teacher led students towards his way of thinking. The other novice teacher in that study, however, with a weaker background in mathematics was better at listening to students and shaping her lessons to take advantage of what the students were learning. This novice teacher, however, often presented incorrect mathematics to her students. That study showed that a strong pedagogical background does not always lead to facilitating robust student learning when the teacher's content knowledge is deficient.

Assessing the Quality of Mathematical Understanding

In trying to identify subject matter knowledge, researchers have used a number of indicators of subject matter knowledge, including formal education in the subject matter, knowledge of the concepts, algorithmic operations, and connections among different algorithmic procedures as well as the understanding of classes of student errors, and knowledge of the curriculum (Grossman et al., 1989; Leinhardt & Smith, 1985). Another measure of teacher knowledge is often assessed by examining college transcripts. Employers look at the number of college mathematics courses taken and the grades that were recorded. The NRC (2001) found that teacher knowledge cannot be accurately measured by simply looking at the number of mathematics courses taken. This finding supports the need to study the nature of the mathematical knowledge, specifically the
knowledge necessary to teach and to develop ways to measure this knowledge more accurately.

While looking for a framework upon which to base this study, studies were found in which researchers used a variety of ways to assess understanding. Studies looking at the mathematical knowledge of preservice and inservice teachers have used tasks, tests, and follow-up interviews as methods of determining the degree of understanding exhibited in the subjects (Ball, 1990a, 1990b; Baturo & Nason, 1996; Bryan, 1999; Even, 1993; Hill et al., 2004; Menon, 1998; Van Dooren et al., 2002). The NAEP, which conducts periodic assessments of American students, has developed questions to test the mathematical abilities of conceptual understanding, procedural knowledge, and problem solving. Additional specifications include reasoning, connections, and communication. These abilities and specifications closely align with the NCTM standards (NCTM, 2000).

Since 1969, the NAEP has periodically measured student achievement in grades 4, 8 and 12. In the subject area of mathematics, test questions were written in the content areas of number properties and operations, measurement, geometry and spatial sense, data analysis and probability, and algebra. The NAEP also categorized its questions according to the mathematical ability being tested: conceptual understanding, procedural knowledge, and problem solving (National Center for Education Statistics [NCES], 2006). These three abilities parallel three of the strands of mathematical proficiency found in Adding It Up (NRC, 2001), conceptual understanding, procedural fluency, and strategic competence.
Kinach (2002b) studied 21 secondary preservice teachers who were students in a secondary mathematics methods course. There were 15 undergraduates and six graduate students. Of these, 19 were mathematics majors, one history major and one engineering major, but all were planning to be certified to teach secondary mathematics. The study was a teaching experiment where students learned how to explain integer addition using a number line or algebra tile unit squares before attempting to explain integer subtraction. Thorough discussions were conducted making it clear what constituted a good explanation versus non-explanations. Good explanations included mathematical reasons of why a procedure was followed, but would not include explanations that would be classified as tricks or steps that told how to do a procedure.

Kinach (2002b) analyzed the data using four levels of relational understanding that compare to the conceptual understanding, strategic competence, and adaptive reasoning strands of mathematical proficiency (NRC, 2001) being used in this study. Her findings bring into question the common assumption that secondary teachers know their mathematics. Her study also showed that if preservice teachers are allowed to use only instrumental (procedural) explanations, their levels of understanding will not necessarily develop as they continue to explain concepts to others. Kinach believed that preservice teachers may be better served if methods courses focused more on teaching for understanding rather than teaching of procedures.

The teacher knowledge test that Hill et al. (2004) developed is meant to be used for testing the measures of teachers’ content knowledge necessary to teach elementary mathematics. Two content areas were chosen—number concepts and operations and
patterns, functions, and algebra. The number concepts area was chosen because it makes up such a large percentage of the K-6 curriculum. Patterns, functions, and algebra was chosen because it represents a newer content strand in the K-6 curriculum. The researchers were studying how teachers' mathematical knowledge for teaching is organized, and whether the items on the survey could reliably measure the mathematical knowledge for teaching. The survey they developed was found to be reliable and may serve to identify the effect of teacher knowledge on student achievement.

Bryan (1999) studied nine secondary preservice teachers who were classified as either juniors or seniors. He compiled background information on the participants, including their mathematics grade point average, and the number and names of mathematics courses completed, in progress, and left to be completed by each individual. All participants were beyond the halfway point with the mathematics courses required for their degrees. None of the participants had completed the mathematics methods course nor had they done any student teaching. Each participant completed four hour-long interviews over a six-week period. During the interviews participants had access to paper and pencil and they were encouraged to write to support their verbal explanations. The interviews were audiotaped and later transcribed. Topics typically found in the high school curriculum such as exponents, fractions, operations with integers, slope, topics from algebra, geometry and trigonometry were explored during the interviews.

The purpose of the study was to determine the conceptual understanding of the preservice teachers. Participants were first asked to demonstrate procedural knowledge of the topic being explored. Next, they were asked if that idea could be explained in a
conceptual manner. Often the participant claimed that the idea just had to be memorized, demonstrating their lack of conceptual knowledge about that topic. This is similar to Ball’s (1990a) study of prospective elementary and secondary teachers where participants were asked if specific ideas could be explained, needed to be memorized, or if they were not sure.

Bryan (1999) categorized his responses into five categories: Showed Procedural Proficiency, Considered Conceptual Basis, Offered No Explanation, Offered Flawed Explanation, or Offered Sound Explanation. The participants were most consistent in showing procedural proficiency, but they were only able to offer sound explanation approximately 22% of the time. These mathematical topics were considered ideas about which any secondary mathematics teacher should have expert knowledge. This finding was surprising, yet troubling, to the researcher. As with Kinach’s (2002b) study, Bryan found a lack of conceptual understanding in his participants and questioned whether that conceptual understanding would be deepened once the preservice teachers began their teaching careers. While his study did not look at ways of improving undergraduate mathematics education programs, he suggested that his study should make mathematics educators question the current practices and take an initial step towards improving the preparation of secondary mathematics teachers.

Even (1993) researched subject matter knowledge as well as pedagogical content knowledge regarding the function concept. Open-ended questionnaires were administered to 152 secondary preservice teachers (all juniors, seniors, or post baccalaureate students) to determine their understanding of functions. These questionnaires consisted of nine
nonstandard problems about functions and six problems where student mistakes or misunderstandings were to be analyzed. Ten participants in this study took part in follow-up interviews the day after the questionnaires were completed. The interview questions were based on the answers and the reasoning behind the answers given on the questionnaires. Even found that the preservice teachers had a limited and underdeveloped concept of functions. She believed that subject matter preparation needs to be improved, not by changing the number of required courses, but rather the courses should be constructed differently, more in the line of constructivist views of teaching.

Wood (1993) used in-depth interviews of eight secondary preservice teachers and found that incorrect statements were made by participants with weak, moderate, and strong mathematical backgrounds. A strong mathematical background did not improve the accuracy of their statements about topics commonly found in high school mathematics curricula.

**Problem Solving**

This study did not focus on problem solving but rather used problem solving as a vehicle for determining the mathematical understanding of secondary preservice teachers. Research on problem solving is vast, defining problem solving (Brownell, 1942; Lesh & Zawojewski, 2007; Schoenfeld, 1992), describing models and procedures for problem solving (Begle, 1979; Kilpatrick, 1985; Polya, 1957), becoming a good problem solver (Graeber & Tirosh, 1988; Van Dooren et al., 2002), and using problem solving to increase mathematical understanding (Ben-Chaim, Keret, & Ilany, 2007; Bransford, Brown, & Cocking, 2000; Hiebert & Wearne, 2003; Rittle-Johnson & Alibali, 1999).
Many researchers have used tasks or problem solving to determine the mathematical understanding of their participants (Baturo & Nason, 1996; Ben-Chaim et al., 2007; Bryan, 1999; Even, 1993; Kinach, 2002b; Menon, 1998; Von Mindon, Walls, & Nardi, 1998). Problem solving in mathematics requires versatility in thinking, persistence and curiosity, and can lead to confidence in unfamiliar situations (NCTM, 2000).

Graber and Tirosh (1988) compared the problem solving abilities of elementary preservice teachers. Participants who were able to accurately write expressions for word problems were also more likely to check the accuracy and reasonableness of their calculations and were better able to describe their thinking than participants who had difficulty writing correct expressions.

Van Dooren et al. (2002) studied the problem solving strategies and skills of 97 elementary and secondary preservice teachers in Belgium. Participants were given 12 word problems that could each be solved in three ways—one algebraic and two arithmetic. They found that secondary preservice teachers clearly preferred algebraic methods for solving problems even when an arithmetic solution would be more efficient. When evaluating the solution strategies of students, the secondary preservice teachers rated algebraic solutions higher than arithmetic solutions. Elementary preservice teachers’ evaluations of solutions were more related to the nature of the task than secondary preservice teachers’ evaluations.

Good problem solvers constantly assess what they are doing to determine whether or not their strategies are effective in the problem situation. Adaptive reasoning is seen in good problem solvers as they recognize the structure of a familiar problem and apply it in
a new situation. The strands of mathematical proficiency (NRC, 2001) are evident in good problem solvers. This study will serve to discover the mathematical proficiency of secondary mathematics preservice teachers in the content area of geometry. Problem solving will be used as a means of revealing these strands of mathematical proficiency (NRC, 2001).
CHAPTER 3
METHODOLOGY

To prepare for this study a pilot study was done at a different institution using participants who were secondary mathematics preservice teachers who had already completed their mathematics methods course. This geometry study focused on the quality of content knowledge of secondary preservice teachers in the context of mathematical proficiency (NRC, 2001). The results of this pilot study demonstrated the need for this further study. The following research questions were posed in this study:

1. Do secondary preservice teachers have mathematical proficiency in topics commonly taught in high school geometry courses?

2. What interplay is seen among the strands of mathematical proficiency (NRC, 2001) related to geometric content knowledge of secondary mathematics preservice teachers?

These questions were explored by examining the geometric content knowledge of secondary mathematics majors in a teacher education program. A mixed methodology for qualitative research was used in this study.

Participants

The participants for this study were 15 secondary mathematics preservice teachers enrolled during the fall term of 2006 in “The Teaching of Middle School/Junior High Mathematics,” a mathematics methods class at a Midwestern state university. The participants for this study were selected using a convenience sampling technique. The ages of the participants were typical of students nearing the end of their undergraduate
college education, 20-24 years of age. There were 12 women and three men who participated in this study and all appeared to be Caucasian. It is a requirement of all secondary mathematics education majors at this institution to take two mathematics methods courses: “The Teaching of Middle School/Junior High Mathematics” and “The Teaching of Secondary Mathematics.” Both methods courses focus on detailed lesson planning, communication, integrating appropriate technology, and meeting the needs of diverse learners.

The middle school methods course “is designed to prepare future teachers to diagnose student needs, develop a dynamic teaching/learning environment, use assessment as a teaching/learning tool, and understand current curriculum issues and trends in middle school mathematics” (Course instructor, personal communication, Fall 2006). Students in this course are required to complete 20 hours of field experience in a middle school/junior high classroom. Through the field experience and coursework, students become knowledgeable about the mathematics content found in the middle school/junior high curriculum. They also are provided instruction in technology, manipulatives, models, materials, and strategies used to develop mathematical understanding for middle school/junior high students. Problems of the Week (POW), commonly used in middle school classrooms, are utilized to help familiarize preservice teachers with the middle school/junior high mathematics curriculum and to develop pedagogical content knowledge at that level.

This study took place during the first week of the middle school/junior high mathematics methods course, which was the first methods course taken by six of the 15
students. The other nine students had already completed the high school methods course. The students were not randomly assigned to this methods course but rather enrolled in the course when it fit into their schedules and prior to student teaching. The self-reported cumulative grade point averages for the 15 participants ranged from 2.7 to 3.8 with a mean of 3.43. (see Table 2). Test numbers were given to all participants and pseudonyms were assigned to participants completing the interview phase of this study.

Testing Instrument

The initial phase of this study required participants to complete a fifteen-item test designed to reveal each participant's procedural knowledge, conceptual understanding, and problem solving ability. In order to create the test an item analysis was completed using published questions from the NAEP. The NAEP has been periodically measuring student achievement in grades 4, 8 and 12 since 1969. These assessments are done in several content areas including mathematics. The results are reported for groups within the populations (for example, gender and ethnicity) of students in grades 4, 8 and 12. In the subject area of mathematics, test items are written in the following mathematics content areas: number properties and operations, geometry and spatial sense, data analysis and probability, algebra, and measurement.

For the item analysis, the search was narrowed to items in the mathematical content area of geometry and spatial sense. Since the published NAEP questions are intended for grades 4, 8, and 12 and this test was to be given to secondary preservice teachers, all grade four questions were eliminated. The NAEP questions are further categorized as easy, medium, and difficult, so the easy questions from grade 8 were also
eliminated. Initially, all medium and hard questions for grades 8 and 12 were included. The NAEP also labels questions according to the mathematical ability being tested: conceptual understanding, procedural knowledge, and problem solving. These three abilities parallel three of the strands of mathematical proficiency described in *Adding It Up* (NRC, 2001). On the scoring key included in the NAEP results, percentages are given for the number of students answering each question correctly, incorrectly, or whether the item was omitted. Because the questions selected would be given to preservice teachers who are mathematics majors rather than students in grades 8 or 12, any questions that were answered correctly by at least half the students were eliminated.

After completing this item analysis, a list of questions meeting the criteria was compiled. Consultations with several mathematics educators with expertise in geometry were held regarding the selection of questions for the test. These experts aided in selection of a set of questions which included procedural knowledge, conceptual understanding, and problem solving ability. Questions from each of the NAEP categories were chosen. Questions 1-3 were categorized and scored on the NAEP as requiring conceptual understanding, questions 4-6 as involving procedural knowledge, and questions 7-15 as necessitating problem solving skills. These questions would serve to differentiate among students’ mathematical abilities.

The NAEP items used for this test covered three strands of mathematical proficiency, conceptual understanding, procedural fluency, and strategic competence. Adaptive reasoning would become apparent during the interview phase as students showed their ability to demonstrate logical thinking about the relationships among the
concepts and procedures. The test questions were refined as a result of the pilot study and the final version of the test is included in Appendix A. Refer to Table 1 below for each test item’s content, knowledge type, grade level, difficulty, and the accuracy percentages by NAEP participants.

Table 1

*Categories for Test*

<table>
<thead>
<tr>
<th>Question</th>
<th>Content</th>
<th>Knowledge Type</th>
<th>Grade level/Difficulty</th>
<th>% Correct (NAEP)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Construct 45° angle</td>
<td>Conceptual understanding</td>
<td>8th/Hard</td>
<td>24%</td>
</tr>
<tr>
<td>2</td>
<td>Diameter/chord</td>
<td>Conceptual understanding</td>
<td>8th/Hard</td>
<td>33%</td>
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<tr>
<td>3</td>
<td>Similar triangles</td>
<td>Conceptual understanding</td>
<td>12th/Hard</td>
<td>24%</td>
</tr>
<tr>
<td>4</td>
<td>Distance</td>
<td>Procedural knowledge</td>
<td>12th/Hard</td>
<td>32%</td>
</tr>
<tr>
<td>5</td>
<td>Perpendicular diagonals</td>
<td>Procedural knowledge</td>
<td>12th/Hard</td>
<td>19%</td>
</tr>
<tr>
<td>6</td>
<td>Similar triangles—same angle</td>
<td>Procedural knowledge</td>
<td>12th/Hard</td>
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<td>7</td>
<td>Open box</td>
<td>Problem solving</td>
<td>8th/Hard</td>
<td>N/A</td>
</tr>
<tr>
<td>8</td>
<td>Angle bisectors</td>
<td>Problem solving</td>
<td>8th/Hard</td>
<td>23%</td>
</tr>
<tr>
<td>9</td>
<td>Radio station</td>
<td>Problem solving</td>
<td>8th/Hard</td>
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</tr>
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<td>10</td>
<td>Side length</td>
<td>Problem solving</td>
<td>12th/Hard</td>
<td>21%</td>
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<td>11</td>
<td>Betweenness</td>
<td>Problem solving</td>
<td>8th/Hard</td>
<td>23%</td>
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<td>Right prism</td>
<td>Problem solving</td>
<td>12th/Hard</td>
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<td>13</td>
<td>Coordinates of circle</td>
<td>Problem solving</td>
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</tr>
<tr>
<td>14</td>
<td>Quadrilateral properties</td>
<td>Problem solving</td>
<td>8th/Hard</td>
<td>25%</td>
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<tr>
<td>15</td>
<td>Midpoint</td>
<td>Problem solving</td>
<td>8th</td>
<td>38%</td>
</tr>
</tbody>
</table>
Grouping Participants

A test was administered to the participants in order to group them according to their test performance. This grouping was done to look for similarities and differences among the different skill levels. Three individuals correctly answered all 15 questions on the test and were designated as the high group. Three individuals incorrectly answered 7 out of 15 questions and were designated as the low group. The remaining nine individuals constituted the middle group. Follow-up interviews were conducted with two participants from the high group, two from the low group and one from the middle group.

Semi-structured Interviews

After careful analysis of the results of the test, questions were constructed that would be used in the interview phase of this study (see Appendix B for the semi-structured interview questions used in this phase of the study). While questions 1-3 were categorized and scored on the NAEP as requiring conceptual understanding, it was possible that a student may have been thinking in a procedural manner when answering these questions correctly. Question 1 is an example of a problem whose category may be ambiguous and thus was included as an interview question. (“Which of the following constructions at point P...will produce an angle of 45°?”)

![Figure 1 Construction question](image-url)
A) Constructing only the bisector of $\angle APB$ ($\angle APB$ is a straight angle (measuring $180^\circ$), thus the bisector would form an angle measuring $90^\circ$.)

B) Constructing only a circle with center at $P$ (constructing a circle with center at $P$ does not even form an angle at $P$.)

C) Constructing one perpendicular line only (constructing one perpendicular line only would lead to a $90^\circ$ angle.)

D) Constructing a circle with center at $P$ and a perpendicular line (constructing a circle with center at $P$ and a perpendicular line would again lead to a $90^\circ$ angle.)

E) Constructing a perpendicular line and an angle bisector (this is the only logical choice.)

Simply knowing the definition of perpendicular (lines forming $90^\circ$ angles) and angle bisector (divides an angle into two angles with equal measure), $90 \div 2 = 45$ confirmed choice E as being correct. While all the multiple choice distracters have a geometric basis, a conceptual understanding of construction was not necessary to arrive at the correct solution. Since the problem used a multiple choice format, it was necessary to see if the participants knew the procedures used in constructing a $45^\circ$ angle or if they simply chose the answer that was most reasonable. Even though the participants all answered this question correctly, an interview question was designed to disclose misconceptions regarding constructions.

The procedural knowledge test items were questions 4-6. Using the distance formula for #4 would show procedural knowledge; however, if the participant did not remember the distance formula, the problem might require additional types of
understanding in order to correctly solve this problem. In the interview, participants were asked how they solved the problem. Additionally, they were asked if they knew any other strategies that could have been used to solve the problem. Question #5, also classified by the NAEP as requiring procedural knowledge, could show evidence of conceptual understanding and strategic competence when participants were asked to justify that their drawings were parallelograms and to prove that the diagonals shown in their figures were perpendicular. A possible procedural approach to this problem would be to first draw perpendicular lines (to become the diagonals of the parallelogram) and then draw in the sides of the parallelogram. An explanation demonstrating adaptive reasoning would be to state a property that was previously learned in another situation applicable to this problem and explain why it was appropriate. (A quadrilateral with perpendicular diagonals that bisect each other is a rhombus).

If the figure did not look like a parallelogram, then the participant was asked what adjustments needed to be made to the diagonals so that the resulting figure would be a parallelogram. Following the completion of drawing a figure resembling a parallelogram, the participants were asked to use reasoning and/or informal proof to confirm that the figure was a parallelogram and that the diagonals were perpendicular.

The problem solving test items 7-15, were examined to determine which questions showed a variety of solution methods. For example, question #13 was solved using three distinct methods by the participants. Most participants were able to state alternate methods that could be employed besides their initial solution method. Question
#15 was chosen as an interview question because the common procedural solution method of using the midpoint formula was misused by so many participants.

Interview questions were based on selected test items. Responses were used as prompts to probe the thinking processes of the participants. Interviewing techniques, such as explaining the project, developing rapport with the participants, and tape recording the interviews were used (Spradley, 1979). The subsequent verbatim transcription represented a portion of the field notes that were collected (Bogdan & Biklin, 1992; Spradley, 1979). Other field notes were taken during the interview that depicted non-verbal responses from the participants and explanations of written work done by the participants during the interviews. Additional field notes were composed after reflection upon listening to the tape recordings while studying the participants’ written work.

*Adding It Up* (NRC, 2001) and four of its five strands of mathematical proficiency served as the framework for the analysis of both the test and the interviews. These research-based coding categories were the basis for the initial coding (Bogdan & Biklin, 1992; NRC, 2001; Strauss, 1987). Using the descriptions of each of the strands of mathematical proficiency, the interviews were color-coded to reflect the specific strand that was represented by the statements and explanations made by the participants during the interview phase. For example, conceptual understanding was shown when participants were able to connect ideas, were able to show similarities and differences in a variety of representations, and were able to know why an idea was important and be able to use it in the correct context. Procedural fluency was shown in the knowledge and skill used in performing appropriate procedures. Demonstrating a variety of solution
strategies leading to the correct solution to a problem was an example of strategic competence. Adaptive reasoning would be exhibited when students were able to logically verbalize their thoughts about the relationships among concepts and strategies and apply them in new situations.

**Procedure**

**Phase 1 Assessment**

All fifteen students enrolled in the single section of “The Teaching of Middle School/Junior High Mathematics” voluntarily chose to participate in the initial phase of this study and did not receive any incentive for completing the test. The methods class met for 75 minutes twice weekly. The test was given the first week of fall term 2006 on the second scheduled class day. The participants completed the test during the last half hour of this regularly scheduled class period. Students were allowed to leave when they completed the test. Most students completed the test in 20-30 minutes although there were a few students who took more time to finish. Students were allowed to decline further participation in the interview phase of the study; 11 of the 15 consented to being interviewed.

The frequency of students with correct answers was tabulated, as well as those with partially correct solutions to the free response items. Each participant was given a score based on total number of correct responses. Participants were then ranked according to this score, and divided into three groups—low, middle, and high. Three participants out of the fifteen answered each question correctly and were, therefore,
placed in the *high* group. There were also three participants who correctly answered only eight out of the fifteen questions and they formed the *low* group.

Tests were examined to see which questions were missed most frequently and which questions were answered correctly by all participants. The work that was shown was inspected to try to determine the solution methods that were used in problems in which the answers were correct but also when the answers were incorrect. For incorrect solutions, the method used was analyzed to see whether the method was appropriate and an arithmetic error was made or if the method itself was inappropriate or flawed in some way other than simple computation. In addition, four of the five intertwining strands of mathematical proficiency (NRC, 2001) were used to further categorize solution methods observed on the test.

In algebra, it is fairly easy to design questions that will show procedural fluency without really having to understand the conceptual basis for the algorithms. In geometry, however, the strands seem more difficult to unravel. For example, an item that was categorized on the NAEP as procedural knowledge (What is the distance between the points (2, 10) and (-4, 2) in the xy-plane?) may require conceptual understanding to solve unless the student has memorized the distance formula and merely substitutes the x and y values into that formula. Moreover, a procedural knowledge question is procedural to students only if they possess the prior knowledge that the test writers are assuming. Thus, selecting items for the test that were purely procedural or purely conceptual was difficult. Therefore, the NAEP categorizations were used for the questions that were selected.
When examining the work shown on the completed tests, the following were criteria that indicated conceptual understanding: (1) setting up a correct proportion, (2) using an appropriate diagram, (3) using the Pythagorean Theorem in a slightly different problem context, and (4) using algebra to solve geometric problems involving angles. A lack of conceptual understanding was noted when incorrect assumptions or inaccurate problem set-ups were made, incorrect formulas were used, a lack of number sense was shown in unreasonable answers, and a misunderstanding of terminology was evidenced. Procedural fluency was shown when formulas, proportions, and algebra were used accurately and appropriately. A lack of procedural fluency was evident when algorithms were used incorrectly or inefficiently. Using diagrams in addition to traditional algorithms showed strategic competence, whereas a lack of strategic competence was demonstrated when participants were unable to properly use strategies such as guess and check or steps towards a solution were begun but did not lead to a final solution.

Phase 2 Interviews

Using purposeful sampling, five participants from the Assessment Phase were selected for interviews. All five of these participants had expressed a willingness to take part in a follow-up interview. Two participants from the high group (one man and one woman) and two participants from the low group (also one man and one woman) were chosen for follow-up interviews. One additional woman participant from the middle group was also chosen for a follow-up interview. Appointments for individual interviews were made with each participant at a time that was mutually convenient. The hour-long interviews were audio taped and subsequently transcribed.
Before each interview, photocopies of the participant’s test were made after first removing any markings that would indicate errors. This allowed the participant to see and be reminded of the work that she or he had done. For question #1, several tools were available from which participants could choose to complete the construction: two types of compasses, a straight edge, a protractor, a Mira, paper, and scissors. While completing the construction, each participant explained what was being done and why those steps were used. Some participants were asked why those steps would produce a valid construction. The use of accurate and precise mathematical language in the explanations was recorded. It was also noted whether steps were algorithmic or if a conceptual understanding of geometry was shown. An example of algorithmic steps is shown below:

1. Center the compass on point $P$ and make markings on the line that are the same distance from $P$ but on opposite sides.
2. Center the compass at each marking and draw arcs that intersect above (and/or below) the line.
3. Use the straight edge to connect the intersection of the two arcs with point $P$. This line segment is perpendicular to the original line.

*Figure 2* Typical compass and straight edge construction of perpendicular line
(4) Now bisect this right angle by centering the compass at point $P$ and drawing an arc through both lines. (5) Center compass at each intersection point (of original line and perpendicular line) and draw intersecting arcs. (6) Connect that intersection point with point $P$. This line segment bisects the right angle so it forms angles that are each $45^\circ$.

![Figure 3 Constructing an angle bisector](image)

An example (using the same steps as above) of an explanation showing conceptual understanding might include the following: (1) Equal segments are marked off because both are endpoints ($X$ and $Y$) of radii of the same circle. $PX$ and $PY$ are radii of the same circle. (2) Using a larger radius on the compass, arcs drawn from $X$ and $Y$ will intersect and the intersection point ($Z$) will be equidistant from $X$ and $Y$ because $XZ$ and $YZ$ are radii of circles with equal radii. (3) Drawing in $XZ$, $YZ$, and $PZ$, gives us two congruent triangles [$XZ \equiv YZ$ (equal radii); $XP \equiv YP$ (equal radii); $PZ \equiv PZ$ (reflexive property); so $\triangle PXZ \equiv \triangle PYZ$ (SSS triangle congruence theorem); then $\angle XPZ \equiv \angle YPZ$.
(corresponding parts of congruent triangles are congruent); \( m \angle X P Z + m \angle Y P Z = 180^\circ \) (straight angle); \( m \angle X P Z = m \angle Y P Z = 90^\circ \). Steps 4-6 are similar.

All paper-and-pencil work that was done by the participants during the interview was analyzed along with the transcriptions of the audio taped interviews. Responses to each question asked in the interview were coded according to whether they were judged to reflect procedural knowledge, conceptual understanding, strategic competence, or adaptive reasoning (NRC, 2001), as well as precision of mathematical language.

After all the interviews were completed, the field notes were transcribed (Bogdan & Biklen, 1992; Spradley, 1979). The transcripts were read, question by question, looking for the variety of solution methods that were used, whether the participants could justify why the particular methods used were appropriate, and noting the vocabulary they used when describing their solution methods. The responses were then coded based on conceptual understanding, procedural fluency, strategic competence, and adaptive reasoning. Because there were indications of variations in productive disposition, coding for productive disposition was included, as well. The coding was examined for similarities and differences among participants. Finally, the test results were scrutinized doing an item by item analysis of the responses on the test questions looking for evidence of conceptual understanding, procedural fluency, and strategic competence.

Later, group comparisons were made of the participants who were interviewed—the two in the high group and the two in the low group. Coincidentally, the high group consisted of one male and one female as did the low group. Further comparisons were
done with the high group male with the low group male and the high group female with
the low group female.
CHAPTER 4

FINDINGS

This study focused on secondary preservice teachers and their content knowledge of geometry topics typically taught in the high school curriculum. The conceptual framework was the intertwining strands of mathematical proficiency (NRC, 2001). The conceptual understanding, procedural fluency, strategic competence, and adaptive reasoning of secondary preservice teachers needs to be brought to the attention of the mathematics community and documented so that improvements in teacher education can be made. The following research questions were posed in this study:

1. Do secondary preservice teachers have mathematical proficiency in topics commonly taught in high school geometry courses?

2. What interplay is seen among the strands of mathematical proficiency (NRC, 2001) related to geometric content knowledge of secondary mathematics preservice teachers?

These questions were explored by examining the content knowledge of geometry held by mathematics majors in a teacher education program. This chapter will present the findings on each of the strands of mathematical proficiency (NRC, 2001) that were explored in this study.

The literature reviewed in Chapter 2 fairly consistently showed the lack of conceptual understanding in many mathematical topics. This study confirmed that lack of conceptual understanding in some participants, specifically, in the area of geometry. Some studies gave brief attention to procedural knowledge but this type of knowledge
sometimes seemed to be assumed, especially for secondary preservice teachers (Eade, 2003; Even, 1995; Kinach, 2002a, 2002b; Wilson, 1994). This study did not assume procedural fluency but rather took an objective look at the presence or possible absence of procedural fluency. Careful examination of the tests revealed instances where the lack of procedural fluency was evidenced. Because of the intertwining nature of the strands of mathematical proficiency, strategic competence and adaptive reasoning were sometimes touched upon in the literature regarding content knowledge but were generally not studied individually. This study considered the strands individually while acknowledging their interconnectivity.

Secondary preservice teachers in this study completed a pencil-and-paper test and selected participants were subsequently interviewed. The purpose of the pencil-and-paper instrument was to group participants according to the knowledge exhibited on the test. Additional analysis revealed the participants' application of some or all of the strands of mathematical proficiency as reflected in the work shown when participants completed the items on the test.

Follow-up interviews delved more deeply into the preservice teachers' understanding of geometry topics. Interviews were coded according to whether the data were judged to reflect each of the intertwining strands of mathematical proficiency (NRC, 2001). While there was a wide range of scores on the geometry test, the students who scored highest on the test also showed a high level of conceptual understanding in the interview phase of this study. Participants were able to solve problems in a variety of ways and draw relationships among concepts and procedures.
Of the fifteen preservice teachers who completed the 15-item geometry test, only three individuals solved all problems correctly (see Table 2). The scores ranged from $53 \frac{1}{3} \%$ to 100%. In order to maintain anonymity, pseudonyms were assigned to participants who completed the Phase 2 interviews and were used when referring to specific participants.

Table 2

Results of test

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<th>#11</th>
<th>#12</th>
<th>#13</th>
<th>#14</th>
<th>#15</th>
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X refers to a question answered incorrectly
While the tests were a good source for determining error patterns and seeing common strategies, the interviews provided valuable insight into the ways the participants represent their knowledge. By talking with these individuals, it was possible to delve more deeply into their mathematical thinking and determine what strands of mathematical proficiency were evident in the participants' responses. Some interview questions elicited more strands of mathematical proficiency than others because of the nature of the question or the individual being interviewed.

**Conceptual Understanding**

Research into conceptual understanding is prevalent in the literature and is reviewed in Chapter 2. Findings from these studies show a lack of conceptual understanding in many populations, including preservice elementary and secondary teachers. This study confirms the lack of conceptual understanding in some secondary preservice teachers, specifically in the area of geometry.

As discussed earlier, conceptual understanding requires comprehension of mathematical ideas, operations, and understanding relationships among those ideas. Differing amounts of conceptual understanding were demonstrated depending upon the question being asked and the participant's own mathematical background.

**The Test**

By doing a careful analysis of the test, it was possible to discern several strands of mathematical proficiency in the answers given to items on the test. The work that was shown offered insight into the students' thinking even without conducting an interview with each student. Evidence of conceptual understanding was shown through using an
appropriate diagram, correctly setting up proportions, and using the Pythagorean Theorem in contexts other than being given a right triangle with the measurement of one side missing.

The first three items on the test were classified on the NAEP as testing conceptual understanding. The first was a multiple choice question about constructing a 45° angle. All participants answered this multiple choice question correctly, but the follow-up interviews disclosed misunderstandings about this question.

Question #2 showed two chords of a circle (one being a diameter) and asked about the relationship between their lengths. Only one participant answered this question incorrectly on the test stating that the chord and the diameter were the same length. This showed a lack of conceptual understanding regarding properties of chords of circles. The final question involving conceptual understanding dealt with similar triangles. For this question, most students used their knowledge that the corresponding sides of similar triangles are proportional.

A problem solving item on the test asked participants to draw nets of an open box whose volume was 8 cubic units. Conceptual understanding was shown on this item when participants used factors of eight to help determine possible dimensions for the box.

The lack of conceptual understanding was seen on the test in a multitude of ways. Proportions were set up incorrectly, as shown in Figures 4 and 5 below. In a procedural knowledge question dealing with triangle similarity, all five of the participants who missed this problem gave the same incorrect answer. Instead of using the proportional relationship \( \frac{BC}{DE} = \frac{AB}{AD} \), participants used an incorrect proportional relationship
such as $BC : DE = BD : AD$ (In similar triangles, the corresponding sides are proportional; but proportionality is not present when one side is subdivided into two sections). Figure 4 shows one incorrectly used proportion where the participant compared the bases of the triangles (12 and $x$), but then compared only part of the side of the large triangle (4 instead of $4 + 2$) to the whole side of the small triangle (2).

In Figure 5, the participant did not recognize which pairs of sides were corresponding and therefore proportional to each other.
Another participant chose the correspondences correctly but in one proportion compared the small triangle to the large triangle but in the other proportion reversed the relationship.

False assumptions were also made by participants when looking at the diagrams that were included on the test. In one case, a participant assumed the triangles shown in Figure 5 were right triangles and therefore used the Pythagorean Theorem to solve for the missing side. Using the Pythagorean Theorem on the top triangle, however, would have shown that the top triangle was not a right triangle. Further, the vertical angle theorem would then confirm that the bottom triangle was also not a right triangle, eliminating the Pythagorean Theorem as an effective strategy for solving this problem.

Conceptual understanding can help avoid computational errors, especially in orders of magnitude (NRC, 2001). Computational errors included solving equations incorrectly, erroneous basic facts, and faulty decimal placement resulting in unrealistic answers that remained unchecked. Figure 6 shows an incorrectly set up proportion with misplacement of the decimal point which resulted in an answer that was clearly not reasonable.

\[
\frac{8}{12.8} = \frac{4.8}{x}
\]

\[
x = 37.5
\]

*Figure 6 Lack of number sense in question 3*
A deficit in conceptual understanding involved remembering formulas incorrectly. One participant failed to square the differences when implementing the distance formula. This resulted in an unreasonable answer of $\sqrt{14}$. Looking at the ordered pairs, (2, 10) and (-4, 2), showed that the horizontal distance was 6 and the vertical distance was 8. Clearly, the distance between these two points could not be less than 4. Actually, the distance would have to be greater than 8 and less than 14.

Incorrect midpoint formulas were also used, again giving unreasonable answers. (most common error was to subtract the coordinates instead of adding them—one participant forgot to divide the sum by two). Participants also seemed sure enough of their memories that they neglected to check the reasonableness of their answers by drawing a simple picture. This midpoint problem was also included in the follow-up interviews.

![Figure 7 Incorrect use of the midpoint formula](Image)
Question #5 was an interesting problem where the participants were asked to draw a parallelogram that had perpendicular diagonals. Figure 8 shows an example of a figure that satisfies the problem requirements.

Figure 8 A correct response for question 5

Approximately one-third of the participants were unable to answer this question correctly on the test. Figures that did not appear to be parallelograms or parallelograms whose diagonals were clearly not perpendicular were the errors found when analyzing this problem. Two participants drew figures that were parallelograms but no diagonals were drawn. These participants apparently did not fully understand the meaning of diagonals.

There were additional examples where it seemed that the participants did not entirely understand the terminology used. One participant drew the net of an open box whose surface area, rather than volume, was 8. Another participant seemed to be confused about the definition of midpoint and found the distance between two given points rather than finding the coordinates of the midpoint of the segment connecting the two endpoints.

One participant noted that the question regarding angle bisectors was poorly worded. When examining his work on this problem, it seems that although he found the
correct answer, he neglected to use the concept of angle bisectors in his solution. He simply used the fact that the sum of the two angles was 90° and half of that would be 45°. This contrasts with the algebraic solution of a different participant who used the concept of angle bisectors in the solution.

An item dealing with spatial sense was the most frequently missed problem on the test. A two-dimensional representation of a pentagonal prism was given. The question asked for the intersection of the prism with a plane containing three non-collinear vertices of that prism—this intersection would be a pentagon. The most commonly chosen incorrect answer (75% incorrect responses) was a triangle, probably because three vertices were chosen. Spatial sense, as well as number sense, is an aspect of conceptual understanding. Conceptual errors, such as making incorrect assumptions, basing answers solely on visual clues, and using an incorrect set-up for a problem, suggest serious flaws in mathematical understanding, which would need to be addressed through instruction.

The Interviews

The initial interview question referred to the first problem on the test—constructing an angle of 45°. While all participants answered this question correctly on the test, it was included on the follow-up interview to see whether the participants actually knew how to do the construction or if the chosen answer was simply the only choice that was reasonable.

During the interview with Cara (low group), it became apparent that she did not know how to do this construction, but she selected the correct answer because it was the only choice that sounded like it would make an angle of 45°. "I looked at the choices to
give me what would sound right for a correct answer. So then I narrowed them down … and part E sounded the best” (Cara’s interview). None of the other interviewees revealed how they chose the correct answer, but instead proceeded to do the construction. In retrospect, it might have provided an added dimension into their thinking if they had been asked to explain how they decided upon their answer for this multiple choice test item.

After Cara used the protractor to draw a 45° angle, she was asked if there would be another way to make it a more exact construction. She replied, “I think that the protractor would make it the most exact.” This demonstrated that Cara and the others using only a protractor did not understand the mathematical meaning of construction.

Adam (high), however, immediately chose a compass and straight edge in order to do his construction, and in doing so, his explanation was much more conceptual than procedural because he didn’t remember exactly how the construction should be done. Although a demonstration of conceptual understanding was lacking in 60% of the interviewed participants regarding compass and straight edge constructions, they did seem to understand the meaning of perpendicular and angle bisector and could use that knowledge to create an angle measuring 45°.

An interview question asked participants to find the distance between two points in the xy-plane. While most participants used a procedure to find this distance, four out of the five interviewees mentioned the relationship between the distance formula and the Pythagorean Theorem. This shows a degree of conceptual understanding regarding these topics.
Cara (low), had difficulty finding a parallelogram whose diagonals were perpendicular. On her test, she had drawn a non-rhomboid parallelogram and had also drawn in the diagonals which were clearly not perpendicular (see Figure 9). She had, however, drawn in right angle symbols where the diagonals intersected.

During the interview, Cara was asked if the diagonals in her picture were perpendicular. She said, "Well, for the most part. Obviously when I was doing it, I didn’t have a straight edge or anything" (Cara’s interview). So she was provided with a straight edge and asked to try it again. This time she drew a more elongated parallelogram.

After drawing the second parallelogram, she explained what she was doing as she drew in the diagonals. "I can make my diagonals ... a line connecting one corner to the opposite
corner. ... then I just need to make sure that they form a 90° angle, which according to my drawing, again, they don’t” (Cara’s interview). When asked to measure the angle that was formed by the diagonals, it measured about 166°. She did not seem to realize that by making the parallelogram longer, the angle between the diagonals also increased. Even though Cara’s second attempt at drawing this figure had its diagonals farther from being perpendicular, she did not consider trying to draw a parallelogram that was shorter than her original figure in an effort to move the diagonals closer to 90°. This indicates a lack of strategic competence. She then gave up on the problem rather than exploring other options for her parallelogram.

Adam (high), on the other hand, had a much deeper understanding of the properties of quadrilaterals mentioning that the diagonals of the parallelogram needed to bisect each other. When asked why that fact was important for this figure, he wasn’t sure but proceeded to try to draw a counterexample—a parallelogram with perpendicular diagonals where the diagonals did not bisect each other. The figure he drew was a kite and he realized that in order for the figure to satisfy the problem constraints, a special kite—a rhombus—would need to be drawn.

The fourth interview question related to finding the points of intersection of a line parallel to the x-axis intersecting the y-axis at (0, 4) and a circle centered at the origin with radius 5. Conceptual understanding was reflected when participants noted properties of a circle, properties of a horizontal line, and the symmetric nature of the intersection points when having a circle centered at the origin.
Participants also seemed to understand the relationship between the Pythagorean Theorem and the distance formula and could generally use those concepts interchangeably. When listening to their informal proofs using congruence, parallelism, and perpendicularity, evidence of conceptual understanding was present.

**Procedural Fluency**

Procedural fluency seems to be assumed in the literature particularly when addressing secondary preservice teachers (Cooney, 1995; Floden & Meniketti, 2005; Frykholm, 1999; Kinach, 2002b). Some studies have revealed insufficiencies in this area even though participants show confidence in their knowledge (Ball, 1990a; Bryan, 1999; Eade, 2003; Even, 1999; Wilson, 1994; Wood, 1993). This study found varying degrees of procedural fluency among its participants.

Procedural fluency is evident when participants show their skill in carrying out the procedures they have learned and by using appropriate procedures. While only three items on the test were classified on the NAEP as requiring procedural knowledge, procedural knowledge was exhibited in most of the items on the test. Aside from some careless arithmetic errors, it seemed that most participants showed procedural fluency in the areas that were tested.

**The Test**

The first of the items that was categorized on the NAEP as requiring procedural knowledge asked participants to find the distance between two points in the $xy$-plane. This could be done using the Pythagorean Theorem (Figure 11) or more directly by using the distance formula (Figure 12).
Careful inspection of Figure 11 disclosed an error in notation. $C^2$ actually equals 100, making $C$ equal to 10 units.

$$d = \sqrt{(2+4)^2 + (10-2)^2} = \sqrt{36 + 64} = \sqrt{100} = 10$$

Figure 11 Using the Pythagorean Theorem

Figure 12 Using the distance formula

Most participants appeared to be comfortable using the Pythagorean Theorem and distance formula even though there were some instances where errors were found in their use. Two-thirds of the participants used the distance formula, while the remaining third used the Pythagorean Theorem. The work shown in the two figures above demonstrate procedural fluency; the correct numbers were inserted into known formulas and then solved correctly for the unknown values. Of the two students who missed this problem, one student used an incorrect distance formula and the other made a computational error while using the distance formula, both examples showing a lack of procedural fluency. Throughout the test, procedural fluency was shown when participants were able to set up proportions or algebraic equations and then solve them accurately.
The question asking participants to draw a parallelogram with perpendicular diagonals was categorized on the NAEP as requiring procedural knowledge. If the participants had remembered that squares and other rhombi have perpendicular diagonals, this would be considered a procedural knowledge question. For the participants who did not remember this property, however, the problem was no longer purely procedural knowledge. Adam (high) began this problem by drawing perpendicular diagonals and then connected the endpoints of these diagonals in a way that would form a parallelogram. He appeared to be the only participant who began with the diagonals, a procedural method for drawing this figure. Beth (high) remembered a theorem from her geometry classes (diagonals of a square are perpendicular) that led her to draw a square. Erin (middle) also remembered that the diagonals of a rhombus were perpendicular, so she used that fact to make her drawing.

Some of the work seen on the test showed a surprising lack of procedural fluency. One participant used an inefficient algorithm when doing long division and also made an error in the decimal placement in the answer (see Figure 6). Another participant used common numerators when solving a proportion problem. While this is not a commonly used method for solving proportions, it is appropriate; however, when using this method, the participant thought that the product of 6 and 8 was 42 and thus, her answer was inaccurate. This participant also used the square root symbol when doing the long division that was necessary to solve the problem. Using this symbol incorrectly will undoubtedly cause confusion in her future students when discussing the two completely different topics of long division and square roots.
Other instances showing lack of procedural fluency included algebraic equations being simplified incorrectly and misidentifying the coordinates of points on a graph. The procedural errors, such as faulty arithmetic or incorrectly remembering formulas or procedures, however, could be the result of simple carelessness or being hurried in the testing situation.

The Interviews

For the first interview question, further investigation into the item dealing with constructing a 45° angle was done. Even though all the participants chose the correct answer (constructing a perpendicular line and an angle bisector) on the test, it was important to find out whether the participants actually knew the procedure for completing the construction. In the interview, a variety of tools were available for the participants to use for constructing an angle of 45°: ruler, two types of compasses, Mira, protractor, scissors, pen, and pencil. Although the word, construction, was used deliberately, four out of the five interviewees chose to use the protractor to construct the 45° angle.

Some participants were very exact in their descriptions of how to use the protractor and most drew a perpendicular line first and then a 45° angle to match the multiple choice answer. Beth (high) stated, in referring back to the method she used on the test, “I guess in the drawing originally, I had a perpendicular line, and then I had my angle bisector, but really I wouldn’t need to do that if I had this tool [the protractor]” (Beth’s interview). It was noted that Cara (low) chose to approximate a perpendicular line even though she was using a protractor at the time. She also approximated the 45° angle by trying to bisect the angle formed between the original line segment and her
perpendicular line. When asked to measure her angle, she noticed that it was not 45°. She then went back to her picture, used her protractor to measure 90°, drew a new line segment, then measured 45°. Cara was able to use protractor procedures to accurately draw a 45° angle.

Figure 13 Cara's construction

Another interview question referred to the item where participants were asked to find the distance between two ordered pairs in the xy-plane. This question was categorized on the NAEP as requiring procedural knowledge, and this is what was seen during the interview as well, especially with the participants who chose to use the distance formula rather than the Pythagorean Theorem.

It seemed that participants were more confident using their procedural knowledge. This is not surprising since schools find procedural knowledge easiest to teach and assess and often this type of knowledge is the focus of the mathematics curriculum.
Strategic Competence

As discussed earlier, strategic competence is shown in the ability to formulate, represent, and solve problems (NRC, 2001). More specifically, I looked for evidence that the participants were flexible enough in their thinking to find multiple ways of solving the problems. The tests in this study did not lend themselves to looking at multiple solution strategies, but the interviews were designed to determine whether the participants were able to think flexibly in their solutions.

The Test

On question 3 dealing with triangle similarity, Cara (low) did a reflection to reorient the upper triangle in order to determine the corresponding sides of the two given triangles. While this showed a lack of spatial sense, it, nonetheless, showed strategic competence in that she was able to overcome this disability.

Lack of strategic competence was shown by participants using the inefficient strategy of guess and check. Other participants attempted items on the test but were stopped when they were unable to find an additional strategy to allow them to complete the problem. For example, when asked to find the midpoint of a line segment, Erin (middle) found the length of the segment (10 units) and knew that the midpoint would be 5 units from each end point. The problem arose when she did not know how to proceed to find the coordinates of the point that was 5 units from the endpoints. While this was not the most efficient way to find the midpoint, it did show some understanding of the problem and could lead to the correct answer if completed.
One way of representing a problem is by drawing a picture or diagram. Diagrams were used both successfully and unsuccessfully by the participants in this study. More successful participants used the drawings to better visualize the problem constraints so that formulas could be used accurately (see Figure 14).

\[ 4^2 + x^2 = 5^2 \]
\[ 16 + x^2 = 25 \]
\[ x^2 = 9 \]
\[ x = 3 \]

*Figure 14 Solution using a diagram and the Pythagorean Theorem*

Less successful participants used the drawings as an aid to visually approximate the answers. At least four participants found their answers solely by looking at their drawings (see Figure 15).

*Figure 15 Incorrect solution using a diagram and inspection*
Sometimes the correct diagrams were misinterpreted when deciding upon a final answer (see Figure 16).

While there were several ways of solving this problem, over half of the correct solutions used the Pythagorean Theorem. It was assumed that some carelessness was exhibited by participants who missed this problem; these errors were both in computations and in misreading the coordinates of the intersection points found by inspection. Thus, this was another question included for more in-depth questioning in the interview portion of this study.

The Interviews

Adam (high), Beth (high), and Erin (middle) all suggested alternate methods for solving the problem shown in Figures 14-17. Drew (low) used the Pythagorean Theorem (with some errors in computation) and Cara (low) used visual inspection of her drawing (initially incorrectly identifying the coordinates and finally using graph paper to make a more accurate representation). Neither Drew nor Cara thought there would be alternate ways to solve this problem. Erin initially used inspection to determine the coordinates of
the ordered pairs. When asked if there was a way she could check her answer, she said that she could use the distance formula. When she found her coordinates to be in error, she again used the distance formula to find the correct coordinates as shown in Figure 17.

\[
\begin{align*}
\text{Distance} & = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\
& = \sqrt{(3 - 1)^2 + (4 - 4)^2} \\
& = \sqrt{4 + 0} \\
& = \sqrt{4} \\
& = 2
\end{align*}
\]

Figure 17 Use of distance formula to find coordinates

For the construction problem, Beth (high) was the only interviewee who used a protractor first and then was able to additionally construct an angle of 45° using a compass and straight edge. One of the participants in the pilot study used a Mira to do this construction, but she appeared to be the only participant who was familiar with Miras.

The parallelogram question elicited remarks showing strategic competence in some of the participants. Both Beth (high) and Adam (high) were able to approach this problem from several directions to help them discover the types of figures that would satisfy the conditions of being a parallelogram with perpendicular diagonals. Adam did
an informal proof using congruence theorems learned in geometry. Beth used grid paper to show that the sides of her figure were parallel. Erin (middle) explained how she would measure the distances between the sides of the parallelogram to justify that the sides were parallel.

For the midpoint question, Adam (high) had not shown any work for this problem on his test, so in the interview portion, he was unable to remember how he had initially solved the problem. He first thought that he had used the distance formula and since he used his graphing calculator, he had no work to be shown. As he continued to look at the problem and his sketch, he saw that he had drawn a right triangle, approximated the midpoints of the legs and noticed that the intersection would be about where the midpoint of the given segment was. He also conjectured that he could “add half the difference from the $x$ to 2 and half the difference from the $y$ to 1 to get $(5, 5)$” (Adam’s interview). Adam initially solved this problem in what he called a “purely mathematical way.” He was skeptical of his solution though and stated, “I am not that confident in my ability to remember obscure [formulas], so I sketched out the problem and found an easier way of looking at the problem” (Adam’s interview).

Drew (low) correctly used the midpoint formula to find the answer to the last question. When he was asked if he knew of another way to solve the problem, he suggested graphing it, measuring the segment, and then finding the middle. When asked what he would have done if the middle had been in between whole number values on his graph, he said he would have to guess. The answer would not be exact but it would be close.
Beth (high), like Drew (low), used the midpoint formula as her second method for solving this problem. She did not, however, call it the midpoint formula and it was only after much questioning that she came to realize that her solution method was indeed the midpoint formula. She was then able to explain and justify why this strategy would work for this problem.

Cara (low) was one of the participants who used an incorrect midpoint formula, arriving at an answer of \((3, 4)\) rather than \((5, 5)\). She thought that the difference in the \(x\) coordinates divided by 2 would give her the correct answer for the \(x\)-coordinate. Without knowing that her initial answer was incorrect, she was asked if she could find the answer in another way. Her first response was to use graph paper to illustrate the problem. She knew by inspection that the correct ordered pair would be in the middle of the line segment. So she attempted to use the distance formula to find the length of the segment and then find half that distance. After finding that the length of the segment was 10, she was not sure how to find the ordered pair associated with the point that was a distance of 5 from each endpoint. When she was asked how she would complete the problem using this method, she explained that she would probably use guess and check and substitute those guessed values into the distance formula to see if the square root would end up being 5. After graphing the problem, Cara determined the correct answer by inspection. When it was brought to her attention that the two answers she had given were different, she realized that she had incorrectly memorized the midpoint formula. Cara did show strategic competence, though, when she initially began with an incorrect midpoint formula but knew to change her answer after drawing a picture.
Erin (middle) said that she had tried to remember the midpoint formula when she did the problem on the test, but in the interview she decided that she could use the distance formula. She correctly found the length of the line segment and knew that the midpoint would be half that distance. She then attempted to use the distance formula again but ran into a problem when trying to solve the quadratic equation with two unknowns. Seeing that she would need another equation, and that calculations might get somewhat messy, she decided to try to find a point halfway between the xs and halfway between the ys. After finding the coordinates of (5, 5), she used the distance formula to confirm that the distance between this proposed midpoint and an endpoint was 5.

The participants seemed to know several methods for finding solutions to these problems. Not all their methods were elegant or expedient but all had merit. They were based on sound mathematical principles and could be used to find a correct solution.

**Adaptive Reasoning**

Adaptive reasoning in this study was evidenced by students being able to think logically among concepts and situations and apply prior knowledge in new situations. Being able to justify solution methods, using logical thought, is an element of adaptive reasoning. Some of the solution methods used by participants were reminiscent of Greeno’s (1991) description of knowing your way around an environment. These students were able to take the information they knew and apply it to a new and different problem. These are aspects of adaptive reasoning.
Using algebra in a geometric context (see Figure 18 below) is a demonstration of adaptive reasoning in that a logical choice of strategies was used and prior knowledge was utilized in an unrehearsed situation.

Evidence of adaptive reasoning in this study is, for example, when a participant did not remember the formula for finding the midpoint of a line segment, the participant used the line segment as the hypotenuse of a right triangle and then drew a vertical leg and a horizontal leg to complete the triangle. The midpoints of those legs were found. Drawing a horizontal line through one midpoint and a vertical line through the other midpoint, the participant found that intersection of those two lines met on the midpoint of the hypotenuse, thus solving for the midpoint of the given line segment.

Adam mentioned that he usually tries to memorize one fairly general formula and then derives other things from that formula. This is another example of adaptive reasoning and using prior knowledge to solve a problem in a different context.
The second interview question delved more deeply into the parallelogram problem. Three out of the five interviewees accurately drew a picture that satisfied the conditions. One drew a square and the other two drew rhombi. Adam (high) said that he could have left his drawing as a square but decided to draw a more general parallelogram instead. After some sketching of different parallelograms and further consideration, Adam decided that it was necessary to draw a special parallelogram where all sides are congruent (thus limiting the choices to a rhombus or square).

The other two participants were not able to draw a figure that would satisfy the stated conditions. Drew (low) did not seem to understand what it meant to have perpendicular diagonals. On his test, he drew a rectangle but no diagonals were shown in his drawing. When asked to draw in the diagonals and see if the diagonals were perpendicular, he asked, “Where are they supposed to be perpendicular, to what? ... I just don’t understand what they are asking” (Drew’s interview). It was necessary, then, to back up and try to figure out what Drew was misunderstanding. He seemed to know that diagonals connect opposite vertices in a rectangle and he also was able to give a vague definition of what it means for lines to be perpendicular (he knew that perpendicular lines intersected to form 90° angles). He did not seem to be able, during the interview, to draw any parallelogram and show the diagonals. He could not seem to draw on prior experiences and apply them in this new situation. This is an example of a lack of adaptive reasoning.
A reconsideration of Figure 19 suggests that Drew was confusing the terms parallel and perpendicular although that conclusion was not apparent while the interview was in progress.

Further evidence of adaptive reasoning was found in the parallelogram problem. Erin (middle) measured the distance between each pair of sides (knowing the distance between parallel lines is constant) to justify that her figure was a parallelogram. She initially measured the length of the line segments connecting the sides she was trying to prove to be parallel but later remembered that she must measure the perpendicular distance. To justify that her diagonals were perpendicular, she said she would use a protractor to measure the angle formed by the diagonals.

Beth (high) had drawn a square to satisfy the requirements of having a parallelogram with perpendicular diagonals, having remembered a theorem from high school geometry. After further probing, Beth decided that a rhombus would also have perpendicular diagonals. To validate her conjecture, Beth used an informal proof to justify that the figure was indeed a parallelogram and that the diagonals were perpendicular.
Much of the work done on the problem of finding the intersection of a circle and a horizontal line was procedural in nature—drawing a diagram, dropping a perpendicular from the intersection point of the circle and the line, taking an equation and substituting in values, manipulating the equations to find a solution. After making a sketch, most of the participants used the Pythagorean Theorem for finding the intersection points, many noting the 3-4-5 right triangle that is commonly used in mathematics problems. Adam and Beth, both in the high group, used or mentioned the possibility of solving this problem by using the equation of a circle, the equation for the line, and then solving the two equations simultaneously. Beth could not immediately recall the formula for a circle but noted that she could look that up. She also remarked that if she were teaching this problem to a class, she would have refreshed her memory beforehand since this would be an appropriate solution method to use. Again, the participants in the high group continued to show evidence of adaptive reasoning.

These findings illustrate the mathematical understanding exhibited by the participants of this study. Most participants showed varying degrees of conceptual understanding. Likewise, procedural fluency was revealed with some instances of carelessness when carrying out procedures. The participants in the low group showed marked deficiencies in both strategic competence and adaptive reasoning.
CHAPTER 5
DISCUSSION

Review of Study

This story began as I journeyed through my mathematical teaching career beginning in K-12 education and leading into higher education where I have the opportunity to impact not only the lives of my own students—preservice teachers—but also the K-12 environment where my journey began. In the process, I have learned the importance of teaching for deeper understanding giving my students the opportunity to become mathematically proficient. Through my study, I have become acutely aware of the capabilities and deficiencies in secondary preservice teachers. My goal is to bring awareness to the mathematics community of the possible shortcomings in our preservice teachers, so that those weaknesses can be addressed within their programs of study.

Secondary mathematics preservice teachers are graduating with differing levels of mathematical proficiency (NRC, 2001). This is not surprising, and it happens in every field. As teacher educators, we want to continue to improve our programs so that future teachers are well prepared to be effective in their classrooms. This study suggests that some gaps exist in the mathematical knowledge of our preservice teachers. As stated earlier, my research questions were:

1. Do secondary preservice teachers have mathematical proficiency in topics commonly taught in high school geometry courses?
2. What interplay is seen among the strands of mathematical proficiency (NRC, 2001) related to geometric content knowledge of secondary mathematics preservice teachers?

To answer these questions required a written test and follow-up interviews with secondary preservice teachers. Analysis was done using the framework found in Adding It Up (NRC, 2001), specifically looking at the strands of conceptual understanding, procedural fluency, strategic competence, and adaptive reasoning related to topics commonly found in the high school geometry curriculum.

In this chapter, I discuss the interpretations and implications of the data collected and analyzed, the need for future study, and I will finish with some concluding remarks.

Interpretations

In this section, I will discuss the conclusions I have drawn from this study, the similarities and differences in the chosen groups, and how mathematical understanding is judged in relation to the strands of mathematical proficiency considered in this study. Looking back at the items that were included in the test, we must consider that these problems were released items from the NAEP and designed for students in grades 8 and 12. My initial reaction to the selections made for this test was that the items might be too easy, that the students would have no trouble accurately completing the problems, and that my study would simply be a confirmation that mathematics majors are adequately prepared to teach high school mathematics. The pilot study, however, showed that the selected problems would provide a challenge for the participants. While the actual
performance of these students led to a more interesting study, it was still surprising to me that students did not achieve, as anticipated, at a higher level.

In answering my first research question, I will discuss my findings related to each strand of mathematical proficiency addressed in my study. We will then look at the interplay among the strands and the resultant mathematical proficiency of the secondary preservice teachers in this study.

**Question 1**

Do secondary preservice teachers have mathematical proficiency in topics commonly taught in high school geometry courses?

**Conceptual Understanding**

The three items on my test that were categorized on the NAEP as Conceptual Understanding were answered correctly more often (88.9% correct) than the three questions that were categorized as Procedural Knowledge (71% correct). I was initially surprised by this finding because I felt that conceptual understanding required a deeper knowledge of the material than procedural knowledge.

The participants in the high group were able to do a compass and straight edge construction of the $45^\circ$ angle. They were also able to draw a parallelogram having perpendicular diagonals. Even though this item was categorized on the NAEP as requiring procedural knowledge, the follow-up interview disclosed the presence of conceptual understanding about this topic. They were able to convince me that their figures were parallelograms and that the diagonals were perpendicular by using informal proofs as justification. They had a conceptual understanding of the terms parallel and
perpendicular, and also had familiarity with the various special quadrilaterals. The high
group also displayed conceptual understanding when discussing the Pythagorean
Theorem, the symmetric nature of a circle, and when finding the midpoint of a line
segment.

Conversely, the low group participants did not seem to know how to do a
mathematical construction using a compass and straight edge. Both participants were
able, however, to draw a $45^\circ$ angle when using a protractor. These participants were
likewise unable to draw a parallelogram with perpendicular diagonals. Both had a
conceptual understanding of what properties were necessary to draw a parallelogram but
could not discover a parallelogram with the additional property of having perpendicular
diagonals. In fact, Drew did not seem to understand the meaning of perpendicular or what
constituted a diagonal. These are basic concepts in Euclidean geometry.

There was a marked disparity in conceptual understanding between the high
group and the low group in this study. This finding is consistent with findings of other
researchers regarding the conceptual understanding of secondary preservice teachers
(Ball, 1990a; Bryan, 1999; Cooney, 1995; Eade, 2003; Kinach, 2002b; Wilson, 1994).
While these preservice teachers have a major in the content area, not all have achieved an
adequate level of conceptual understanding.

**Procedural Fluency**

My follow-up interview question regarding the construction of a $45^\circ$ angle
revealed that the majority of the responses were procedural in nature and generally not
well connected to the multiple choice answer that was correctly chosen by the
participants. They all knew (to varying degrees) the procedures necessary for using protractors to draw a 45° angle.

The distance problem, unlike the other items, elicited the most procedural explanations. In the low group, some procedural errors were detected and subsequently corrected during the interview sessions.

Drawing a parallelogram with perpendicular diagonals was classified on the NAEP as Procedural Knowledge, and for the participants who remembered properties of the diagonals of quadrilaterals, this problem would be procedural. Some participants began with a parallelogram and worked with the parallelogram to find one whose diagonals were perpendicular. I found the comments from the participants who did not remember the properties of quadrilaterals to be more beneficial in understanding their thinking, however, since I was able to listen to their reasoning and justification for the figures they had drawn. Procedural fluency was sometimes exhibited in explanations that were used to justify that the drawings satisfied the requirements of the problem.

Drawing a picture should be a routine procedure used to begin many problems. Most participants did draw a picture when solving the problem of finding the intersection of a line and a circle. By dropping perpendicular line segments and forming a right triangle, the Pythagorean Theorem could easily be used to solve this problem. Some participants identified the ordered pairs of the solution by inspection and then checked their answers using the Pythagorean Theorem.

The most significant differences I saw between the groups in procedural fluency were arithmetic errors and a lack of number sense in the low group. This study serves to
bring attention to a possible deficit in procedural fluency of some secondary preservice teachers. As in my study, Bryan (1999) also found deficiencies in procedural fluency. Other researchers acknowledge the part procedural fluency plays in mathematical understanding. Rather than investigating the presence or absence of procedural fluency, though, researchers seemed to assume its presence and focused more attention on the relationship between conceptual and procedural knowledge (Carpenter, 1986; Hiebert & Lefevre, 1986; Rittle-Johnson & Alibali, 1999; Skemp, 1978).

Strategic Competence

When I was looking for the strategic competence strand of mathematical proficiency, I tried to see if the interviewees could think of multiple approaches to solving the problems. The problem dealing with the midpoint of a line segment was the item that produced the most strategies, though not all strategies were equally effective. The midpoint formula was the most often used strategy, although that formula was not always remembered correctly.

I was very surprised to see the number of participants who used inspection to solve the problem of finding the intersection points of a circle and a horizontal line. My preconceived notions about mathematics majors led me to believe that they might be more inclined to use computation to find a precise answer. Inspection would have been a more common approach if a grid had been provided. I was encouraged, however, to see that all but one participant did make a sketch to aid in the solution.

For finding the distance between two ordered pairs in the xy-plane, four out of the five interviewees initially used the distance formula and all but one participant found the
Pythagorean Theorem and the distance formula to be interchangeable in doing this problem. It seemed that the participants were very familiar with both of these formulas and could use them correctly. This is one area in geometry where the participants seem to have strategic competence.

While the participants in my study showed strategic competence regarding the Pythagorean Theorem and its relationship to the distance formula, they lacked strategic competence in other topics in Euclidean geometry. My study suggests that when students are given ample opportunity to explore a topic, they are able to learn it with understanding and be able to use that knowledge flexibly and appropriately. With less experience with a topic, strategic competence is reduced. An educational goal should be to determine what measures will most efficiently lead to an increase in strategic competence in preservice teachers.

Adaptive Reasoning

For the adaptive reasoning strand, I listened and looked for evidence of the student using concepts, procedures, and strategies from previous problems in new situations. During the interview portion of this study, the high and middle groups demonstrated adaptive reasoning in many topics. They were able to draw on their conceptual understanding of mathematics in order to solve problems that may have been unfamiliar to them. They were flexible in their thinking when they were justifying the accuracy of their parallelogram drawings. They remembered formulas and theorems from other classes and were able to fairly effectively use those ideas in new problem situations. The high group and Erin, in the middle group, often exhibited adaptive reasoning when
they shared their logical reasoning by emphasizing relationships and connections with situations previously encountered.

High group participants were able to reason their way through problems by drawing on past experiences even when they do not remember specific formulas. This gave me more insight into their mathematical knowledge, though, as I listened to their reasoning. It was interesting to note, however, that simply being in the high group did not guarantee that important connections among topics were always present. For example, when Beth (high) did not remember the midpoint formula, she was able to use prior experience and logical reasoning to determine the midpoint of the line segment. She did not, however, notice the connection between finding the midpoint of a segment and finding an average (mean) of two numbers. She compartmentalizes her knowledge and did not immediately see the connection, stating that she categorizes the concept of averages in statistics rather than in geometry. Part of the beauty of mathematics is the connections that can be found among various topics. These connections are essential in adaptive reasoning.

The low group, on the other hand, did not display adaptive reasoning and often made errors that pointed out their lack of conceptual understanding of the items being discussed. It was disconcerting to note that the two participants in the low group had no items coded as adaptive reasoning during their interviews. Their explanations were either procedural in nature or absent altogether. They did not seem to be able to draw on previously learned concepts and then apply them in these unfamiliar circumstances.
The test used in this study revealed the presence or lack of procedural fluency and conceptual understanding in the participants. Follow-up interviews with grouped participants more clearly demonstrated those strands, but also the strands of strategic competence and adaptive reasoning were discernable. The most marked difference among the groups was found in the absence of adaptive reasoning in the low group. This suggests the need for increased attention to problem solving or other activities that will foster the development of strategic competence and adaptive reasoning in preservice teachers.

Question 2

What interplay is seen among the strands of mathematical proficiency (NRC, 2001) related to geometric content knowledge of secondary mathematics preservice teachers?

The intertwining nature of these strands of mathematical proficiency strongly came into play when coding the interview questions. I considered the way the interviewees talked about their methods and tried to determine whether they were drawing from and connecting to learned mathematical ideas or if they were employing rote procedures. It was refreshing to see that the students who performed the best on the test also showed high levels in all the intertwining strands of mathematical proficiency (NRC, 2001).

The participants in the high group showed evidence of all the strands of mathematical proficiency. For questions that required conceptual understanding, the high group was able to make the necessary connections. They showed fluency in their procedural knowledge by carrying out appropriate procedures with accuracy. They also
used mathematical terminology related to the procedures they were doing as if it were second nature to them. They were able to show strategic competence by solving problems using a variety of methods. Adaptive reasoning also seemed to come naturally to them. They were able to draw on their prior knowledge and apply it in different situations to find their solutions. They talked about their reasoning as an integral part of their solution process. The determination and perseverance displayed by this group as they approached more difficult questions was impressive. Even when they were not sure of the exact path they would take with a problem, they seemed certain they could eventually find an answer. A productive disposition was evident in both participants.

In contrast, the low group showed deficiencies in their mathematical knowledge as evidenced by a lack of or complete absence of several of the strands of mathematical proficiency. Although they did show some knowledge of mathematics, their responses were almost entirely procedural in nature. They have learned some geometry content and evidence of other mathematical content was also seen. On the other hand, a lack of conceptual understanding was shown when these participants used formulas incorrectly and when they were unfamiliar with geometric terminology. Strategic competence was shown to be lacking when these participants relied on a single, perhaps inefficient, strategy or when they did not even know how to begin a problem. Given these deficiencies, it was not surprising that they would lack adaptive reasoning. They were not able to solve a problem that was unfamiliar to them because they were unable to draw on prior knowledge and make some mathematical connections that could help in solving these problems.
They were confident, however, when solving problems that they had seen before, although at times inaccurate formulas were used and/or arithmetic errors were made. An inefficient guess-and-check strategy was used more often than I would expect from mathematics majors, and it was disappointing to see their lack of alternate strategies. Unlike the high group, these participants needed probing questions from me in order to attempt to justify their solutions.

The students in the low group, however, were willing to participate in the interview portion of this study, suggesting that they did have an interest in mathematics education. They also seemed eager to teach mathematics upon completion of their program of study. One participant in the low group wanted to teach upper level high school mathematics classes such as statistics and pre-calculus demonstrating the confidence in mathematical understanding that was held even by participants in the low group. The most disturbing aspect of these students, however, was their lack of tenacity when they came across a difficult problem. It was surprising that they were so ready to just dismiss not knowing how to do the problem and then to simply move on to another problem.

The participant in the middle group was actually more like the participants in the high group than participants in the low group. She showed conceptual understanding of the geometry topics and procedural fluency spanning geometry and other mathematical areas. She occasionally used incorrect terminology, but I think that was due to carelessness rather than not actually knowing the correct terms. While she was able to explain her reasoning, her explanations were not as detailed as the explanations from the
high group. She also displayed adaptive reasoning by using previously learned concepts in a new situation. She certainly displayed a productive disposition by being self-efficacious, curious, and willing to recheck her work to be sure of its accuracy.

While all strands of mathematical proficiency (NRC, 2001) are important, mathematics educators generally agree that knowledge of concepts is the foundation for procedures and that a teacher should focus on the development of both conceptual and procedural knowledge (Hiebert & Lefevre, 1986). The adaptive reasoning strand, however, seems to be the glue that holds the other strands together. Having the ability to reason adaptively allows a person to effectively navigate throughout the mathematical environment (Greeno, 1991). This strand goes beyond what is expected from procedural and conceptual understanding and may provide a clearer picture of one's mathematical proficiency. Furthermore, by specifically including adaptive reasoning, this study adds an important new dimension to the body of research on mathematical understanding.

**Implications**

The conclusions from this study are important for raising the awareness of the general population, and more specifically, the mathematics community regarding the mathematical content knowledge of secondary preservice teachers. This study leads us to question whether all secondary preservice teachers have gained the depth of understanding necessary to effectively provide a rich learning environment for their future students. Rather than focusing only on conceptual understanding and procedural fluency, this study used problem solving to look at two additional strands of mathematical proficiency, strategic competence and adaptive reasoning.
Conceptual understanding and procedural fluency, well-researched topics, are already a focus in many mathematics content courses. The less researched strands of strategic competence and adaptive reasoning may be somewhat neglected. These two strands, however, seem to be essential in deepening understanding so that mathematics can be useful, and seen to be useful, outside of the classroom. Strategic competence assures us of having multiple approaches and it allows us to think flexibly about problems. Adaptive reasoning makes it possible for us to think logically about a problem and then reach back to prior experiences and make the connections that are necessary to solve a problem encountered in a different situation. Through this study, I have found that the strands of strategic competence and adaptive reasoning (NRC, 2001) create the depth of knowledge needed in highly competent mathematics teachers.

When Greeno (1991) referred to knowing one’s way around an environment, he surely assumes strategic competence and adaptive reasoning in addition to conceptual understanding and procedural fluency. The participants in the low group in this study had difficulty navigating through the geometry test items and often could not apply the knowledge they did have to new situations. The high group, in contrast, did have a deeper understanding of geometry and thus were able to find their way around the geometry environment.

Likewise, the high group not only demonstrated their knowledge about geometry but also showed that their knowledge was readily available when it was needed to solve a problem. Knowing-about as well as knowing-to are both marks of deep understanding (Mason & Spence, 1999; Ryle, 1949) that were seen in the high group participants. The
low group knew about geometry; they were familiar with procedures and some concepts. They lacked the deeper understanding that would be characterized as knowing-to because of their deficiencies in the strategic competence and adaptive reasoning strands. As with knowing-to, these strands might be difficult to teach directly, but it is important to create opportunities for students to experience non-routine problems requiring strategic competence and adaptive reasoning. Viewing mathematics in this broader perspective and adapting teaching to reflect this viewpoint most assuredly would be valuable for deepening students' mathematical understandings.

All of the participants who were interviewed believed that they would be effective teachers, and it remains to be seen whether there will be any differences in the teaching effectiveness of these two different groups of teachers. I suspect that both groups will be able to accurately present mathematics instruction to their high school students if the emphasis is merely on symbol manipulation. It is important to remember, however, that students need to learn conceptually before being taught a procedure (Pesek & Kirshner, 2000; Rittle-Johnson & Alibali, 1999). Therefore, preservice teachers need to develop the deeper understandings of mathematics that will allow them to teach their students conceptually. They need to have experience working with and solving non-routine problems which will, in turn, give them confidence to provide their students with activities that will enable the students to also develop strategic competence and adaptive reasoning.

Teaching for conceptual understanding may be more difficult for the preservice teachers who were in the low group. These preservice teachers may not possess the
necessary mathematical understanding required to teach effectively. Another reason they may teach procedurally is because they do not feel the need to teach any differently than they had been taught, especially if they do not recognize their deficiencies. While both groups may be able to teach the content, the participants from the high group may be better able to anticipate trouble areas and more effectively differentiate instruction to the wide range of abilities that are present in most classrooms. Added emphasis on strategic competence and adaptive reasoning may also encourage teachers to include non-routine problem solving in their classrooms.

In addition, this study may help to inform mathematics educators about the strengths and weaknesses in secondary preservice teachers and guide them when planning curricular programs.

Need for Future Study

Productive Disposition

A large body of literature is devoted to attitudes and beliefs making productive disposition beyond the scope of this study. While coding the interviews, however, many instances were found in which the participants' attitudes and beliefs about mathematics were evident. Following are some examples of productive disposition seen during the interviews. Beth showed her perseverance when she was stuck on a problem but insisted on going back to it when she had completed the rest of the interview questions. While Adam did not exhibit overconfidence in his abilities, his self-efficacy was evident during his interview. At one point he said, "I'm pretty sure I don't know the right way to do this [problem], but I am going to do it some way" (Adam's interview). Erin also believed she
would be able to solve a problem she was working on, stating, "If I thought about it, I could certainly figure it out...I am certainly curious now" (Erin's interview). These contrast with the comments from Drew and Cara, in the low group, who both gave up, saying they just didn't know how to do the problems.

While the analysis in this study was done through the lens of the five intertwining strands of mathematical proficiency (NRC, 2001), the focus of the study was on the first four strands: conceptual understanding, procedural fluency, strategic competence, and adaptive reasoning. The fifth strand, productive disposition, was not formally investigated in this study but the data suggest that it would be worthwhile investigating in a future study.

A study of dispositions and the relationship between those dispositions of fluid, adaptive problem solvers in geometry versus those whose expertise is more rigid would also be an interesting topic. My findings seemed to show that the participants who could more easily adapt to multiple solution strategies also seemed more enthusiastic about mathematics. They were willing to spend time looking at problems from a multitude of directions, eagerly offering explanations, and exploring new ways of thinking about the problems. The more rigid problem solvers generally found one way to solve a problem and then wanted to move on rather than trying to find multiple strategies. It would be interesting to see if the students' expertise in problem solving gave them a productive disposition or if their disposition was responsible for their persistence.
Mathematical Language

A topic that emerged during an examination of my data was the use of accurate mathematical terminology. A relationship between the way the participants were able to talk about their reasoning and their level of understanding was suggested by the data in this study. The literature also shows evidence of language usage being an indicator of mathematical understanding (Batro, Nason, 1996; Borko, et al., 1992; Even, 1993; Kinach, 2002b; Leinhardt & Smith, 1985). Accurate and precise mathematical language seemed to be used more regularly with the participants in the high group than in the low group. Adam and Beth, in the high group, both used accurate and specific mathematical language in their explanations and justifications and there were five or fewer instances of incorrect use of mathematical terminology. Erin from the middle group, used language more closely aligned with the participants in the high group than the low group. Drew and Cara (the low group) each used fewer accurate mathematical terms during the interview and used informal terminology and/or incorrect mathematical terminology more frequently than the other participants who were interviewed.

Although this language factor is beyond the scope of this study, it would be worthwhile to explore how this usage is acquired and if it has an impact on student learning. It would be interesting to see if usage of accurate/precise mathematical language could be used as an assessment device to determine mathematical understanding. If accurate/precise mathematical language is a factor in determining mathematical understanding, then it would be important to find a way to develop and encourage those language habits. Perhaps more importantly, we should investigate
whether teachers using accurate/precise mathematical language positively impacts student learning.

Large Scale Study

This study was an exploratory study and limited by its small sample size. Nonetheless, I was still able to find a range of abilities and investigate the mathematical understanding of high and low students. Similar findings were seen in my pilot study and may be typical with respect to preservice teachers at the methods level of their education. The small sample size in my study did allow for in-depth interviews, the kind that would be impractical in a much larger study. Larger quantitative studies could be designed using a testing instrument similar to the one used in my study to investigate the mathematical understanding of preservice teachers in different types of institutions, for example public versus private.

The institution used in this study requires two mathematics methods courses for its secondary mathematics teaching majors. Four out of the five participants interviewed in my study were in their second methods course. It may have given me a more accurate view if I had also interviewed an equal number of participants who were in their first methods course. The similarities between the two participants in the low group (one was in his first methods course and the other was in her second methods course), however, led me to believe that their lack of understanding was not dependent upon the number of methods courses they had taken. In retrospect, I think it was fortunate that I chose a participant from the low group who was also in the second methods course. If both low-group participants had been in their first methods course, it might have been tempting to
place too much credit on the methods course for increasing student understanding. My results do corroborate results found elsewhere in the literature, but additional study with a more varied population would be worthwhile. Comparisons could also be made by looking for differences in mathematical proficiency of participants in programs requiring one versus two methods courses. Additional studies looking at the mathematical content knowledge of secondary preservice teachers with similar programs would also assist in validating results.

Content Course Delivery

Mathematical content courses vary in the ways they are delivered. A study comparing traditionally-taught mathematics content courses and content courses taught in a more standards-based (NCTM, 2000) approach might shed light on how students can best achieve mathematical proficiency. Looking at how content courses are delivered may give us insight into practices that may deepen the mathematical understanding that students develop. It would be interesting to determine what distinguishes a highly effective content course from one that is less effective in developing understanding. From talking with the participants, it was apparent that being able to explain one’s thinking aids in deepening the mathematical understanding. This suggests to me that content courses in which value is placed on explaining one’s reasoning in addition to finding an answer would be more effective in student learning than a class that merely values accurate computation.
Connections Course

Students often complain that they do not see the connections between their advanced mathematics content courses and the topics taught within the high school curriculum. Conference Board of the Mathematical Sciences (CBMS, 2001) sees the value in making these connections and recommends a capstone course that would make connections between the college mathematics courses taken and the high school mathematics they will be teaching.

Whether these connections are made in a methods course or a capstone course, the preservice teachers would benefit from actively finding those connections. As is always the problem with adding a new course or even just new content, consideration needs to be given to where to insert a course, or part of a course, that would explore these connections in enough depth to make it worthwhile for the students. Most college majors are already at their limit for credit hours and adding additional hours could make the major less appealing. We cannot allow that to happen since secondary mathematics teachers are already in short supply. Instead of adding new courses, perhaps we should look at the focus of our current mathematics methods courses. These methods courses might be an appropriate place to look at the connections between the mathematics content courses and the mathematics courses commonly taught in high school. It would be worthwhile to study the effectiveness of this type of course in increasing mathematical understanding.
Modeling Standards-based Teaching

Listening to concerns raised by preservice teachers as they embark on student teaching raises awareness that they are feeling less prepared than they would like to be. Ideally, preservice teachers would see standards-based teaching in their content and methods courses since teachers are likely to teach as they have been taught. Unfortunately, this is not the case, especially in secondary mathematics content courses. Modeling communication in the classroom and drawing connections among mathematical topics facilitates the deepening of student understanding of how to teach (NCTM, 2000).

Further, during student teaching cooperating teachers should model standards-based teaching in their own classrooms. The student teaching experience has the potential to reinforce what the preservice teacher has learned or, unfortunately, it can also undermine what was taught as best practice. It may be the case that an effective cooperating teacher is crucial to the student teaching experience, but finding these highly qualified cooperating teachers with whom to place preservice teachers is not always possible. It would be advantageous to find classroom teachers who would be willing to undergo specialized training to become highly qualified cooperating teachers who connect concepts and skills in a non-routine problem solving curriculum and use accurate mathematical language in order to increase the likelihood of having model placements for our student teachers. A study to see how modeling standards-based teaching in the classroom and/or during student teaching affects a preservice teacher's effectiveness in the classroom would also be worthwhile.
A somewhat related Professional Development School pilot study is being conducted at the institution where my study originated. This study is offering professional development to teachers who host field experience students from the university. This project is multi-beneficial in that inservice teachers receive professional development to become more effective teachers, field experience students receive mentoring from these teachers, and the university is increasing the number of teachers who are willing and qualified to be mentors for field experience students.

**Longitudinal Study**

This study reveals a snapshot of what secondary preservice teachers knew about Euclidean geometry when they were enrolled in a mathematics methods course. A longitudinal study documenting the mathematical understanding of preservice teachers as they progress from novice to experienced teachers could reveal factors that lead to a deeper understanding of the mathematical content that is being taught or whether the teachers merely retain the knowledge they acquired in their undergraduate studies. This longitudinal study could also look at student achievement in relationship to the teacher’s growth in understanding.

**Parallel Studies**

A parallel study of the mathematical understanding of elementary education preservice teachers who have a mathematics minor (area of concentration) might give insights into the effectiveness of the ways that content courses are delivered. Since secondary preservice teachers complete a mathematics major, their content courses are usually taught by mathematics content specialists and not necessarily by persons who
have studied teaching strategies. Often elementary preservice teachers receive instruction in their mathematics minor courses from mathematics educators whose interests include effective teaching strategies. This type of study would reveal the mathematical understanding of elementary education preservice teachers. Then a comparative study of the mathematical understanding of elementary education preservice teachers with mathematics minors versus the mathematical understanding of secondary preservice teachers for similar content areas could also be completed.

My study focused on the content area of geometry. A parallel study in the content area of data, probability, and statistics would also be informative. Data, probability, and statistics is another under-researched content area in mathematics but one that is worthy of study. This content area is one that is often neglected in both elementary schools and high schools, yet it is of crucial importance, both as a building block for higher mathematics and statistics, and for understanding mathematical information encountered in media in daily life.

**Summary**

This study confirmed results found in other studies of secondary preservice teachers in regard to the lack of conceptual understanding exhibited by some participants. This study looked Euclidean geometry, an area where performance of U.S. students is shown to be inadequate, and results suggest that some preservice teachers remain deficient in this area. Procedural fluency was tested in this study and found to vary among the participants. A larger scale study might help clarify whether the error patterns seen were due to not understanding the necessary procedures or if they were simply a
result of carelessness. It does appear, however, that these two strands of mathematical proficiency are being addressed in undergraduate education.

Strategic competence and adaptive reasoning were seen in varying degrees in each of the participants. The participants in this study all seemed to have the ability to work efficiently with problems involving the Pythagorean Theorem and the distance formula. The remaining topics included in the test, such as triangle similarity, properties of quadrilaterals, and constructions, were less skillfully completed. The strands of strategic competence and adaptive reasoning seem to be receiving less attention but are certainly areas that need to receive consideration by mathematics educators. If we expect to produce secondary mathematics teachers who will be able to implement high quality instruction, they must be given the opportunities to experience teaching that stresses the importance of making connections, using multiple strategies, and drawing on previous knowledge to solve non-routine problems. They should no longer be satisfied knowing how to do mathematics, but must insist upon knowing about mathematics (Mason & Spence, 1999).

Concluding Remarks

What I have learned from this study is what it really means to be proficient. When I began the study using the framework from Adding It Up (NRC, 2001), I just accepted the authors’ definition of mathematical proficiency in terms of the five intertwining strands: conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition. The meaning of each strand made sense to me and I could see how the strands were related and interconnected. It wasn’t until I was near the
completion of the write-up of my dissertation that I fully understood how accurate and all encompassing this description of mathematical proficiency is. It wasn’t until I was able to use my own adaptive reasoning, apart from mathematics, and apply these strands to proficiency in different areas that I realized the significance of each strand.

I began thinking about the development in my proficiency regarding my dissertation—a project that was insurmountable at the beginning. Looking at my progression through this writing project, I clearly see each of the five strands of proficiency in various stages of development. To begin, conceptual understanding is essential because without that, we have no place to begin. I needed to understand the make-up of the dissertation—the chapters that must be included and their importance to a cohesive document. I considered procedural fluency to be the mechanics necessary for writing—spelling, grammar, sentence structure. Once the ideas and concepts are in place, procedural fluency can take over.

Equating this to an example in mathematics follows: once a formula is presented, numbers can be plugged in, procedures followed, and an answer is found. I remember my frustration, however, when I did not have the conceptual understanding that would allow me to proceed. This is much like a student who becomes frustrated when trying to do mathematics problems without the necessary conceptual background.

The other strands, not talked about as much in the literature, appear in more complex situations—like the writing of a dissertation. Strategic competence is part of problem solving. In my dissertation, for example, I have all this rich data. What do I do with it? What parts of it will best illustrate and support the points I am trying to make
with the readers? We then cycle back to the conceptual understanding and procedural fluency strands to get these points expressed in a meaningful way.

Adaptive reasoning seems to be a key component in this problem solving process. With adaptive reasoning we can look back at our prior experiences and see how we can use those experiences in this new situation. In fact, I am using adaptive reasoning now by applying what I have learned about mathematical proficiency to this example of dissertation proficiency. Please note that I realize I have not reached the goal of dissertation proficiency. This dissertation has merely given me opportunities to begin to develop the strands that could lead towards that proficiency. Likewise, mathematical proficiency is a goal for which we should strive.

The final strand, not formally addressed in this dissertation, is certainly an important aspect of proficiency. Without seeds of productive disposition—perseverance, self-efficacy, a feeling of purpose—this dissertation would not be completed. As I reflect over my experiences in completing this dissertation and especially remembering the difficulties I encountered, I must remind myself of similar difficulties students have when learning mathematics.

It is abundantly clear to me that we need to present opportunities in our classrooms that will allow our students access to all five strands that together lead toward mathematical proficiency. We must not be satisfied when our students exhibit only conceptual and procedural knowledge, but rather we need to make sure they are willing and able to use that knowledge strategically and adapt it to new and different situations. How we accomplish this, however, is a matter for future study. Personally, though, when
I am teaching, I will be more aware of the components that are necessary to progress towards mathematical proficiency and try to provide opportunities and experiences in my classroom that will set the stage for my students’ continued growth and deepening of their mathematical understanding.
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APPENDIX A: GEOMETRY ASSESSMENT

Test Number ______
Please do not put your name on this test. Show any work that was necessary to complete the problem.

1. Which of the following constructions at point P in the figure below will produce an angle of 45°?

A) Constructing only the bisector of ∠ APB
B) Constructing only a circle with center at P
C) Constructing one perpendicular line only
D) Constructing a circle with center at P and a perpendicular line
E) Constructing a perpendicular line and an angle bisector

2. Point O is the center of the circle below. Line segment AC is a diameter of the circle. Line segment BC does not pass through the center of the circle. Which of the following is true?
A) AC is longer than BC
B) BC is longer than AC
C) AC and BC are the same length
D) BC is twice as long as OA
E) Not enough information is given

3. In the figure below, the two triangles are similar. What is the value of x?

4. What is the distance between the points (2, 10) and (-4, 2) in the xy-plane?
5. In the space provided, use a ruler to draw a parallelogram that has perpendicular diagonals. Show the diagonals in your sketch.

6. If the triangles ADE and ABC shown in the figure below are similar, what is the value of x?

When the open box shown above is cut along the four darkened edges and then flattened, the result is shown below.

7. Using the grid paper, draw two flattened boxes that will fold up into different open boxes. Each box should have a volume of 8 cubic units. Be sure to label your drawings with numbers that show the length, width, and height for each box. Each square on the grid has a side length 1.
8. The sum of the measures of angles 1 and 2 in the figure below is 90°. What is the measure of the angle formed by the bisectors of these two angles?

9. Radio station KMAT in Math City is 200 miles from radio station KGEO in Geometry City. Hwy 7, a straight road, connects the two cities. KMAT broadcasts can be received up to 150 miles in all directions from the station and KGEO broadcasts can be received up to 125 miles in all directions. Radio waves travel from each radio station through the air, as represented below.

Draw a diagram that shows the following:
- Hwy 7
- The location of the two radio stations
- The part of Hwy 7 where both radio stations can be received

Be sure to label the distances along the highway and the length in miles of the part of the highway where both stations can be received.
10. In the figure below, ABDG is a parallelogram and CDEF is a rectangle. If EF = 9 and CG = 10, what is AB to the nearest hundredth?

11. Jaime knows the following facts about points A, B, and C.
   - Points A, B, and C are on the same line, but might not be in that order.
   - Point C is twice as far from point A as it is from point B.
Jaime concluded that point C is always between points A and B. Is Jaime’s conclusion correct? In the space provided, use a diagram to explain your answer.

12. In the figure below, points A, E, and H are on a plane that intersects a right prism. What is the intersection of the plane with the right prism?
   A) A line
   B) A triangle
   C) A quadrilateral
   D) A pentagon
   E) A hexagon
13. In the xy-plane, a line parallel to the x-axis intersects the y-axis at the point (0, 4). This line also intersects a circle in two points. The circle has a radius of 5 and its center is at the origin. What are the coordinates of the two points of intersection?

14. A certain 4-sided figure has the following properties:
- Only one pair of opposite sides are parallel
- Only one pair of opposite sides are equal in length
- The parallel sides are not equal in length
Which of the following must be true about the sides that are equal in length?

A) They are perpendicular to each other
B) They are each perpendicular to an adjacent side
C) They are equal in length to one of the other two sides
D) They are not equal in length to either of the other two sides
E) They are not parallel

15. The endpoints of a line segment are the points with coordinates (2, 1) and (8, 9). What are the coordinates of the midpoint of this line segment?
Your gender  Male / Female

Your GPA  ______

Would you be willing to participate in a follow-up interview?  Yes / No

Please rank order 1-5 (with 1 being your first choice) the five middle school and/or high school mathematics courses you would most like to teach.

____ 7th grade mathematics
____ 8th grade mathematics
____ Pre-Algebra
____ General mathematics
____ Consumer mathematics
____ Algebra I
____ Geometry
____ Algebra II
____ Trigonometry
____ Pre-Calculus
____ Statistics
____ Calculus
____ Computer programming
____ Other ____________________
APPENDIX B: INTERVIEW QUESTIONS

1. In question #1, you knew that you would have to construct a perpendicular line and an angle bisector in order to construct an angle of 45 degrees. I have some construction tools available here and I would like you to show me how to actually construct that 45 degree angle through point P.

How do you know that the line is perpendicular?

How do you know that the angle is bisected?

2. Question #4—Can you tell me why you decided to do the problem this way?

What other strategies could you use other than the one you tried?

3. Question #5—When you did this problem, can you tell me what you were thinking about as you were attempting to draw this parallelogram.

Can you convince me that the diagonals are perpendicular?

Can you convince me that this figure is a parallelogram?

4. Question #13—Please explain how you solved this problem.

Is there another way you could solve this problem?

5. Question #15—Can you show me how you found your answer?

Is there another way you could have found the answer?

6. This is a question that was not on the questionnaire. Please solve this problem and then tell me how you decided where and how to place the letters.